Stability of a viscoelastic thread immersed in a confined region

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Abstract

In this paper, we consider an infinitely long viscoelastic thread in a tube filled with a Newtonian fluid. We apply Jeffreys model as the rheological relation for the thread. The thread moves due to a constant pressure gradient, but so slowly that the quasi-static creeping flow approximation is applicable. We investigate the effect of the ratio of viscosities, fluid elasticity, confinement and prescribed flow on the stability of the immersed thread. The stability is characterized by the growth rate of a small perturbation. The more viscous the thread is, the more time it takes to break up. A viscoelastic thread breaks up faster than a Newtonian one, and the wave number, which is responsible for the break-up, is smaller for a viscoelastic thread than for a Newtonian one. The latter implies that a viscoelastic thread will break up in larger droplets than a Newtonian one. The thread breaks up slower when the degree of confinement is higher. A critical gap width beyond which the presence of the wall of the tube has no longer an effect on the stability of the thread is found. In case of a Newtonian thread the prescribed flow only causes the thread to be oscillatory unstable with the growth rate equal to the one within a fluid at rest. Moreover, in case of a viscoelastic thread the prescribed flow contributes to both the real and the imaginary part of the growth rate of the perturbation. So, a viscoelastic thread will be oscillatory unstable and will break up faster than the one within a fluid at rest.

1 Introduction

Nowadays, modern developments in the design and utilization of microfluidic devices for fluid transport have found many applications such as drug design and diagnostic devices in biomedicine and microdrop generators for image printing (Stone & Kim (2001) and Lee (2003)). The most important new themes introduced by the small length scales of microfluidic devices are the significant role of surface forces (surface tension, electrical effects, etc.), complicated three-dimensional geometries, and the possibility that suspended particles have dimensions comparable to the microchannels (Stone & Kim (2001)). The comparable size between the suspended particles and the gap width (confinement) implies that the wall effects on suspended particles motion are crucial, making this problem important from a scientific perspective.

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It is well known (Tomotika (1935) and Mikami et al. (1975)) that an immersed cylindrical liquid thread in an infinite region will develop small radial disturbances which grow until the eventual break-up into a small array of droplets. Many authors have studied the interplay of viscosity, surface tension, and flow, which causes the break-up of small droplets in shear or extensional flow. The stability of a droplet in nonuniform shear flow was considered by Chin & Han (1980). They investigated the stability of a long threadlike viscoelastic droplet suspended in a viscoelastic medium through a cylindrical tube. Fixing the ratio of thread and tube radius, Chin & Han found the regions of stability in terms of fluids and rheological properties, and they found that the larger this ratio is (smaller gap width), the more stable the system becomes. Stability of a viscoelastic jet driven by surface tension has been numerically studied by Bousfield et al. (1986). Using finite element methods, Bousfield et al. found the initial growth rate of the perturbation to be in agreement with linear stability theory, whereas at longer time scales, the growth rate decreased dramatically, due to the build-up of extensional stresses, and the filament evolved to a bead-on-string configuration. Palierne & Lequeux (1991) considered the relation between the wavelength and the growth rate of the sausage instability of a thread embedded in a matrix, in the limit of vanishing Reynolds number and for incompressible fluids. The theory applied to general viscoelastic fluids, and took into account the dynamic properties of the interfacial tension due to adsorbed contaminants or interfacial agents, as well as the capability of the interface to resist a shear deformation. Recently, microscope observations have shown that the immersed thread can be elongated into long 'string' in steady states of a polymer solution under strong shear flow (Hashimoto et al. (1995)). In a shear field, the stability of the thread is determined by the ratio of viscous and interfacial stresses, i.e. the capillary number $\text{Ca}$. Frischknecht (1998) considered the stability of a long cylindrical domain in a phase-separating binary fluid under influence of shear flow, and showed that the shear flow suppresses and sometimes completely stabilizes the instability of the cylinder against varicose perturbations. The results were consistent with a 'string phase' behaviour in phase-separating fluids in shear observed by Hashimoto et al. (1995). However, this study focussed on the immersed thread in an infinite region. Migler (2001) performed experiments by placing a sample (with dispersed drops) between two parallel quartz disks and rotating the upper one at a controlled rate. The shear rate was defined as the ratio between the upper plate velocity at the radial point of measurement and the gap width between the two disks. Results showed that a droplet-string transition in concentrated polymer blend occurs when the size of the dispersed droplets becomes comparable to the gap width between the shearing surfaces. Upon reduction of the shear, the transition proceeded via the coalescence of droplets in four stages. First, droplets coalesce with each other and so increase the average droplet size. Second, the large droplets self-organize into pearl necklace structures (chaining). Third, the aligned droplets coalesce with each other to form strings. And fourth, the strings then coalesce with each other to form ribbons. In the string structures regime, Migler found an unexpected phenomenon, in which the structures were still stable after the shear was stopped. Typically, at rest (after cessation of shear) the strings will break up (Tomotika (1935)). Migler proposed two stability mechanisms for the string structures. First, the wall effect, and second, the shear flow. The wider string, i.e. the string whose size is comparable to the gap width of the shearing surfaces, is stabilized by a suppression of
the instability due to finite size effects (confinement), while the narrower one is stabilized by shear flow. Some experimental and theoretical works on the microstructure in a polymer blend arising through the interaction of the deformation due to flow, retraction, coalescence, and break-up were reviewed by Tucker III & Moldenaers (2002). The effect of confinement on the stability of a liquid thread has recently been studied by Pathak & Migler (2003), Son et al. (2003) and Hagedorn et al. (2003).

