Bounds on Optimally Triangulating Connected Subsets of the Minimum Weight Convex Partition

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Abstract

Given a set S of n points, we show that the length of
1) the minimum weight triangulation (MWT) of the minimum weight convex partition ($MWCP$) of S (T_{MWCP}) is at most $\Theta(n)$ longer than the MWT of S if collinearity of two or more edges is allowed and $\Theta(\log n)$ otherwise,
2) the MWT of the minimum spanning tree (MST) of the $MWCP$ of S ($T_{mst(MWCP)}$) is at most $\Theta(n)$ longer than the MWT of S if collinearity of two or more edges is allowed and $\Theta(\log n)$ otherwise,
3) the MWT of any connected subset G of the $MWCP$ of S ($T_{MWCP(G)}$) is at most $\Theta(n)$ longer than the MWT of S if collinearity of two or more edges is allowed.

1 Introduction

A triangulation of a set S of n points in the plane is a maximal set of non-intersecting edges connecting the points in S. The minimum weight triangulation MWT of S is a triangulation of minimum total edge length. It is unknown whether the MWT problem is NP-complete or solvable in polynomial time [2].

However, since the MWT of a simple polygon can be found in $O(n^3)$ time [3], it sounds reasonable to approximate the MWT of a point set by first connecting the set of points into a single component (a polygon). If the polygon is convex and no three vertices are collinear, a triangulation of weight $O(\log n)$ times the polygon's perimeter can be found by the *ring heuristic* of repeatedly connecting every second vertex [8]. Using this heuristic and a complicated method to partition the input into convex polygons, it was shown in [9] that a triangulation of $O(\log n)$ times the MWT length can be achieved.

Notation: We use the following abbreviations:

$MWCP$:	minimum weight convex partition
T_{MWCP}:	MWT of the $MWCP$
$mst(MWCP)$:	minimum spanning tree (MST) of the $MWCP$
$T_{mst(MWCP)}$:	MWT of $mst(MWCP)$
$MWCP(G)$:	a connected subset of the $MWCP$
$T_{MWCP(G)}$:	MWT of the $MWCP(G)$

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New Results:
1) The length of the T_{MWCP} of S is at most $\Theta(n)$ greater than the MWT of S if collinearity of two or more edges is allowed and $\Theta(\log n)$ otherwise.
2) The length of the $T_{mst(MWCP)}$ of S is at most $\Theta(n)$ greater than the MWT of S if collinearity of two or more edges is allowed and $\Theta(\log n)$ otherwise.
3) The length of the $T_{MWCP(G)}$ of S is at most $\Theta(n)$ greater than the MWT of S if collinearity of two or more edges is allowed.

2 Tight Bounds on T_{MWCP} and MWT of S

Theorem 1 *For any $n \geq 9$, there is a set S of n points in the plane, such that the T_{MWCP} of S can be $\Theta(n)$ longer than the MWT of S if collinearity of three or more vertices is allowed.*

Proof. For the lower bound we consider the set S of n points in Figure 1. S is symmetric and compressed w.r.t. the y-axis by a larger factor than shown in Figure 1 s.t. each diagonal between the convex hull pieces from v_7 to v_* and from v_{*+1} to v_n is of length at most $\frac{1}{n^2}$. The length of the diagonal connecting v_3 to v_4 is 1 and the length of the diagonals between (v_1, v_3), (v_2, v_3), (v_4, v_5), (v_4, v_6) are $\frac{1}{n}$. Consequently, the diagonals between (v_3, v_7), (v_3, v_{*+1}), (v_*, v_4), (v_4, v_n) have a length of about $\frac{1}{2}$ each for larger n.

The *only single* diagonal that can eliminate concavity at v_3 and v_4 after the insertion of diagonals between (v_1, v_3), (v_2, v_3), (v_4, v_5), (v_4, v_6) is the diagonal from v_3 to v_4. Let C be the convex hull piece from v_7 to v_*, and C' be the convex hull piece from v_{*+1} to v_n. (C and C' are straight lines.) An alternative elimination of the concavity at v_3 (resp. v_4) after the insertion of the diagonals (v_1, v_3), (v_2, v_3) (resp. (v_4, v_5), (v_4, v_6)) is to insert two diagonals, one from v_3 (resp. v_4) to a vertex on C, and the other from v_3 (resp. v_4) to a vertex on C'.

