Numerical analysis of the motion of glass under external pressure.

K. Laevsky, R.M.M. Mattheij
Department of Mathematics and Computer Science,
Eindhoven University of Technology,
PO Box 513, 5600 MB The Netherlands

Abstract

We give a mathematical model of the forming of a glass product in a mould under pressure. It turns out that the equations of motion are the Stokes equations. One part of the boundary is given, another part is free. The latter means that the velocity there comes from an external force, in particular from a piston that drives a moving part of the mould (the plunger) into the glass. This provides for an additional (kinematic) boundary condition. The complication here is that the movement of this piston on one hand and the counter force from the glass on the other are coupled. The equation of motion are the Stokes equations. The boundary condition couples these with the motion of the plunger, being an ordinary differential equation. It turns out that the resulting equation for the plunger velocity is stiff, so it should be solved by an implicit method. However, due to the above mentioned coupling a straightforward implementation of such an implicit scheme is impossible. We give a solution to this problem.

1 Introduction

Glass is a simple material and is available in all sorts of applications. Yet production and forming are matters that still pose questions the answers to them relying more and more on mathematical modelling and simulation, cf. [2], [3], [4]. In this paper we consider the motion of molten glass by pressure, which is an important step in the production of container glass. In particular we model the pressing of a preform or parison in a mould used in the mechanical production of container glass. In Figure 1.1 we have sketched the various parts making up for the mould. The actual mould consists of the baffle, the blank, and the neckring. Initially the baffle part is removed and the mould is open from above (cf. Figure 1.1a). Once a gob of glass is inside the mould, the baffle is closed and the plunger moves up by the force of a piston (cf. Figure 1.1b,c). This parison (see Figure 1.1d), is then blown into its final shape in the next stage (see Figure 1.2).

Although the temperature plays an important role in this modelling [5], it can be shown that during the pressing phase the temperature changes are rather small because of the short time the pressing takes. Hence we consider the problem to be isothermal. We shall model the process assuming all parts of the mould and the plunger to have axisymmetric geometry. An appropriate choice for the coordinate system to be used in order to solve the equations numerically are then cylindrical coordinates. The motion of a fluid can be described by Navier Stokes. By dimension analysis it can be shown that they simplify to Stokes equations, cf. [4] The Stokes equations in cylindrical coordinates can be formulated as follows, cf [1]. Find the velocity field $\mathbf{v} := (u_r(r,z,\varphi), u_z(r,z,\varphi), u_\varphi(r,z,\varphi))^T$ and pressure field $p := p(r,z,\varphi)$, which satisfy
\[ \nabla \cdot \sigma(v, p) = 0, \]  
\[ \nabla \cdot v = 0, \]  
where \( \sigma(v, p) \), the stress tensor, is given by
\[ \sigma(v, p) = -p I + \eta (\nabla v + \nabla v^T). \]  
Here \( I \) is the identity tensor.

Using the formula for the gradient in cylindrical coordinates we obtain
\[ \sigma = \begin{pmatrix} -p + 2\eta \frac{\partial u_r}{\partial r} & \eta \left( \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} \right) & \eta \left( \frac{1}{r} \frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} \right) \\ \eta \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) & -p + 2\eta \frac{\partial u_z}{\partial z} & \eta \left( \frac{1}{r} \frac{\partial u_z}{\partial \varphi} + \frac{\partial u_\varphi}{\partial z} \right) \\ \eta \left( \frac{1}{r} \frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} \right) & \eta \left( \frac{1}{r} \frac{\partial u_z}{\partial \varphi} + \frac{\partial u_\varphi}{\partial z} \right) & -p + 2\eta \left( \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\varphi}{r} \right) \end{pmatrix}. \]  

Equations (1.2), (1.1), rewritten in terms of cylindrical coordinates, read as
\[ \begin{align*} 
\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{2}{r^2} \frac{\partial u_\varphi}{\partial r} - \frac{u_r}{r^2} &= \frac{1}{\eta} \frac{\partial p}{\partial r}, \quad (1.5) \\
\frac{\partial^2 u_z}{\partial r^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} &= \frac{1}{\eta} \frac{\partial p}{\partial z}, \quad (1.6) \\
\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{\partial^2 u_\varphi}{\partial z^2} + \frac{1}{r^2} \frac{\partial u_\varphi}{\partial \varphi} + \frac{2}{r} \frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi}{r^2} &= \frac{1}{\eta r} \frac{\partial p}{\partial \varphi}, \quad (1.7) \\
\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} &= 0. \quad (1.8) 
\end{align*} \]