In this paper, we consider an infinitely long viscoelastic thread in a tube filled with a Newtonian fluid. We apply Jeffreys model as the rheological relation for the thread. The thread moves due to a constant pressure gradient and does not bend or twist, but it moves so slowly that the creeping flow approximation is applicable. Assuming that the thread surface is perturbed by small perturbations, we investigate the stability of the thread, which is characterized by the (real part of) growth rate of the perturbations. If the growth rate is positive, then the thread is unstable, whereas if the growth rate of all perturbations is negative, then the thread is stable. Much attention is paid to the effects of the ratio of viscosities, fluid elasticity, confinement, and prescribed flow on the stability of an immersed thread. The fluid elasticity of the thread is represented by two parameters, i.e. the Deborah number De and the material constant Λ. The presence of the confined region and the prescribed flow are represented by the dimensionless length h, i.e. the ratio between the radius of the thread and the radius of the tube, and the uniform constant pressure gradient C_p, respectively. As for the effect of the ratio of viscosities, the more viscous the thread is, the more time it takes to break up. The effect of the fluid elasticity reveals that a viscoelastic thread breaks up faster than a Newtonian one. The confinement turns out to stabilize the thread, whereas the flow does not. In case of a Newtonian thread, the flow only affects the imaginary part of the growth rate. This implies that a Newtonian thread will be oscillatory unstable and break up as fast as the one within a fluid at rest. In case of a viscoelastic thread, the flow affects both the imaginary and the real parts of the growth rate, implying that the thread will be oscillatory unstable and break up faster than the one within a fluid at rest.

2 Mathematical formulation and solution methodology

Let us consider a long, neutrally buoyant linear viscoelastic thread of radius a_1 with zero shear viscosity \( \eta^d_0 \), immersed in a tube of radius a_2 filled with a Newtonian fluid of viscosity \( \eta^c_0 \). We take cylindrical coordinates \((r, \phi, z)\) with the \(z\)-axis along the thread axis. The indices \(c\) and \(d\) stand for the continuous phase (the surrounding fluid) and the dispersed phase (the thread), respectively. The fluids are assumed to be incompressible. Both phases are assumed to be mutually immiscible, and no external body forces are present. The thread moves due to a constant pressure gradient in the tube. Then, the system is governed by the quasi-static creeping flow,

\[
\text{div}\ u = 0 ,
\]

\[
\text{div}\ \tau = 0 ,
\]
where \( \mathbf{u} \) is the velocity field and \( \mathbf{\tau} \) the total stress tensor. Since we are interested in the stability of the system, we write the solution in the form

\[
\mathbf{u} = \mathbf{V} + \epsilon \mathbf{v}, \quad \mathbf{\tau} = \mathbf{\Pi} + \epsilon \mathbf{\pi},
\]

\[
\mathbf{\Pi} = -P\mathbf{\delta} + \mathbf{\Gamma}, \quad \mathbf{\pi} = -p\mathbf{\delta} + \mathbf{\tau},
\]

(2.2)

where \( \mathbf{V} = (U, W) \) is the unperturbed velocity, with \( U \) and \( W \) the velocity components in radial and axial directions, respectively, \( \mathbf{\Pi} \) the unperturbed total stress tensor, \( P \) the unperturbed pressure, \( \mathbf{\delta} \) the unit tensor and \( \mathbf{\Gamma} \) the unperturbed extra stress tensor. Variables \( \mathbf{v} = (u, w) \), \( \mathbf{\pi} \) and \( \mathbf{\tau} \) are the perturbations of \( \mathbf{V} \), \( \mathbf{\Pi} \) and \( \mathbf{\Gamma} \). Here, we assume that the thread does not bend or twist, so the azimuthal velocity vanishes and the equations are independent of the azimuthal direction. The domain of system is sketched in Figure 1.

2.1 Mathematical formulation

For the thread, we take as constitutive relation a linear viscoelastic model, the so called Jeffreys model (\( T \) indicates the transpose):

\[
\left( 1 + \lambda_1 \frac{\partial}{\partial t} \right) \mathbf{\tau}^d = \eta_0^d \left( 1 + \lambda_2 \frac{\partial}{\partial t} \right) \left( \text{grad} \mathbf{v}^d + (\text{grad} \mathbf{v}^d)^T \right).
\]

(2.3)

Here, \( \lambda_1 \) is the stress relaxation time, and \( \lambda_2 \) is the deformation retardation time of the thread. Note that \( \lambda_1 > \lambda_2 \) (this is necessary from thermodynamic considerations, since there are no energy sources within the fluid (see Harris (1977))). In the steady unperturbed state, the constitutive equation (2.3) reduces to the constitutive equation for a Newtonian fluid. For the surrounding Newtonian fluid, we have

\[
\mathbf{\tau}^c = \eta_0^c [\text{grad} \mathbf{v}^c + (\text{grad} \mathbf{v}^c)^T],
\]

(2.4)

and a similar relation holds for the unperturbed extra stress \( \mathbf{\Gamma}^c \).