An $MWCP$ algorithm will always choose the diagonal between (v_3, v_4) of length 1 and the diagonals between (v_1, v_3), (v_2, v_3), (v_4, v_5), (v_4, v_6), since they give the minimum edge length convex partition. Including the convex hull CH of length about 2. This the total length of this convex partition is approximately $3 + \frac{4}{n}$. Any alternative convex partition which inserts two edges incident to v_3 and to v_4 results in an edge length of $(4 \pm \epsilon) + \frac{4}{n}$.

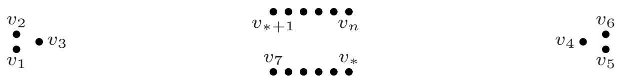

Figure 1: An approximate illustration of the set of points which shows the lower bound.

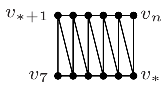

Figure 2: Approximate illustration of an optimal triangulation of area bounded by C and C'.

The T_{MWCP} of S includes the optimal triangulation of the sub-polygon q containing the vertices $(v_1, v_3, v_4, v_5, v_*, \ldots, v_7)$ and its symmetric counterpart q' containing vertices $(v_2, v_3, v_4, v_6, v_n, \ldots, v_{*+1})$. The sub-polygon q is triangulated by adding edges between v_3 and vertices on C and/or edges between v_4 and vertices on C. Each of these edges has a length of approximately $\frac{1}{2}$. For larger n there are about $\frac{n}{2}$ vertices on C (there are at least $\frac{2n}{9}$ vertices on C, since $n \geq 9$). Thus the total length of the edges needed to triangulate q is $\frac{1}{2} \cdot \frac{n}{2} = \frac{n}{4}$, and since q' is symmetric to q, the total length of the edges needed to triangulate both q and q' is $2\frac{n}{4} = \frac{n}{2}$. Adding the total edge length $3 + \frac{4}{n}$ for the $MWCP$ of S obtained above, we have that the T_{MWCP} of S has a total edge length of approximately $\frac{n}{2}$, for larger n.

The MWT of S, however, includes the diagonals between the convex hull CH, (v_1, v_3), (v_2, v_3), (v_4, v_5), (v_4, v_6), (v_3, v_7), (v_3, v_{*+1}), (v_*, v_4), (v_4, v_n) and diagonals going between C and C'. The optimal triangulation T of the area bounded by C and C' approaches zero for larger n, because each edge going between C and C' in T has length at most $\frac{1}{n^2}$ and there are $O(n)$ edges (see Figure 2). The MWT of S thus has a total edge length of about 4 for larger n. Hence $\frac{|T_{MWCP}|}{|MWT|} \approx \frac{n}{8}$.

For the upper bound, we draw on a result in [4], where it was shown that for a point set S any triangulation achieves a total edge length $O(n)$ times the MWT of P. Therefore the $\Theta(n)$ bound is tight. □

Theorem 2 *For any n, there is a set S of n points in the plane, such that the T_{MWCP} of S can be $\Theta(\log n)$ longer than the MWT of S if collinearity of three or more vertices is disallowed.*

Proof. To show the lower bound, we modify the set S of points in Figure 1 such that (1) on the convex hull piece C from v_7 to v_* the vertices lie on a circular arc so that no three vertices are collinear, likewise on the convex hull piece C' from v_{*+1} to v_n; (2) each edge between adjacent vertices on C and C' has length $\frac{1}{n}$. C and C' are both of length about 0.3; (3) the distance from each v on C (resp. C') to the closest vertex on C' (resp. C) is at most $\frac{1}{n^2}$; (4) the diagonals between (v_3, v_7), (v_3, v_{*+1}), (v_*, v_4), (v_4, v_n) have length of about 0.35 each.

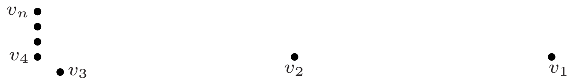

Figure 3: An approximate illustration of a point set S of points showing the lower bound.

