Contents

	Fore	word, by Henk Barendregt	page xiii
	Prefe	ace	XV
	Ackn	nowledgements	xxvii
	Gree	$k \ alphabet$	xxviii
1	Unt	yped lambda calculus	1
	1.1	Input-output behaviour of functions	1
	1.2	The essence of functions	2
	1.3	Lambda-terms	4
	1.4	Free and bound variables	8
	1.5	Alpha conversion	9
	1.6	Substitution	11
	1.7	Lambda-terms modulo α -equivalence	14
	1.8	Beta reduction	16
	1.9	Normal forms and confluence	19
	1.10	Fixed Point Theorem	24
	1.11	Conclusions	26
	1.12	Further reading	27
	Exer	cises	29
2	Simply typed lambda calculus		33
	2.1	Adding types	33
	2.2	Simple types	34
	2.3	Church-typing and Curry-typing	36
	2.4	Derivation rules for Church's $\lambda \rightarrow$	39
	2.5	Different formats for a derivation in $\lambda \rightarrow$	44
	2.6	Kinds of problems to be solved in type theory	46
	2.7	Well-typedness in $\lambda \rightarrow$	47
	2.8	Type Checking in $\lambda \rightarrow$	50
	2.9	Term Finding in $\lambda \rightarrow$	51

viii Contents

	2.10	General properties of $\lambda \rightarrow$	53
	2.11	Reduction and $\lambda \rightarrow$	59
	2.12	Consequences	63
	2.13	Conclusions	64
	2.14	Further reading	65
	Exer	cises	66
3	Seco	ond order typed lambda calculus	69
	3.1	Type-abstraction and type-application	69
	3.2	Π -types	71
	3.3	Second order abstraction and application rules	72
	3.4	The system $\lambda 2$	73
	3.5	Example of a derivation in $\lambda 2$	76
	3.6	Properties of $\lambda 2$	78
	3.7	Conclusions	80
	3.8	Further reading	80
	Exer	cises	82
4	Types dependent on types		85
	4.1	Type constructors	85
	4.2	Sort-rule and var-rule in $\lambda \underline{\omega}$	88
	4.3	The weakening rule in $\lambda \underline{\omega}$	90
	4.4	The formation rule in $\lambda \underline{\omega}$	93
	4.5	Application and abstraction rules in $\lambda \underline{\omega}$	94
	4.6	Shortened derivations	95
	4.7	The conversion rule	97
	4.8	Properties of $\lambda \underline{\omega}$	99
	4.9	Conclusions	100
	4.10	Further reading	100
	Exer	cises	101
5	\mathbf{Typ}	es dependent on terms	103
	5.1	The missing extension	103
	5.2	Derivation rules of λP	105
	5.3	An example derivation in λP	107
	5.4	Minimal predicate logic in λP	109
	5.5	Example of a logical derivation in λP	115
	5.6	Conclusions	118
	5.7	Further reading	119
	Exer	cises	121
6	The Calculus of Constructions		123
	6.1	The system λC	123
	6.2	The λ -cube	125

		Contents	ix
	6.3	Properties of λC	128
	6.4	Conclusions	132
	6.5	Further reading	133
	Exer	cises	134
7	The	encoding of logical notions in λC	137
	7.1	Absurdity and negation in type theory	137
	7.2	Conjunction and disjunction in type theory	139
	7.3	An example of propositional logic in λC	144
	7.4	Classical logic in λC	146
	7.5	Predicate logic in λC	150
	7.6	An example of predicate logic in λC	154
	7.7	Conclusions	157
	7.8	Further reading	159
	Exer	cises	162
8	Defi	nitions	165
	8.1	The nature of definitions	165
	8.2	Inductive and recursive definitions	167
	8.3	The format of definitions	168
	8.4	Instantiations of definitions	170
	8.5	A formal format for definitions	172
	8.6	Definitions depending on assumptions	174
	8.7	Giving names to proofs	175
	8.8	A general proof and a specialised version	178
	8.9	Mathematical statements as formal definitions	180
	8.10	Conclusions	182
	8.11	Further reading	183
	Exer	cises	185
9	Exte	ension of λC with definitions	189
	9.1	Extension of λC to the system λD_0	189
	9.2	Judgements extended with definitions	190
	9.3	The rule for adding a definition	192
	9.4	The rule for instantiating a definition	193
	9.5	Definition unfolding and δ -conversion	197
	9.6	Examples of δ -conversion	200
	9.7	The conversion rule extended with $\stackrel{\Delta}{\rightarrow}$	202
	9.8	The derivation rules for λD_0	203
	9.9	A closer look at the derivation rules of λD_0	204
	9.10	Conclusions	206
	9.11	Further reading	207
	Exer	cises	208