As unperturbed state, we take the thread to be a perfect cylinder. Its stability is tested by applying a small perturbation. Since the thread does not bend or twist, we apply an axisymmetric perturbation. Thus, the perturbed thread surface is represented by

\[
R(z, t) = a_1 \left[ 1 + \epsilon f(z, t) \right], \quad \text{where} \quad f(z, t) = \Re[\epsilon_0 e^{ikz+qt}],
\]

(2.5)
with \( \varepsilon_0 \) the initial amplitude of the perturbation and \( \varepsilon \) a small parameter \((0 < \varepsilon \ll 1)\). The parameter \( k \) represents the wave number of the perturbation, \( i = \sqrt{-1} \) is the imaginary unit number, \( t \) is the time, and \( q \) is the growth rate of the perturbation. The growth rate determines the (linear) stability of the thread. Note that \( k \) must be real since the solution is periodic in the \( z \)-direction, whereas \( q \) can be complex. In the sequel we shall not write the real part symbol ‘\( \Re \)’ explicitly.

### 2.2 Boundary conditions

As for the boundary conditions at the tube wall, we assume the no-slip condition,

\[
\mathbf{V} = 0 = \mathbf{v}, \quad \text{at } r = a_2. \quad (2.6)
\]

At the interface, we apply continuity of velocity, the dynamical conditions for the stresses, and kinematic conditions expressing that the thread surface is a material surface.

The detailed evaluation of the boundary conditions is as follows. The continuity of the velocity is written as

\[
\begin{bmatrix} u \end{bmatrix} = 0. \quad (2.7)
\]

Here, \( [g] = g^d - g^c \) denotes the jump of an arbitrary function \( g \) across the interface. Evaluating \( \mathbf{u} = \mathbf{u}(r, z, t) \) at the interface \( r = a_1 + \varepsilon a_1 f \) (suppressing the dependence on \( z \) and \( t \) for the moment)

\[
\begin{align*}
\mathbf{u}(a_1 + \varepsilon a_1 f) &= (\mathbf{V} + \varepsilon \mathbf{v})(a_1 + \varepsilon a_1 f) \\
&= (\mathbf{V} + \varepsilon \mathbf{v})(a_1) + \varepsilon a_1 f \frac{\partial}{\partial r} \left[ \mathbf{V} + \varepsilon \mathbf{v} \right](a_1) + O(\varepsilon^2) \\
&= \mathbf{V}(a_1) + \varepsilon \left[ \mathbf{v}(a_1) + a_1 f \frac{\partial \mathbf{V}}{\partial r}(a_1) \right] + O(\varepsilon^2). \\
\end{align*}
\]

Substitution of (2.8) into (2.7) yields for the unperturbed \((O(\varepsilon^0))\) and the perturbed \((O(\varepsilon^1))\) terms,

\[
\begin{align*}
\varepsilon^0 & : \begin{bmatrix} \mathbf{V}(a_1) \end{bmatrix} = 0, & (2.9a) \\
\varepsilon^1 & : \begin{bmatrix} \mathbf{v}(a_1) + a_1 f \frac{\partial \mathbf{V}}{\partial r}(a_1) \end{bmatrix} = 0. & (2.9b)
\end{align*}
\]

Next, we formulate conditions for the stresses. The outward unit normal \( \mathbf{n} = \mathbf{n}_r \mathbf{e}_r + \mathbf{n}_z \mathbf{e}_z \) at the interface is given by

\[
\begin{align*}
\mathbf{n} &= \frac{1}{\sqrt{1 + (\varepsilon a_1 \frac{\partial f}{\partial z})^2}} \left[ \mathbf{l} \mathbf{e}_r - \varepsilon a_1 \frac{\partial f}{\partial z} \mathbf{e}_z \right] \\
&= \mathbf{e}_r - \varepsilon a_1 \frac{\partial f}{\partial z} \mathbf{e}_z + O(\varepsilon^2). \\
\end{align*}
\]
Here, $e_r$ and $e_z$ are the unit base vectors in radial and axial direction, respectively. Let $t = t_r e_r + t_z e_z$ be the unit tangent vector on the perturbed interface, orthogonal to the vector $n$. We find

$$t_r = \epsilon a_1 \frac{\partial f}{\partial z} + O(\epsilon^2), \quad t_z = 1 + O(\epsilon^2). \quad (2.11)$$

The stress vector $g$ at the interface is given by

$$g = \tau n = (\Pi + \epsilon \pi)n. \quad (2.12)$$

Substituting (2.10) into (2.12), we find the $r-$ and $z-$ components of $g$ at the interface:

$$g_r(a_1 + \epsilon a_1 f) = \Pi_{rr}(a_1) + \epsilon \left[ \pi_{rr}(a_1) + a_1 f \frac{\partial \Pi_{rr}}{\partial r} (a_1) - a_1 \Pi_{rz}(a_1) \frac{\partial f}{\partial z} \right] + O(\epsilon^2), \quad (2.13a)$$

$$g_z(a_1 + \epsilon a_1 f) = \Pi_{rz}(a_1) + \epsilon \left[ \pi_{rz}(a_1) + a_1 f \frac{\partial \Pi_{rz}}{\partial r} (a_1) - a_1 \Pi_{zz}(a_1) \frac{\partial f}{\partial z} \right] + O(\epsilon^2). \quad (2.13b)$$