An $MWCP$ algorithm always chooses the diagonal between (v_3, v_4) of length 1 and the diagonals between (v_1, v_3), (v_1, v_3), (v_4, v_5), (v_4, v_6), since they give the minimum edge length convex partition (similar explanation as in the proof of Theorem 1).

The T_{MWCP} of S includes the triangulation of the convex sub-polygon q containing vertices $(v_1, v_3, v_4, v_5, v_*, \ldots, v_7)$ and its symmetric counterpart q' containing vertices $(v_2, v_3, v_4, v_6, v_n, \ldots, v_{*+1})$. From [5, 6] we know that the *greedy triangulation*[1] of a convex polygon P is an $O(1)$ approximation of the MWT of P. The greedy triangulation of the $MWCP$ of S adds the diagonals between (v_7, v_*) and (v_{*+1}, v_n) before the diagonals (v_3, v_7) and (v_3, v_{*+1}) (resp. (v_*, v_4) and (v_4, v_n)) in q (resp. q'). The sub-polygon containing the circular arc C (resp. C') and the diagonal between (v_*, v_7) (resp. (v_{*+1}, v_n)) is referred to as a *semi-circular polygon* in [7]. [7] showed that the MWT of such semi-circular polygons has length $\Theta(\log n)$ times its perimeter. Thus the triangulations of such resulting sub-polygons have length $\Theta(\log n)$ plus the length of the perimeters of the sub-polygons. The length of the MWT of the two semi-circular polygons of S is $\Theta(\log n)$ (since the greedy triangulation of P is $O(1)$ of the MWT of P). Thus the total edge length of the T_{MWCP} of S is $\Omega(\log n)$.

A much shorter triangulation of S includes the diagonals between the vertices stated for the MWT in the proof of Theorem 1, giving a total edge length of at most $O(1)$. Using results from [9, 8] it can be deduced that given a set S (disallowing collinearity) partitioning the region of the plane enclosed by the CH of S into convex polygons one can achieve an $O(\log n)$ approximation to the MWT by triangulating the convex polygons. Therefore the $\Theta(\log n)$ bound is tight. □

3 Tight Bounds on $T_{mst(MWCP)}$ and MWT of S

Theorem 3 *For any $n > 0$, there exists a set S of n points for which the length of the $T_{mst(MWCP)}$ of S can be $\Theta(n)$ times the length of the MWT of P if collinearity of three or more vertices is allowed.*

[1]The greedy triangulation is obtained by repeatedly adding the shortest edge that does not lead to an intersection.

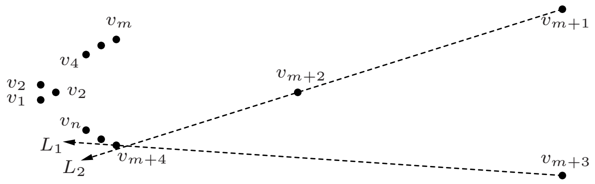

Figure 4: An approximate illustration showing the lower bound for the $T_{mst(MWCP)}$ and MWT ratio.

Proof. Consider the set S of $n \geq 10$ points in Figure 3 and let the distances between pairs of vertices be $d(v_1, v_2) = 1.2$, $d(v_2, v_n) = 1.2$, $d(v_1, v_n) = 2.4$, $d(v_2, v_4) = 1$, $d(v_1, v_3) = 2.3$, $d(v_2, v_3) > 1$, $d(v_3, v_4) \leq \frac{1}{n+2}$, $d(v_n, v_4) \leq \frac{1}{n+1}$, $d(v_n, v_3) \leq \frac{1}{n}$. We observe that the $MWCP$ includes the convex hull of S and the edges (v_1, v_2), (v_2, v_4), since the total edge length of this partition is minimum (concavity at v_2 is removed, since v_1, v_2, v_4 are collinear).

The $mst(MWCP)$ includes all edges in the $MWCP$ of S except (v_1, v_n) and (v_1, v_3). There are at most $n - 5$ vertices between v_n and v_4, and each (including v_n) is connected to the vertex v_2 by an edge of length about 1.2 in the $T_{mst(MWCP)}$.