2 Rotational Symmetry

As was explained in Section 1 both the mould and the plunger are axisymmetric. Since the plunger is moving in vertical direction the velocity, \( V_p(t) \) say, we can write

\[ \mathbf{v}_p(t) = V_p(t) \mathbf{e}_z := (0, V_p(t), 0)^T, \quad (2.1) \]

where \( \mathbf{e}_z \) is the unit vector in \( z \) direction. We may reduce the dimension of the problem and consider (1.1), (1.2) in two-dimensional axisymmetric coordinates. The velocity field then has the components

\[ \mathbf{v} := (u_r(r, z, \varphi), u_z(r, z, \varphi), 0)^T, \quad (2.2) \]

and the pressure field

\[ p := p(r, z). \quad (2.3) \]

From (1.4) we obtain the stress tensor for the axisymmetric case.
\[
\sigma = \begin{pmatrix}
-p + 2\eta \frac{\partial u_r}{\partial r} & \eta \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) & 0 \\
\eta \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) & -p + 2\eta \frac{\partial u_z}{\partial z} & 0 \\
0 & 0 & -p + 2\eta \frac{u_r}{r}
\end{pmatrix}.
\]

The Stokes equations (1.5)-(1.8) take the following form

\[
\begin{align*}
\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - u_r &= \frac{1}{\eta} \frac{\partial p}{\partial r}, \\
\frac{\partial^2 u_z}{\partial r^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} &= \frac{1}{\eta} \frac{\partial p}{\partial z}, \\
\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} &= 0.
\end{align*}
\]

Clearly, the pressure \( p \) is defined up to a constant. One can should notice singularities in (2.4)-(1.7) when \( r = 0 \).

3 Boundary Conditions

As we have an axisymmetric problem we obtain a domain \( \Omega \), as sketched in Figure 3.1. The boundary \( \Gamma := \partial \Omega \) of the domain consists of four parts

\[
\Gamma = \Gamma_s \cup \Gamma_m \cup \Gamma_p \cup \Gamma_f,
\]

where the indices \( s, m, p, f \) represent the symmetric, mould, plunger and free boundaries respectively. Let

\[
n = (n_r, n_z, 0)^T, \quad t = (t_r, t_z, 0)^T
\]

be the normal and tangent unit vectors respectively for the boundary \( \Gamma \) in the directions as displayed in Figure 3.1. Then we find the following boundary conditions. Because of symmetry, the boundary conditions on \( \Gamma_s \) are

\[
\begin{align*}
v \cdot n &= 0, \\
\sigma n \cdot t &= 0.
\end{align*}
\]

It is easy to see that

\[
n = (-1, 0, 0)^T, \quad t = (0, -1, 0)^T, \quad \sigma n = (-\sigma_{rr}, -\sigma_{rz}, 0)^T
\]

on \( \Gamma_s \). Using the expressions for the stress tensor components (2.4) we obtain

\[
u_r = 0, \quad \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0.
\]

Since \( u_r \equiv 0 \) on \( \Gamma_s \), it follows that the derivative along \( \Gamma_s \) is also equal to zero, i.e., \( \partial u_r / \partial z = 0 \). As a result the boundary conditions on \( \Gamma_s \) can be written as
For the mould and the plunger we will allow both slip and no slip boundary conditions and everything in between. A partial slip boundary condition for the mould means that the normal component of the velocity should be zero and the tangential component proportional to the tangential stress, i.e.

\[ v \cdot n = 0, \]
\[ (\sigma n + \beta_m v) \cdot t = 0, \]

where \( \beta_m \) is a friction coefficient. The first equation clearly represents a Dirichlet boundary condition, and the second a Robin boundary condition.