The dynamical boundary conditions require that

$$[g \cdot t] = 0, \quad (2.14a)$$

$$[g \cdot n] = -\sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right). \quad (2.14b)$$

Here, $\sigma$ is the surface tension (in Newton/meter), and $R_1$ and $R_2$ the principle radii of curvature, defined as

$$\frac{1}{R_1} = -a_1 \epsilon \frac{\partial^2 f}{\partial z^2} \left[ (1 + (\epsilon a_1 \frac{\partial f}{\partial z})^2 \right]^{3/2} \right] = -\epsilon a_1 \frac{\partial^2 f}{\partial z^2} + O(\epsilon^2), \quad (2.15)$$

$$\frac{1}{R_2} = \frac{1}{a_1 (1 + \epsilon f)} = \frac{1}{a_1} \left[ 1 - \epsilon f \right] + O(\epsilon^2). \quad (2.15)$$

So, (2.14a) represents continuity of the tangential component of the stress vector $g$, and (2.14b) the discontinuity of its normal component. Note that the jump in the normal stress is balanced by the surface tension. From (2.14a), we obtain

$$\epsilon^0 : [\Pi_{rz}] = 0, \quad (2.16a)$$

$$\epsilon^1 : \left[ \pi_{rz} + a_1 f \frac{\partial \Pi_{rz}}{\partial r} + a_1 \left[ \Pi_{rr} - \Pi_{zz} \right] \frac{\partial f}{\partial z} \right] = 0, \quad (2.16b)$$

and from (2.14b),

$$\epsilon^0 : [\Pi_{rr}] = -\frac{\sigma}{a_1}, \quad (2.17a)$$

$$\epsilon^1 : \left[ \pi_{rr} + a_1 f \frac{\partial \Pi_{rr}}{\partial r} - 2a_1 \Pi_{rz} \frac{\partial f}{\partial z} \right] = \frac{\sigma}{a_1} \left[ f + a_1^2 \frac{\partial^2 f}{\partial z^2} \right]. \quad (2.17b)$$
Note that from now on the jump \([\cdot]\) is evaluated at \(r = a_1\). Using (2.2) and (2.4), we can rewrite (2.16) and (2.17) in terms of the pressure and the velocities, as will be done in the next section.

The kinematic condition requires that at the perturbed interface \(R(t, z)\), being a material surface, the radial velocity is given by the material derivative \((dR/dt)\) following a thread particle:

\[
\mathbf{u}^d \cdot \mathbf{e}_r = \frac{dR}{dt} = \frac{\partial R}{\partial t} + (\mathbf{u}^d \cdot \mathbf{e}_z) \frac{\partial R}{\partial z}.
\]  

(2.18)

2.3 The steady unperturbed solution

The steady unperturbed solution is derived from the solution of the Poiseuille flow. Let \(C_p = dP/dz\) (in Newton/meter\(^3\)) be a constant pressure gradient for both phases. In the steady state, the relation (2.3) reduces to the constitutive relation for a Newtonian thread with viscosity \(\eta^d_0\). So, the general solution is given by \(V = (0, W(r))\) (a fully developed flow), where

\[
W(r) = \frac{1}{4\tilde{\eta}} C_p r^2 + A \ln \left( \frac{r}{a_2} \right) + B.
\]  

(2.19)

Here, \(\tilde{\eta} = \eta^d_0\) for the thread and \(\tilde{\eta} = \eta^c_0\) for the surrounding fluid. The constants \(A\) and \(B\) are to be determined from the boundary conditions, and take different values in the two phases. At the origin the solution must remain bounded. At the wall and the interface we have

\[
W(a_2) = 0, \tag{2.20a}
\]

\[
[W] = 0, \tag{2.20b}
\]

\[
[\Pi_{rz}] = \left[ \tilde{\eta} \frac{\partial W}{\partial r} \right] = 0, \tag{2.20c}
\]

\[
[\Pi_{rr}] = \left[ -P \right] = -\frac{\sigma}{a_1}. \tag{2.20d}
\]

Evaluating the boundary conditions, we find for (2.19)

\[
W = \begin{cases} 
-\frac{a_2^2}{4\eta^d_0} C_p \left[ 1 + \frac{1 - \mu_0}{\mu_0} \left( \frac{a_1}{a_2} \right)^2 - \frac{1}{\mu_0} \left( \frac{r}{a_2} \right)^2 \right], & 0 \leq r \leq a_1, \\
-\frac{a_2^2}{4\eta^d_0} C_p \left[ 1 - \left( \frac{r}{a_2} \right)^2 \right], & a_1 \leq r \leq a_2.
\end{cases}
\]  

(2.21)

and

\[
P = \begin{cases} 
C_p z + \frac{\sigma}{a_1}, & 0 \leq r \leq a_1, \\
C_p z, & a_1 \leq r \leq a_2.
\end{cases}
\]  

(2.22)

Here, we have defined

\[
\mu_0 = \frac{\eta^c_0}{\eta^d_0}.
\]  

(2.23)
as the ratio of the zero shear-rate viscosities of both fluids. For convenience, the expressions are brought into dimensionless form. Since we are interested in the influence of the surface tension \( \sigma \) on the stability of the system, we may use it as a scaling factor. So, as units of distance, velocity, and stress components, we take \( a_1, \sigma/\eta_0^c \), and \( \sigma/a_1 \), respectively. For instance, we have

\[
r = a_1 r^*, \quad u = \frac{\sigma}{\eta_0^c} u^*, \quad \tau = \frac{\sigma}{a_1} \tau^*, \quad k^* = \frac{k^*}{a_1}, \quad \text{and} \quad t = \frac{a_1 \eta_0^c t^*}{\sigma}.
\]