However, the MWT includes edges (v_2, v_n), (v_2, v_3) and edges from v_3 to each of the vertices from v_5 and v_n. Each of the $n - 5$ edges from v_3 to vertices $(v_5, v_6, \ldots, v_n)$ has length at most $\frac{1}{n}$. The total length of the MWT of S is $O(1)$. Thus the ratio of the lengths of the $T_{mst(MWCP)}$ and the MWT is $\Omega(n)$. This proves a lower bound for the above problem.

For the upper bound we know that every triangulation has length $O(n)$ times the optimum (MWT) [4, 1]. Therefore the $\Theta(n)$ bound is tight. □

Theorem 4 *For any $n > 0$, there exists a set S of n points for which the length of the $T_{mst(MWCP)}$ of S can be $\Theta(\log n)$ times the length of the MWT of S if collinearity of three or more vertices is disallowed.*

Proof. We construct a set S of n points, $n \geq 15$, which is sketched in Figure 4. We assume that S is compressed w.r.t. the y-axis s.t. the y-coordinate of each point is multiplied by $\frac{1}{n^2}$ and S has the following properties: (1) All vertices except v_3 and v_{m+2} (which lie on the x-axis) lie on the convex hull CH. (2) On the CH the vertices $v_{m+4}, v_{m+5}, \ldots, v_n$ lie on a circular arc. (3) Let $\delta(u, v)$ denote the *vertical distance* between any two given vertices u and v, and $d(u, v)$ the *distance* between u and v. Then $d(v_1, v_2) = \frac{1}{2n}$, $d(v_1, v_3) = \frac{2}{n}$, $\delta(v_3, v_n) = 0.2$, $\delta(v_n, v_{m+4}) = 0.1$ $\delta(v_{m+4}, v_{m+2}) = 0.7$, $d(v_3, v_{m+2}) = \delta(v_3, v_{m+2}) = 1$ $\delta(v_{m+2}, v_{m+3}) > 1$, $d(v_{m+3}, v_{m+1}) = 2$. The shortest diagonal from any vertex on the convex hull piece (v_n, v_{m+4}) to any vertex on the convex hull piece (v_4, v_m) is no longer than 1. (4) Let L_1 (see Figure 4) be the half-line extension of v_{m+1} to v_{m+2} and L_2 the half-line extension of v_{m+3} to v_{m+4}. The vertices v_{m+4} to v_n lie above the intersection of L_1 and L_2.

In any convex partition of S: The two diagonals (v_{m+2}, v_{m+1}) and (v_{m+2}, v_{m+3}) must both be present to remove the concavity at v_{m+2}, because even adding all remaining diagonals incident to v_{m+2} does not remove the concavity at v_{m+2} (this follows from property 4 above). To remove the concavity at v_3, if the diagonal (v_3, v_{m+2}) is added, at least two other diagonals incident to v_3 are needed. However, if the diagonal (v_3, v_{m+2}) is not added, either (v_1, v_3) or (v_2, v_3) together with at least two other diagonals are needed, namely one incident to v_3 and going to the left of v_3 and one incident to v_3 and going to the right of v_3.

To find the $MWCP$ of S, inserting the diagonal (v_3, v_{m+2}) removes the concavity at v_{m+2} and requires inserting either (v_1, v_3) and (v_2, v_3) to remove the concavity at v_3 giving a total edge length of $1 + \frac{4}{n}$. If we do not, however, add the diagonal (v_3, v_{m+2}), a possible solution is the insertion of either (v_{m+2}, v_{m+4}) or (v_{m+2}, v_m) at v_{m+2} and (v_n, v_3), (v_3, v_4) and either (v_1, v_3) or (v_2, v_3) to remove the concavity at v_3 giving a total edge length of $0.7 + 0.2 + 0.2 + \frac{2}{n}$. We conclude that the $MWCP$ of S includes the convex hull of S and edges (v_1, v_3), (v_2, v_3), (v_3, v_{m+2}), (v_{m+2}, v_{m+3}) and (v_{m+2}, v_{m+1}).

An $mst(MWCP)$ of S includes the edges in the $MWCP$ of S with the exception of (v_{m+4}, v_{m+3}), (v_m, v_{m+1}) and either (v_{m+2}, v_{m+1}) or (v_{m+2}, v_{m+3}).