For the plunger which moves with velocity \( v_p \) (see (2.1)), we find

\[ (v - v_p) \cdot n = 0, \]
\[ (\sigma n + \beta_p (v - v_p)) \cdot t = 0. \]

Note that \( v_p \) does not depend on \( r, z \), and \( \beta_p \) is again the friction coefficient. The physical meaning of these conditions is the same as for (3.8), (3.9), with the only difference that here we consider...
the velocity relative to \( v_p \), i.e., \( v - v_p \). Also we are using the fact that \( \sigma(v - v_p, p) = \sigma(v, p) \). Let \( V_p > 0 \) be the absolute velocity of the plunger, then

\[
v_p = V_pe_z := (0, V_p, 0)^T.
\] (3.12)

Actually, the velocity of the plunger \( V_p \) is an unknown function of time \( t \), so we should write \( V_p(t) \). Nevertheless, for the boundary conditions below and the Stokes problem as such, we view this as just a parameter. Hence, the boundary conditions read as follows

\[
v \cdot n = V_pe_z \cdot n,
\] (3.13)

\[
(\sigma n + \beta_p v) \cdot t = \beta_p V_pe_z \cdot t.
\] (3.14)

Finally the boundary conditions at the free boundary \( \Gamma_f \) are defined as the vector relation

\[
\sigma n = -p_0 n,
\] (3.15)

where \( p_0 \) is the external pressure. We can take the inner product of (3.15) with \( n, t \) and obtain the boundary conditions in the form of two scalar equations

\[
\sigma n \cdot n = -p_0,
\] (3.16)

\[
\sigma n \cdot t = 0.
\] (3.17)

Note that the velocity field found from (1.1), (1.2) with the boundary conditions (3.3) – (3.17), is independent of the value of \( p_0 \). From a physical point of view this can be explained by the incompressibility of the fluid.

### 4 An Ordinary Differential Equation for the Plunger Velocity

The velocity \( V_p(t) \) of the plunger is not known beforehand and in fact coupled to the motion of the glass itself. Indeed, the plunger movement is the result of a certain pressure \( p_p \) applied to its bottom. Let \( F(t) \) denote the total force on the plunger and \( m_p \) be the mass of the plunger. Then

\[
\frac{dV_p(t)}{dt} = \frac{F(t)}{m_p}.
\] (4.1)

This total force is the sum of

\[
F(t) = F_p + F_g(t),
\] (4.2)

where \( F_p \) remains constant through the whole process and \( F_g(t) \) is the force on the plunger from the glass. The constant force can be computed as

\[
F_p = S_p p_p = \text{being some constant}.
\] (4.3)

Here \( S_p \) is the area of the surface where pressure \( p_p \) is applied. The second term \( F_g(t) \), is the force on the plunger from the glass. The force from the glass can be expressed in terms of the stress tensor (2.4)

\[
F_g(t) = \int_{S(t)} \sigma n \cdot e_z \, dS,
\] (4.4)

where \( \sigma \equiv \sigma(t) \) is the stress tensor, and \( S(t) \) the part of the plunger surface which is in contact with the glass at time \( t \). The formula requires integration of the second component of \( \sigma n \) only,
Consider Figure 4.1 which depicts one half of the plunger (cf. Figure 4.1) turned by 90 degrees. If \( z \) is the axial variable and \( R(z) \) denotes the form of the plunger we can derive (cf. [7])

\[
dS = 2\pi R_p(s) \, ds = 2\pi \sqrt{1 + R_p'(z)^2} R_p(z) \, dz, \tag{4.5}
\]

where \( s \) represents the length over the plunger profile. The two dimensional surface \( S(t) \) is related to the interval \([z_0, z_1]\) on the \( z \) axis. Then (4.4) can be written as follows

\[
F_g(t) = 2\pi \int_{z_0}^{z_1} \sigma n \cdot e_z \sqrt{1 + R_p'(z)^2} R_p(z) \, dz. \tag{4.6}
\]

The values of \( \sigma n \) can be obtained as follows The normal components \( n_r, n_z \) (see Figure 4.1) are computed as follows

\[
n = -\frac{1}{\sqrt{1 + R_p'(z)^2}} (1, R_p'(z), 0)^T. \tag{4.7}
\]

Using the expressions (2.4) for the stress tensor components, (4.6) reads

\[
F_g(t) = 2\pi \int_{z_0}^{z_1} \left( p - 2\eta \frac{\partial u_z}{\partial z} \right) R_p'(z) + \eta \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \right) R_p(z) \, dz. \tag{4.8}
\]