With (2.24), the dimensionless pressure gradient \( C^*_p \) and the dimensionless growth rate \( q^* \) become

\[
C^*_p = \frac{a_1^2}{\sigma} C_p, \quad \text{(2.25a)}
\]

\[
q^* = \frac{a_1 \eta_0^c}{\sigma} q. \quad \text{(2.25b)}
\]

In the sequel, we omit the stars since confusion is not possible. In non-dimensional form we have

\[
W = \begin{cases} 
-C_p \frac{r^2}{4} \left[ h^2 + \frac{1 - \mu_0 - r^2}{\mu_0} \right], & 0 \leq r \leq 1, \\
-C_p \frac{r^2}{4}, & 1 \leq r \leq h.
\end{cases}
\]

(2.26)

and

\[
P = \begin{cases} 
C_p z + 1, & 0 \leq r \leq 1, \\
C_p z, & 1 \leq r \leq h.
\end{cases}
\]

(2.27)

Here, we have introduced the dimensionless length \( h \) by

\[
h = \frac{a_2}{a_1}. \quad \text{(2.28)}
\]

The smaller \( h \), the narrower the gap, i.e. the smaller distance between the outer surface of the thread and the wall of the tube.

We calculate the derivative of \( W \) with respect to \( r \) and we obtain

\[
\frac{\partial W}{\partial r} = \begin{cases} 
\frac{C_p}{2 \mu_0} r, & 0 \leq r \leq 1, \\
\frac{C_p}{2} r, & 1 \leq r \leq h.
\end{cases}
\]

(2.29)

Note that (2.29) has a discontinuity at \( r = 1 \), except for \( \mu_0 = 1 \).

2.4 The perturbed solution

The perturbed equations are

\[
\text{div} \ v = 0. \quad \text{(2.30a)}
\]

\[
\text{grad} \ p = \text{div} \ \tau. \quad \text{(2.30b)}
\]
In dimensionless form, we may write (2.3) as

\[ \left( 1 + \text{De} \frac{\partial}{\partial t} \right) \mathbf{\tau}^d = \mu_0 \left( 1 + \Lambda \text{De} \frac{\partial}{\partial t} \right) \left( \text{grad} \, \mathbf{v}^d + (\text{grad} \, \mathbf{v}^d) \cdot \mathbf{T} \right), \]  

(2.31)

with

\[ \text{De} = \frac{\lambda_1 \sigma}{a_1 \eta_0^c}, \quad \Lambda = \frac{\lambda_2}{\lambda_1}. \]  

(2.32)

Here, the Deborah number De expresses the ratio of the rheological time scale, i.e. the stress relaxation time \( \lambda_1 \), and the capillary time scale \( \frac{a_1 \eta_0^c}{\sigma} \). The constant \( \Lambda \) is the ratio of deformation retardation \( \lambda_2 \) and stress relaxation time \( \lambda_1 \). So, these numbers express the contribution of the fluid elasticity to the system. For \( \Lambda = 0 \), (2.31) is the constitutive relation of the Maxwell model, and for \( \Lambda = 1 \), it is the relation of a Newtonian fluid.

We propose as general expressions for the solution in the dispersed phase

\[ p^d = p_0^d(r)e^{i k z + q t}, \]  

(2.33a)

\[ u^d = u_0^d(r)e^{i k z + q t}, \]  

(2.33b)

\[ w^d = -i w_0^d(r)e^{i k z + q t}, \]  

(2.33c)

\[ \mathbf{\tau}^d = \tau_0^d(r)e^{i k z + q t}, \]  

(2.33d)

\[ \dot{\mathbf{\gamma}}^d = \dot{\gamma}_0^d(r)e^{i k z + q t}. \]  

(2.33e)

The rate of strain tensor \( \dot{\mathbf{\gamma}}^d \) is defined as

\[ \dot{\mathbf{\gamma}}^d = \text{grad} \, \mathbf{v}^d + (\text{grad} \, \mathbf{v}^d) \cdot \mathbf{T}. \]  

(2.34)

The factor \(-i\) in (2.33c) is added for later convenience. Substitution of (2.33d) and (2.33e) into (2.31) yields, after rearranging,

\[ \tau_0^d(r) = \eta(q) \dot{\gamma}_0^d(r), \quad \text{with} \quad \eta(q) = \frac{\mu_0}{1 + q \Lambda \text{De}}, \]  

(2.35)

Substitution of (2.33) and (2.35) into (2.30) yields the following equations:

\[ 0 = \frac{1}{r} \frac{\partial}{\partial r} \left[ r u_0^d \right] + k w_0^d, \]  

(2.36a)

\[ \frac{\partial p_0^d}{\partial r} = \eta(q) \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_0^d}{\partial r} \right) - \frac{1 + (k r)^2}{r^2} u_0^d \right], \]  

(2.36b)

\[ -k p_0^d = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial u_0^d}{\partial r} \right] - k^2 w_0^d. \]  

(2.36c)