The greedy triangulation of any $mst(MWCP)$ of S includes the triangulation of the so-called *semi-circular polygon* [7] bounded by the convex hull piece C from v_n to v_{m+4} and the edge (v_n, v_{m+4}), as well as its symmetric counterpart the convex hull piece C' from v_4 to v_m and the edge (v_4, v_m). In [7] it was shown that the MWT of such regular semi-circular polygons has length $\Theta(\log n)$ times its perimeter. Since the greedy triangulation is an $O(1)$ approximation of the MWT for any convex polygon [5, 6], the total edge length of the T_{msp} of S must be also $\Omega(\log n)$.

A shorter triangulation T of S includes the convex hull of S, the edges (v_1, v_3), (v_2, v_3), (v_3, v_4), (v_3, v_n), (v_{m+2}, v_{m+4}), (v_{m+2}, v_m), (v_{m+2}, v_{m+4}), (v_{m+2}, v_{m+1}), and edges from the triangulation of the area bounded by C, C', (v_4, v_n), (v_{m+4}, v_m) (see Figure 2 for a similar triangulation). In all there is a linear number of edges going between the area bounded by edges (v_4, v_n), (v_{m+4}, v_m), C and C', each of which has length at most $O(\frac{1}{n^2})$ giving a total edge length of $O(\frac{1}{n})$. Thus the total edge length in T is $O(1)$.

The set S considered above has an even number of points. For the case when n is odd we add a *dummy vertex* v_d in the area bounded by the triangle with corners v_1, v_2, and v_3 to maintain the symmetric nature of S. The introduction of the dummy vertex gives the

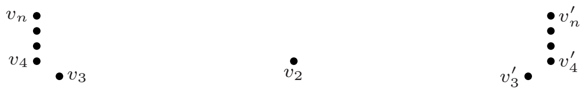

Figure 5: An sketch showing a modification of the point set S in Figure 3. The point sets are symmetric along the x-axis.The figure is not to scale.

same lower bound as shown for the even case since the concavity at v_d can be removed by inserting edges from v_d to v_1, v_2, and v_3.

The $\Theta(\log n)$ bound is tight. We can show this by starting with polygonal regions formed by combining the convex hull with the MST of the point set, and then computing the MWT of these regions using the *ring heuristics* proposed by Lingas [8]. The ring heuristics achieves a $O(\log n)$ approximation to the MWT of polygons. □

Generalization

Theorem 5 *For any n, there exists a point set S for which $\frac{|T_{MWCP(G)}|}{|MWT|} = \Omega(n)$.*

Proof. We show that Theorem 5 holds by modifying the point set S in Figure 3 to be symmetric w.r.t. the y-axis: the point v_2 lies on the y-axis, the points v_3 to v_n are at the same positions relative to v_2, and we add corresponding points v'_3 to v'_n at symmetric positions (see Figure 5). Any connected subset G in the $MWCP$ of S includes either the edge (v'_3, v_2) or (v_2, v_3). Any of these two edges prevents us from getting triangulation edges having length of at most $\frac{1}{n}$ as shown in the proof of Theorem 3. □

Observations

Definition 1 *A vertex of a polygon is strictly convex if its internal angle is strictly less than 180 degrees. Every vertex of a strictly convex polygon is also strictly convex. Similarly every polygon of a strictly convex partition is also strictly convex.*

Observation 1 *For the case where* strictly convex partitions *are required, the $T_{mst(MWCP)}$ of S is of length $\Theta(\log n)$ times the length of the MWT of S if collinearity of three or more vertices is allowed. We can prove the $\Omega(\log n)$ lower bound part of the $\Theta(\log n)$ bound using the same proof as for Theorem 4 (for the non collinear case), since a strictly convex partition means a strictly convex polygon, collinearity of three or more vertices is not allowed in the convex polygons formed from the strictly convex partitions. To show the $O(\log n)$ upper bound part of the $\Theta(\log n)$ bound, the ring heuristics [9] can be used to optimally triangulate all the strictly convex polygons derived from the strict $MWCP$ of S.*

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