Now in order to compute the velocity of the plunger \( V_p(t) \) as a function of time, one should solve the ordinary differential equation

\[
\begin{aligned}
\frac{dV_p(t)}{dt} &= \frac{F_g(t)}{m_p} + \frac{F_p}{m_p}, \\
V_p(0) &= V_0,
\end{aligned} \tag{4.9}
\]

where \( V_0 \) is some initial velocity of the plunger. Note that we can compute \( F_g(t) \), once \( u_r, u_z, p \) (or \( \sigma n \)) are known. The latter are obtained from solving the Stokes equations. In order to solve the Stokes equations one needs some value for the plunger velocity \( V_p \) in (3.13) and (3.14). So, at time \( t = 0 \) we use \( V_0 \) from (4.9) and find \( F_g(0) \). We can thus perform an explicit integration step in (4.9). In general, suppose we use the Euler forward scheme

\[
V_p^{k+1} = V_p^k + \Delta t \frac{F_g^k}{m_p} + \frac{F_p}{m_p}. \tag{4.10}
\]
Having solved the Stokes equations, with the new velocity of the plunger $V_p^{k+1}$, we can complete the boundary conditions for the Stokes problem at $t = t^{k+1}$. To this end the velocity of the plunger obtained from (4.10) is used. However, as illustrated in Figure 4.2, the algorithm turns out to be unstable. Looking more carefully at Figure 4.2 we detect a phenomenon that looks like stiffness. To overcome this we should take recourse to implicit methods. A fully implicit scheme, however, practically impossible as we do not know the plunger velocity at $t^{k+1}$; thus we cannot use it for the boundary conditions (3.13), (3.14). Of course, a predictor-corrector scheme for such an implicit integrator will only converge for infeasible small time steps because of stiffness.

5 A Stiffness Phenomenon

In this section we like to investigate the stiffness of the ordinary differential equation (4.9). Clearly, we need to have a closer look at $F_g(t)$, as derived in (4.8). In general it is impossible to compute it exactly so we take recourse to a thin film approximation. Here we shall approach the problem analytically in order to point out the stiffness phenomenon detected in numerical simulation. For a more detailed discussion see [7]. We shall consider a simple, yet meaningful geometry for the mould and the plunger, see Figure 5.1. Let each of them be defined by a parabola, say

$$R_m(z) = d_m \sqrt{z}, \quad R_p(z) = d_p \sqrt{z - z_0},$$

where coefficients $d_m, d_p$ have positive values and $z_0$ is the position of the plunger.

Let us define $\varepsilon := D/L$ as the ratio between the length scales corresponding to the parison’s wall thickness $D$ and the height of the parison $L$. Since $D$ is smaller than $L$, $\varepsilon$ is a small parameter. The variables can be then scaled as follows

$$r = Dr', \quad z = Lz', \quad u_r = \varepsilon Vu_r', \quad u_z = Vu_z', \quad p = \frac{\eta V L}{D^2} p',$$

where $V$ is the typical flow velocity. Using (5.2) we can make (4.8) dimensionless

$$F_g(t) := 2\pi \eta V L F_g'(t).$$

Then (4.8) can be approximated by the following expression
Figure 5.1: Mould and plunger geometries defined by parabolas.

\[ F'_g(t) = \int_{z_0}^{z_1} \left( \left( p' - 2 \epsilon z \frac{\partial u'_z}{\partial z} \right) R'_p(z') + \left( 2 \epsilon \frac{\partial u'_r}{\partial z} + \frac{\partial u'_z}{\partial r} \right) \right) R'_p(z') \, dz \]

\[ \approx \int_{z_0}^{z_1} \left( p' R'_p(z') + \frac{\partial u'_r}{\partial z} \right) R'_p(z') \, dz. \]  

(5.4)

Using (5.2) it is possible to find the exact solution of the Stokes equations (2.5), (2.6), (2.7) (see [7])

\[ u'_r = \frac{1}{\rho} \frac{d}{dz} \int_{r'}^{R'_m} r' u'_z(r', z') \, dr', \]

\[ u'_z = \frac{1}{4} \rho \frac{dz}{dz} + A(z') \ln r' + B(z'), \]  

(5.5)

where \( A(z') \) and \( B(z') \) can be obtained from the boundary conditions. The eventual dimensional force \( F'_g(t) \) takes then the following form

\[ F'_g(t) \approx 2 \pi \eta V L V'_p(t') \int_{z_0}^{z_1} \frac{c_m - c_p}{(b_m - b_p)^2 - (a_m - a_p)(c_m - c_p)} \, dz. \]  

(5.6)