Expressions (2.33) and (2.36) also hold for the continuous phase with \( d \) being replaced by \( c \) and \( \eta(q) \) by 1. To solve (2.36), we use the same analysis as used in Gunawan et al. (2001).
The solution for the dispersed phase \((r < 1)\) reads
\[
p_0^d(r) = 2\eta(q)A_0 I_0(kr),
\]
\[
u_0^d(r) = A_0 r I_0(kr) - \left[ B_0 + \frac{2}{k} A_0 \right] I_1(kr),
\]
\[
w_0^d(r) = -A_0 r I_1(kr) + B_0 I_0(kr),
\]
and for the continuous phase \((1 < r < h)\),
\[
p_0^c(r) = 2[D_0 K_0(kr) + F_0 I_0(kr)],
\]
\[
u_0^c(r) = D_0 r K_0(kr) + \left[ E_0 + \frac{2}{k} D_0 \right] K_1(kr)
+ F_0 r I_0(kr) - \left[ G_0 + \frac{2}{k} F_0 \right] I_1(kr),
\]
\[
w_0^c(r) = D_0 r K_1(kr) + E_0 K_0(kr) - F_0 r I_1(kr) + G_0 I_0(kr).
\]
Coefficients \(A_0, B_0, \text{etc.}\), are to be determined from the boundary conditions at the interface and at the wall. From Section 2.2, we derive the four interfacial conditions:
\[
[u_0] = 0,
\]
\[
[w_0] = iC_p \frac{\mu_0 - 1}{2\mu_0} \varepsilon_0,
\]
\[
\left\llbracket \hat{\eta} \left( k u_0 - \frac{\partial w_0}{\partial r} \right) \right\rrbracket = 0,
\]
\[
\left\llbracket -p_0 + 2\hat{\eta} \frac{\partial u_0}{\partial r} \right\rrbracket = (1 - k^2)\varepsilon_0,
\]

To conclude this section, we note that from (2.39), (2.40) and (2.41) we can determine an equation for the growth rate \(q\). In general, since \(q\) is a complex quantity this equation will also be complex.
3 Stability analysis

Substituting (2.37) and (2.38) into (2.39) and (2.40), we obtain the linear system

\[ \mathbf{M} \mathbf{z} = \mathbf{e}, \]  

(3.1)

where \( \mathbf{M} \) is a 6 by 6 matrix, \( \mathbf{z} = (A_0, B_0, D_0, E_0, F_0, G_0) \) and

\[ \mathbf{e} = (0, iC_p(\mu_0 - 1)\varepsilon_0/2\mu_0, 0, (1 - k^2)\varepsilon_0, 0, 0)^T. \]  

(3.2)

The expression for \( \mathbf{M} \) is given in the Appendix. Note that \( \mathbf{M} \) is a complex matrix, since \( \mathbf{M} \) depends on \( q \).

According to Cramer’s rule, the solution of (3.1) is given by

\[ [\mathbf{z}]_j \equiv z_j = \left[ i \frac{(-1)^{2+j}C_p(\mu_0 - 1)}{2\mu_0} |\mathbf{M}^{2,j}| + (-1)^{4+j}(1 - k^2)|\mathbf{M}^{4,j}| \right] \frac{\varepsilon_0(t)}{|\mathbf{M}|}, \]  

(3.3)

where \( |\cdot| \) denotes the determinant and \( \mathbf{M}^{i,j} \) is the 5 \( \times \) 5 sub-matrix of \( \mathbf{M} \) that can be found by omitting the \( i \)-th row and the \( j \)-th column of \( \mathbf{M} \). From (2.37b), (2.41) and (3.3) we obtain an implicit equation for \( q \), after rearranging,

\[ q = F_1(k, \eta(q), h) + i C_p F_2(k, \eta(q), h), \]  

(3.4)

with \( \eta(q) = \mu_0(1 + q\Lambda\text{De})/(1 + q\text{De}) \), and \( F_1 \) and \( F_2 \) the complex valued functions,

\[ F_1 = \frac{(k^2 - 1)}{|\mathbf{M}|} \left[ \left( I_0(k) - \frac{2}{k} I_1(k) \right) |\mathbf{M}^{4,1}| + I_1(k)|\mathbf{M}^{4,2}| \right], \]

\[ F_2 = \left[ 1 - \frac{\mu_0}{2\mu_0|\mathbf{M}|} \left( I_0(k) - \frac{2}{k} I_1(k) \right) |\mathbf{M}^{2,1}| + I_1(k)|\mathbf{M}^{2,2}| \right] + \frac{k}{4}(h^2 - 1). \]  

(3.5)

As is clear from (3.4), \( C_p \) contributes to both the real and the imaginary part of \( q \). The real part of \( q \) is responsible for stability (\( \Re[q] < 0 \)) or instability (\( \Re[q] > 0 \)), whereas the imaginary part \( \Im[q] \) represents an oscillatory behaviour. In case the thread is Newtonian, that is for \( \Lambda = 1 \), (3.4) becomes

\[ q = F_1(k, \mu_0, h) + i C_p F_2(k, \mu_0, h). \]  

(3.6)

Since now \( F_1 \) and \( F_2 \) are real, \( C_p \) only contributes to the imaginary part of \( q \).

