Multirate methods for the transient analysis of electrical circuits

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Multirate integration is an important tool to increase the speed of the transient analysis of circuits. This paper shows an approach for the “Compound-Fast” multirate algorithm how to control the errors at the coarse and the refined time-grid by means of the independent stepizes of these grids.

1 Introduction

Analogue electrical circuits are usually modelled by differential-algebraic equations of the following type:

\[
\frac{d}{dt} [q(t, x)] + j(t, x) = 0,
\]

where \( x \in \mathbb{R}^n \) represents the state of the circuit. A common analysis is the transient analysis, which computes the solution \( x(t) \) of this non-linear DAE along the time interval \([0, T]\) for a given initial state.

In the classical circuit simulators, this Initial Value Problem is solved by means of implicit integration methods, like the BDF-methods. Each iteration, all equations are discretized by means of the same stepsize. In contradiction to classical integration methods, multirate methods integrate both parts with different stepizes or even with different schemes. Besides the coarse time-grid \( \{T_n, 0 \leq n \leq N\} \) with stepizes \( H_n = T_n - T_{n-1} \), also a refined time-grid \( \{t_{n-1,m}, 1 \leq n \leq N, 0 \leq m \leq q_n\} \) is used with stepizes \( h_{n,m} = t_{n,m} - t_{n,m-1} \) and multirate factors \( q_n \). If the two time-grids are synchronized, \( T_n = t_{n,0} = t_{n-1,q_n} \) holds for all \( n \). There are a lot of multirate approaches for partitioned systems [1, 4, 5] but we will consider the “Compound-Fast” version of the BDF methods. This method first integrates the complete circuit at the coarse grid. Afterwards only the active part is integrated at the refined grid, while the latent interface variables are interpolated.

Multirate BDF methods need both coarse and refined Nordsieck vectors \( \dot{Y}^n, \ddot{X}^n, \dddot{P}^n, \dddot{Q}^n \) and \( \dddot{Y}^m, \dddot{X}^m, \dddot{P}^m, \dddot{Q}^m \), which represent the coarse and refined Predictor and Corrector polynomials for \( x(t) \) and \( q(t, x(t)) \) [3]. Note that for the refined vectors only the active part is stored. Algorithm (1) shows the structure of Partitioned ’Compound-Fast’ Multirate BDF methods.

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**ALGORITHM 1** Partitioned "Compound-Fast" Multirate BDF algorithm

**Compound phase**

Initialize integration method
While \( T_{n-1} < T \) do:
- Compute coarse predictor Nordsieck vectors
- Initialize coarse step
- Solve \( x^n \) from nonlinear equation
- Update coarse corrector Nordsieck vectors
- Compute (weighted) error norm \( \hat{r}^p_n \) and interpolation error norm \( \hat{r}^q_n \)
  
  If \( \hat{r}^p_n < \text{TOL}_C \) and \( \hat{r}^q_n < \text{TOL}_I \) do:
  \( T_n = T_{n-1} + H_n \)

  **Perform refinement phase**

  Update coarse corrector Nordsieck vectors for active part
  Compute next timestep \( H_{n+1} \) and order \( k_{n+1} \)
  \( n = n + 1 \)
  else do:
  Reduce timestep (or order)
End

**Refinement phase**

Initialize refinement phase
While \( t_{n-1,m-1} < T_n \) do:
- Compute refined predictor Nordsieck vectors
- Initialize refinement step
- Interpolate \( \tilde{x}^{n-1,m}_{L} \) from \( \tilde{X}^n \)
- Solve \( x^{n-1,m}_L \) from nonlinear equation
- Update refined corrector Nordsieck vectors for active part
- Compute error norm \( r^n_{A} \)
  
  If \( r^{n-1,m}_A < \text{TOL}_A \) do:
  \( t_{n-1,m} = t_{n-1,m-1} + h_{n-1,m} \)
  Compute next timestep \( h_{n-1,m+1} \) and order \( k_{n-1,m+1} \)
  \( m = m + 1 \)
  else do:
  Reduce timestep (or order)
End