Here \( V'_p(t') \) is the dimensionless velocity of the plunger scaled with \( V; a_m, a_p, b_m, b_p, c_m, c_p \) denote

\[ a_m = \ln R'_m(z') + s_m/R'_m(z'), \quad a_p = \ln R'_p(z') + s_p/R'_p(z'), \]

\[ b_m = R'_m^2(z')(1 + 2s_m/R'_m(z')), \quad b_p = R'_p^2(z')(1 + 2s_p/R'_p(z')), \]

\[ c_m = R'_m^4(z')(1 + 4s_m/R'_m(z')), \quad c_p = R'_p^4(z')(1 + 4s_p/R'_p(z')), \]  

(5.7)

respectively. Here \( s_m, s_p \) are dimensionless parameters similar to the friction coefficients \( \beta_m, \beta_p \) as defined in Section 3. Note that all defined quantities are dimensionless.

The dimensionless integral in (5.4) can be computed numerically. The graph in Figure 5.2 shows the results of this integration as a function of upper bound \( z'_1 \) in (5.4). Using the same scaling (4.9) reads
The typical values for $L$ and $V$ are $10^{-1}$ m and $10^{-1}$ s respectively. The mass of the plunger device $m_p$ is of order 1. The viscosity coefficient $\eta$ for our problem is a large number $\eta \approx 10^4$ kg/s m. (5.9)

One can see that the coefficient of $V'_p$ on the right-hand side of ??chapter6/section2: equation8) is large. Indeed, taking $I(t) \approx 1$ (see Figure 5.2) we find

$$I(t) \frac{2\pi L^2}{Vm_p} \eta \approx 10^4.$$ (5.10)

This clearly indicates that (5.8) is a stiff equation. One should note that $\eta$ is the dominating quantity. This will also be the case for more complicated geometries. This then shows the inherent stiffness of the plunger motion equation.

### 6 Uncoupling the Flow Equations and the Plunger Velocity

From the preceding analysis it follows that an explicit method leads to numerical instabilities, for not unduly small time steps. We therefore prefer to use an implicit method instead. However, the right-hand side $F(t)/m_p$ of (4.1) depends on the solution of the Stokes equations. In order to apply an implicit step to (4.1) at time $t = t^k$ we need to know $F_g(t^{k+1})$. In this case we would compute

$$V_{p}^{k+1} = V_{p}^k + \Delta t^k \frac{F_g(t^{k+1}) + F_p}{m_p}.$$ (6.1)

Note that $F_g(t^{k+1})$ resulting from the solution of the Stokes equations with $V_p^{k+1}$. Clearly, in this way the Stokes equations and the motion of the plunger are coupled. In order to use the implicit scheme (6.1), we could, for example, predict the velocity of the plunger using (4.10) and then use it for the boundary conditions in the Stokes equations. After having solved the latter, let us compute the value for $F_g(t^{k+1})$ and perform (6.1). Unfortunately this does not work because of the explicit prediction step, which sooner or later cause numerical instabilities.

Below we work out how to overcome the stiffness phenomenon for our problem. The crucial role here is played by regarding the velocity of the plunger $V_p(t)$ uncoupled from the parameter $V_p$ in the boundary conditions for the Stokes problem. We shall make use of the following lemma.
LEMMA 6.1 Let \( v_1, p_1 \) and \( v_2, p_2 \) be the solutions of the Stokes equations (2.5), (2.6) and (2.7) with corresponding plunger velocities \( V_{p_1} \) and \( V_{p_2} \), respectively. Then \( k_1 v_1 + k_2 v_2, p_0 + k_1 (p_1 - p_0) + k_2 (p_2 - p_0) \) is also a solution of these equations with \( V_p = k_1 V_{p_1} + k_2 V_{p_2} \).

**Proof.** From \( \nabla \cdot p_0 l = 0 \), it follows that

\[
\nabla \cdot \sigma(k_1 v_1 + k_2 v_2, p_0 + k_1 (p_1 - p_0) + k_2 (p_2 - p_0)) = 0.
\]

It is simple to see that such a linear combination satisfies Stokes equation. Note that

\[
\nabla \cdot (k_1 v_1 + k_2 v_2) = k_1 \nabla \cdot v_1 + k_2 \nabla \cdot v_2 = 0.
\]

Likewise such a property can be shown for the boundary conditions. Considering the pressure field relative to \( p_0 \), the boundary conditions (3.16), (3.17) are satisfied

\[
\sigma(k_1 v_1 + k_2 v_2, p_0 + k_1 (p_1 - p_0) + k_2 (p_2 - p_0))n = 0
\]

\[
k_1(\sigma(v_1, p_1)n + p_0n) + k_2(\sigma(v_2, p_2)n + p_0n) - p_0n = -p_0n.
\]

This proves the lemma.