3.1 Effect of fluid elasticity

To study specifically the effect of fluid elasticity, we here take \( C_p = 0 \) (no prescribed flow), and \( h \gg 1 \) (no confinement). Since we restrict ourselves to an analysis of instability of the thread, we are only interested in the real part of \( q \). Therefore, we only have to consider the real part of equation (3.4). Figure 2(a) shows curves of \( q \equiv \Re[q] \) versus \( k \) for various values of \( \mu_0 \) for viscoelastic threads with \( \Lambda = 0 \) (Maxwell fluid in a Newtonian matrix), and for \( \text{De} = 10 \). We observe that the values of \( q \) decrease with increasing \( \mu_0 \). Thus, the more viscous the thread, the more time it takes to break up. If the thread is very
viscous, it will elongate to a cylinder of very small radius and remain stable for a long time before finally breaking up into droplets of very small size. Figure 2(b) shows curves of $k_{\text{max}}$ versus $\mu_0$ for a Newtonian (data taken from Tomotika (1935)) and a viscoelastic thread both immersed in a Newtonian matrix. Note that $k_{\text{max}}$ is the wave number related to the fastest growing growth rate $q_{\text{max}}$. For the viscoelastic thread, $q_{\text{max}}$ occurs for smaller values of $k$ than for the Newtonian one. Since $q_{\text{max}}$ is responsible for the break-up of the thread, the viscoelastic thread will break up in larger droplets (smaller $k_{\text{max}}$) than the comparable Newtonian thread.

The effect of fluid elasticity is shown in Figures 3 and 4, in which $q$ as function of $k$ is depicted for various values of $De$ (Figure 3) or $\Lambda$ (Figure 4). In Figure 3, we see that the smaller $De$, the lower the value of $q$ is. As the instability grows faster for greater values of $q$, we note that the thread breaks up faster for increasing $De$, that is when the rheological time scale becomes greater than the capillary time scale (the $\lambda_1$-effect contra the $a\eta^c/\sigma$-effect). This effect is most manifest for $De > 1$, and almost diminished to zero for $De < 1$. Moreover, observing the places of the maxima of $q$, we see that the value of $k_{\text{max}}$ is only very weakly affected by $De$. The latter implies that $De$ has not much influence on the magnitude of the eventual droplets. In Figure 4, we see that the smaller $\Lambda$, the greater the value of $q$ is. When $\Lambda$ decreases, the thread becomes more elastic, i.e. more solid-like. Also here, we see that the value of $k_{\text{max}}$ is only very weakly affected by, in this case, $\Lambda$, and then this also holds for the eventual magnitude of the droplets.

To conclude this section, we note that we have found that a viscoelastic thread immersed in a Newtonian fluid breaks up faster than a Newtonian one, and that this effect becomes stronger if the elasticity of the viscoelastic thread increases. The value of $k$ where $q$ reaches
Figure 3: Curves of $q$ as function of $k$ for viscoelastic threads, for $De = 10$ (dash-dot curve), $De = 1$ (solid) and $De = 0.1$ (dashed). The remaining parameter values are $\mu_0 = 0.1$, $\Lambda = 0$, $C_p = 0$ and $h \gg 1$.

Figure 4: Curves of $q$ as function of $k$ for viscoelastic threads, for $\Lambda = 1$ (dash-dot curve), $\Lambda = 0.5$ (solid) and $\Lambda = 0.01$ (dashed). The remaining parameter values are $\mu_0 = 0.1$, $De = 10$, $C_p = 0$ and $h \gg 1$.

its maximum is hardly influenced by $De$ and $\Lambda$. Hence, the elasticity of the thread has only minor influence on the magnitude of the eventual droplets.

3.2 Effect of confinement

In this section, we study the effect of confinement. For this, we consider a case of no prescribed flow, $C_p = 0$, and a viscoelastic thread with $\Lambda = 0$ (Maxwell model). Figure 5 shows the curves of $q$ as functions of $k$ for a confined geometry, first (part (a)), for $h = 2$ and for the same set of values of $\mu_0$ as in Figure (2)(a)), and, second (part (b)), for a set of three different confinements ($h = 2, 5$, or $20$) and for $\mu_0 = 0.1$. Comparing Figure 5(a) with Figure 2(a), we notice two striking differences: first, the growth rate is drastically smaller (a factor 10) for the confined thread, and, second, the values of $k$ where $q$ is maximal are much less affected by the value of $\mu_0$ for the confined thread than the unconfined thread. Hence,
the confinement causes a slower break-up, in which the magnitudes of the droplets are practically insensible of the viscosity ratio $\mu_0$. In Figure 5(b), we observe that the growth rate decreases with decreasing $h$ (more confinement), implying that the thread breaks up slower when the degree of confinement is higher (smaller $h$), which is in accordance with the first conclusion from Figure 5(a). Figure 6 shows a curve of $q_{\text{max}}$ as function of $h$ for a viscoelastic thread with $\Lambda = 0$, $\text{De} = 10$, $\mu_0 = 0.1$, and $C_p = 0$. In this Figure, we see that from a certain distance $\overline{h}$ on ($\overline{h} \approx 8$) the values of $q_{\text{max}}$ hardly depends on the degree of confinement. Hence, if $h > \overline{h}$, the presence of the tube wall has no longer effect on the thread stability. This implies that $h \gg 1$ effectively means $h > \overline{h} \approx 8$.