x Contents

10	Rules and properties of λD	211
	10.1 Descriptive versus primitive definitions	211
	10.2 Axioms and axiomatic notions	212
	10.3 Rules for primitive definitions	214
	10.4 Properties of λD	215
	10.5 Normalisation and confluence in λD	219
	10.6 Conclusions	221
	10.7 Further reading	221
	Exercises	223
11	Flag-style natural deduction in λD	225
	11.1 Formal derivations in λD	225
	11.2 Comparing formal and flag-style λD	228
	11.3 Conventions about flag-style proofs in λD	229
	11.4 Introduction and elimination rules	232
	11.5 Rules for constructive propositional logic	234
	11.6 Examples of logical derivations in λD	237
	11.7 Suppressing unaltered parameter lists	239
	11.8 Rules for classical propositional logic	240
	11.9 Alternative natural deduction rules for \vee	243
	11.10 Rules for constructive predicate logic	246
	11.11 Rules for classical predicate logic	249
	11.12 Conclusions	252
	11.13 Further reading	253
	Exercises	254
12	Mathematics in λD : a first attempt	257
	12.1 An example to start with	257
	12.2 Equality	259
	12.3 The congruence property of equality	262
	12.4 Orders	264
	12.5 A proof about orders	266
	12.6 Unique existence	268
	12.7 The descriptor ι	271
	12.8 Conclusions	274
	12.9 Further reading	275
	Exercises	276
13	Sets and subsets	279
	13.1 Dealing with subsets in λD	279
	13.2 Basic set-theoretic notions	282
	13.3 Special subsets	287
	13.4 Relations	288

Contents	xi
----------	----

	13.5 Maps	291
	13.6 Representation of mathematical notions	295
	13.7 Conclusions	296
	13.8 Further reading	297
	Exercises	302
14	Numbers and arithmetic in λD	305
	14.1 The Peano axioms for natural numbers	305
	14.2 Introducing integers the axiomatic way	308
	14.3 Basic properties of the 'new' N	313
	14.4 Integer addition	316
	14.5 An example of a basic computation in λD	320
	14.6 Arithmetical laws for addition	322
	14.7 Closure under addition for natural and negative numbers	324
	14.8 Integer subtraction	327
	14.9 The opposite of an integer	330
	14.10 Inequality relations on \mathbb{Z}	332
	14.11 Multiplication of integers	335
	14.12 Divisibility	338
	14.13 Irrelevance of proof	340
	14.14 Conclusions	341
	14.15 Further reading	343
	Exercises	344
15	An elaborated example	349
	15.1 Formalising a proof of Bézout's Lemma	349
	15.2 Preparatory work	352
	15.3 Part I of the proof of Bézout's Lemma	354
	15.4 Part II of the proof	357
	15.5 Part III of the proof	360
	15.6 The holes in the proof of Bézout's Lemma	363
	15.7 The Minimum Theorem for \mathbb{Z}	364
	15.8 The Division Theorem	369
	15.9 Conclusions	371
	15.10 Further reading	373
	Exercises	376
16	Further perspectives	379
	16.1 Useful applications of λD	379
	16.2 Proof assistants based on type theory	380
	16.3 Future of the field	384
	16.4 Conclusions	386
	16.5 Further reading	387

xii Contents

$Appendix A$ Logic in λD	391
A.1 Constructive propositional logic	391
A.2 Classical propositional logic	393
A.3 Constructive predicate logic	395
A.4 Classical predicate logic	396
$Appendix \ B$ Arithmetical axioms, definitions and lemmas	397
Appendix C Two complete example proofs in λD	403
C.1 Closure under addition in \mathbb{N}	403
C.2 The Minimum Theorem	405
$Appendix D$ Derivation rules for λD	409
References	411
Index of names	419
Index of definitions	421
Index of symbols	423
Index of subjects	425