From Lemma 6.1 it follows that we may consider the velocity and pressure fields at some time \( t \) as affine functions of \( V_p \), so

\[
v(t; V_p) = V_p v(t; 1),
\]

\[
p(t; V_p) = p_0 + V_p (p(t; 1) - p_0).
\]

Here \( v(t; \alpha), p(t; \alpha) \) is the solution of the Stokes equations with the velocity of the plunger equal to \( \alpha = \text{const} \). As a consequence we deduce from (4.8) that this then also holds for the glass force

\[
F_g(t; V_p) = F_0(t) + V_p (F_g(t; 1) - F_0(t)),
\]

where \( F_0(t) \) is the force on the glass due to normal air pressure

\[
F_0(t) = 2\pi \int_{z_0}^{z_1} p_0 R_p'(z) R_p(z) \, dz.
\]

Using (6.6) we can reformulate (4.9) as follows

\[
\begin{align*}
\frac{dV_p(t)}{dt} &= V_p(t) \left( F_g(t; 1) - F_0(t) \right) \frac{m_p}{m_p} + F_p + F_0(t) \frac{m_p}{m_p}, \\
V_p(0) &= V_0.
\end{align*}
\]

Equation (6.8) should be reformulated as follows

\[
F_g := F_g(z; V_p), \quad V_p := V_p(z).
\]
\[
\begin{aligned}
\frac{1}{2} \frac{dV_p^2(z)}{dz} &= V_p(z) \frac{F_g(z; 1) - F_0(z)}{m_p} + \frac{F_p + F_0(z)}{m_p}, \\
V_p(0) &= V_0.
\end{aligned}
\]  
(6.10)

Here we used
\[
\frac{dV_p(t)}{dt} = \frac{dV_p(z)}{dz} V_p(z).
\]  
(6.11)

By solving these equations for a evolving glass domains, we can obtain a table with plunger positions, and velocity and pressure fields computed for \(V_p = 1\) in such domains. Hence, the velocity of the plunger can be considered to be a function of the plunger position, but still being unknown as a function of \(t\).

If one applies the Euler explicit method to (6.10),
\[
\begin{aligned}
\frac{1}{2} \frac{V_p^{k+1} - V_p^k}{z^{k+1} - z^k} &= V_p^k \frac{F_g(z^k; 1) - F_0(z^k)}{m_p} + \frac{F_p + F_0(z^k)}{m_p}, \\
V_p^0 &= V_0.
\end{aligned}
\]  
(6.12)

it appears that this approach is identical to one in which the plunger velocity for the boundary conditions at the next time-step were obtained straight from the previous velocity field and pressure field
\[
V_p(t + \Delta t) = V_p(t) + \Delta t \frac{F_g(t) + F_p}{m_p}.
\]  
(6.13)

The boundary conditions (3.13), (3.14) for the next stationary Stokes problem should use \(V_p(t + \Delta t)\). We omit further discussion of (6.13).

Now consider the implicit Euler method instead
\[
\begin{aligned}
\frac{1}{2} \frac{V_p^{k+1} - V_p^k}{z^{k+1} - z^k} &= V_p^{k+1} \frac{F_g(z^{k+1}; 1) - F_0(z^{k+1})}{m_p} + \frac{F_p + F_0(z^{k+1})}{m_p}, \\
V_p^0 &= V_0.
\end{aligned}
\]  
(6.14)
Although (6.14) is implicit, we just have a quadratic equation for $V_p^{k+1}$, which can be solved trivially. The result is in Figure 6.1a. We clearly have a stable calculation now. The velocity of the plunger in Figure 6.1a is a function of $z$. In order to obtain the velocity as a function of $t$ the following approximation can be used

$$\begin{cases}
    z^{k+1} = z^k + \Delta t^k V_p(z^k), \\
    t^{k+1} = t^k + \Delta t^k,
\end{cases}$$

(6.15)

where $t^0 = 0$. The final graph is depicted in Figure 6.1b.

References