Figure 5: Curves of $q$ as function of $k$ for Maxwell model ($\Lambda = 0$, $\text{De} = 10$) for (a) $h = 2$ and for the same set of values of $\mu_0$ as in Figure 2(a), and (b) for several values of $h$.

Figure 6: Curves of $q_{\text{max}}$ as function of $h$ for viscoelastic threads. The remaining parameter values are $\Lambda = 0$, $\text{De} = 10$, $\mu_0 = 0.1$, and $C_p = 0$. 
3.3 Effect of prescribed flow

Figure 7: Curves of $q$ as function of $k$ for viscoelastic threads, for $C_p = 0$ (dashed curve), $C_p = 0.02$ (dash-dot curve), $C_p = 0.2$ (dot curve) and $C_p = 0.5$ (solid curve). The remaining parameter values are $\Lambda = 0$, $De = 10$, $\mu_0 = 0.1$, and $h \gg 1$.

Finally, we investigate the case $C_p \neq 0$, that is we consider the effect of the prescribed (Poiseuille) flow on the break-up of the thread. Formula (3.4) indicates that in general $C_p$ contributes to both the real and imaginary part of $q$. However, in case of a Newtonian thread (see (3.6)), $C_p$ only contributes to the imaginary part of $q$. This implies that then the thread is oscillatory unstable with the growth rate equal to the one within a quiescent fluid. In case the thread is viscoelastic, $C_p$ contributes to both parts of $q$. In this case, the flow causes the growth rate, i.e. $\mathfrak{R}[q]$, to increase as shown in Figure 7, where $q \equiv \mathfrak{R}[q]$ is depicted as function of $k$ for various values of $C_p$ (and for $\Lambda = 0$, $De = 10$, $\mu_0 = 0.1$, and $h \gg 1$). This implies the effect of prescribed flow is that the viscoelastic thread will become oscillatory unstable and will break up faster than the one within a quiescent fluid.

4 Conclusions

In this chapter, we have studied the flow problem of a viscoelastic (Jeffreys model) thread immersed in a Newtonian fluid occupying a cylindrical tube. The thread is placed concentric in the tube. In the steady basic state the fluid-thread system flows, driven by a constant uniform pressure gradient, according to a Poiseuille flow (prescribed flow). The stability, or break-up, of the thread is investigated. Especially, one by one the influences are inventorised of the viscosity ratio $\mu_0$, the degree of confinement $h$, the prescribed flow ($C_p$), the Deborah number $De$, and the material constant $\Lambda$ on the growth rate $q$ of the break-up process and on the wave number $k$ of the perturbed thread. The magnitude of the eventual droplets is related to this wave number.

Some conclusions that are found here, are:

1. In general, i.e. for $C_p \neq 0$, the growth rate $q$ is a complex number; the (sign of the) real part is responsible for stability or instability (break-up) of the thread; its imaginary part for oscillatory behaviour.
2. The wave number $k_{max}$ in break-up is smaller for a viscoelastic thread than for a Newtonian one, implying that a viscoelastic thread will break up in larger droplets than a Newtonian thread.

3. The rate of break-up of a viscoelastic thread immersed in a Newtonian fluid is influenced by both $De$ and $Λ$, in so far that the thread breaks up faster for increasing $De$ and decreasing $Λ$. This also reveals that a viscoelastic thread breaks up faster than a Newtonian one.

4. The magnitude of the eventual droplets is strongly affected by the ratio of viscosity $μ_0$, but practically unaffected by $De$, $Λ$, and $h$.

5. Confinement ($h$) does not make the thread stable (no break-up), but it makes the break-up slower, in so far that $q$ decreases with decreasing $h$.

6. In case of a Newtonian thread, $C_p$ only contributes to the imaginary part of $q$. This implies that the thread is oscillatory unstable with the growth rate equal to the one within a quiescent fluid.

7. In case of a viscoelastic thread, $C_p$ contributes to both the real and the imaginary part of $q$. The growth rate, which is equal to $ℜ[q]$, is always positive and becomes drastically larger for increasing $C_p$. This implies that the thread will break up faster than the one within a quiescent fluid. Moreover, the break-up process will show oscillatory behaviour.

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**Appendix**

$$M = \begin{pmatrix} kI_0(k) - 2I_1(k) & -kI_1(k) & -[kI_0(k) + 2K_1(k)] & -kK_1(k) & -[kI_0(k) - 2I_1(k)] & kI_1(k) \\ -I_1(k) & I_0(k) & -K_1(k) & -K_0(k) & I_1(k) & -I_0(k) \\ η(q)kI_1′(k) & -η(q)kI_1(k) & kK_1′(k) & -kK_1(k) & -kI_1'(k) & kI_1(k) \\ 2η(q)[kI_1(k) - 2I_1′(k)] & -2η(q)kI_1′(k) & -2[2K_1′(k) - kK_1(k)] & -2kK_1′(k) & -2[kI_1(k) - 2I_1′(k)] & 2kI_1′(k) \\ 0 & 0 & khK_0(kh) + 2K_1(kh) & kK_1(kh) & khI_0(kh) - 2I_1(kh) & -kI_1(kh) \\ 0 & 0 & hK_1(kh) & K_0(kh) & -hI_1(kh) & I_0(kh) \end{pmatrix}$$

$$η(q) = \frac{μ_0}{1 + qΛDe}$$

$$\frac{1}{1 + qDe}$$
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