3 Multirate error control

Although the orders \( K_n \) and \( k_{n,m} \) can be variable, from now on they are assumed to be fixed at \( K_n = K, k_{n,m} = k \). The local discretization error \( \delta^n \) of the compound phase still has the same behaviour \( \delta^n = O(H^{n+1}_n) \). This error can be estimated by \( \delta^n \) using the Nordsieck vectors for \( q: \hat{q}^n = \frac{z^n - B^n_{n-1}}{-L} [\tilde{Q}^n - B^n_{0}] \). Now \( \hat{r}^C_n = ||B^*_n \hat{q}^n|| = \| \hat{r}^C_n \| \) is the used weighted error norm, which must satisfy \( \hat{r}^C_n > \text{TOL}_C \). At the refined time-grid the DAE has been perturbed by the interpolated latent variables. The local discretization error \( \delta^{n,m} \) is defined as the residue after inserting the exact solution in the BDF scheme of the refinement phase. However during the refinement instead of \( \delta^{n,m} \) the perturbed local error \( \overline{\delta}^{n,m} \) is estimated. During the refinement each step \( \overline{\delta}^{n,m} \) is computed from the following scheme:

\[
\alpha_{n-1,m} q_A(t_{n-1,m}, x^n_{A}, \tilde{x}^{n-1,m}_L, h_{n,n-1,m}) + h_{n,n-1,m} J_A(t_{n-1,m}, x^n_{A}, \tilde{x}^{n-1,m}_L) + \beta_{n-1,m} = 0.
\]

Here \( \beta_{n-1,m} \) is a constant which depends on the previous values of \( x_A \) and \( \tilde{x}_L \). Let \( \beta_{n-1,m} \) be the constant for exact values of \( x_L \), then we assume that \( \beta_{n-1,m} = \beta_{n-1,m} + \frac{\partial \delta_{n,m}}{\partial \tilde{x}_L} (\hat{x}_L(t_{n-1,m})) \). Using the notation \( t = t_{n-1,m}, h = h_{n,n-1,m} = \alpha = \alpha_{n-1,m}, \beta = \beta_{n-1,m} \), the error \( \delta^{n,m} \) satisfies

\[
B_A \delta^{n-1,m} \equiv \alpha q_A(t, x_A(t), x_L(t)) + h_j A(t, x_A(t), x_L(t)) + \beta = \alpha q_A(t, x_A(t), \tilde{x}^{n-1,m}_L) + h_j A(t, x_A(t), \tilde{x}^{n-1,m}_L) + \beta + h K_{n-1,m} (x_L(t) - \tilde{x}^{n-1,m}_L).
\]

The coupling matrix \( K_{n-1,m} \) can be estimated by

\[
K_n = \frac{1}{h} B_A \frac{\partial \delta^{n-1,m}}{\partial x_A} (T_n, x_A^n) - \frac{\partial \delta^{n-1,m}}{\partial x_L} (T_n, x_L^{n-1}) B_L^T + \frac{1}{h} \frac{\partial q_A}{\partial x_L} (T_n, x_A^n) + \frac{1}{h} \frac{\partial q_A}{\partial x_L} (T_n, x_A^n) + \frac{1}{h} \frac{\partial q_A}{\partial x_L} (T_n, x_A^n).
\]

The perturbed local discretization error \( B_A \delta^{n,m} \) behaves as \( O(H^{n+1}_n) \) and can be estimated in a similar way as \( \delta^n \). Let \( L \) be the interpolation order, e.g. \( L = K \), then it can be shown that \( \| [K_n B_L]^{n-1,m} \| \) has the asymptotic behaviour \( \hat{r}^A_n = O(H^{n+1}_n) \). It follows that \( B_A \delta^{n,m} \) satisfies:

\[
\| B_A \delta^{n-1,m} \| \leq \hat{r}^A_{n-1,m} + \hat{r}^A_n =: \hat{r}^A_{n-1,m}.
\]

If \( \hat{r}^A_n \leq \text{TOL}_A \), then \( r^{n-1,m} \leq \text{TOL}_A + h \text{TOL}_I = \text{TOL}_A \). The weighting factor \( 0 < \beta < 1 \) is chosen such that \( (\hat{r}^C_{TOL}/\text{TOL}_C) \leq (\hat{r}^I_{TOL}/\text{TOL}_I) \). Adaptive stepsize control can be used to keep \( \hat{r}^I_n = O(H^{n+1}_n) \) and \( \hat{r}^A_{n-1,m} = O(h^{k+1}_{n-1,m}) \) close to \( \theta \text{TOL}_A \) and \( \theta \text{TOL}_A \) respectively, where \( 0 < \theta < 1 \) is a safety factor.

**References**


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