The Joukowsky equation for fluids and solids

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Abstract

This report provides an extension to a previous paper (Tijsseling and Anderson, 2004) in which we showed that Johannes von Kries (1883) was the first to derive and validate the "Joukowsky equation" for waterhammer. Since there is a strong analogy between pressure waves in fluids and stress waves in solids, and waterhammer relates to impact mechanics, it is likely that the "Joukowsky equation" for solids already existed before 1883. Also, 19th century's scientists must have been aware of the fluids/solids analogy.

In this historical study we try to answer the question of who was the first to derive the Joukowsky equation in either fluids or solids.

Key words: Water hammer; Transient flow; Pipe flow; Impact mechanics; History

The "Joukowsky equation" for fluids

The fundamental equation in waterhammer theory relates pressure changes, Δp, to velocity changes, Δv, according to

$$\Delta p = \rho c \Delta v \qquad (1)$$

where ρ is the fluid mass density and c is the speed of sound. Korteweg's (1878) formula defines c for fluid contained in cylindrical pipes of circular cross-section:

$$c = \sqrt{K^* / \rho} \qquad \text{and} \qquad K^* = K / [1 + (DK)/(eE)] \qquad (2)$$

where D is the diameter of the pipe, e is the wall thickness, E is the modulus of elasticity for the wall, and K is the bulk modulus of the contained fluid.

Relation (1) is commonly known as the "Joukowsky equation", but it is sometimes referred to as either the "Joukowsky-Frizell" or the "Allievi" equation. Its first explicit statement in the context of waterhammer is usually attributed to Joukowsky (1898). Frizell (1898) and Allievi (1902, 1913), unaware of the achievements by Joukowsky and Frizell, also found equation (1), but they did not provide any experimental validation. Anderson (2000) noted that Rankine (1870) had already derived equation (1) in a context more general than waterhammer. See the Appendix of (Tijsseling and Anderson, 2004). Kries (1883, p. 74) derived relation (1), mentioning – without a particular reference – its existence in the theory of shock waves, but at the same time stating that it had not been validated by experiments, something he would do.

The "Joukowsky equation" for solids

The early investigators of waterhammer had not noticed the analogy with longitudinal waves in solid bars (Boulanger 1913, p. 14) except for Stromeyer (1901) in a rare paper and Gibson (1908, pp. 40-41).

Young (1807, pp. 143-145) found that the strain ε produced by the impact of elastic solid bodies equals v/c. With Hooke's law stating that $\varepsilon = -\sigma/E$, where σ is stress and E is Young's modulus of elasticity, this gives $\sigma = -Ev/c$. Assuming that $c = \sqrt{(E/\rho)}$, one obtains for the solids equivalent of equation (1):

$$\sigma = -\rho c v \tag{3}$$

Young (1808) was the first to find the pressure wave speed for incompressible liquids contained in elastic tubes, and the authors think that Young was also aware of the speed of sound in solid bars, $c = \sqrt{(E/\rho)}$, as explained in the Appendix herein. Young's work is difficult to read, but Timoshenko (1953, pp. 93-94) gives a neat summary of the above expressed in modern terminology. It is noted that the strain ε in liquids contained in tubes equals P/K^*, where K^* is the effective bulk modulus representing fluid compressibility and tube wall elasticity.

Saint-Venant (1867) gives a clear, rigorous and complete treatment of the longitudinal collision of two solid bars. This is analogous to frictionless waterhammer. On the pages 355-357, Eqs. (a), (b) and (c), he derives for a bar of cross section A: $F = \sigma A = -EA\varepsilon$, $v = c\varepsilon$ and $c = \sqrt{(E/\rho)}$, which can be combined into Eq. (3). In later papers Saint-Venant (1870, 1883) gives full credit to Babinet for the first clear derivation of c (oral presentation in 1829, written down by Pierre in 1862, p. 155), although the formula itself goes back to Newton, Euler and Lagrange. The corresponding speed of sound in liquids is $c = \sqrt{(K^*/\rho)}$. Korteweg (1878) derived the proper value for K^* in waterhammer given in Eq. (2). Saint-Venant also employed a graphical method forerunning the Schnyder (1932) - Bergeron (1935) graphical method (this was the standard waterhammer calculation tool in the pre-computer era). It is remarkable to see that it is Rankine (1867) who reviewed Saint-Venant's (1867) paper (with partial translation into English). In earlier work Rankine (1851) had found the wave speed of nearly longitudinal vibration and he already noted the similarity of vibrations in solids and liquids.

The history of this subject is extensively described by Todhunter and Pearson (1886, 1893) and Timoshenko (1953). Timoshenko and Goodier (1970, pp. 492-494) summarise the achievements of Young and Saint-Venant. Bergeron (1950; 1961, pp. 194-233) is probably the first to apply – the other way around – waterhammer theory to the axial vibration of solid bars.

Conclusions

The "Joukowsky equation", $\Delta p = \rho c \Delta v$, its derivation and validation, was published by Joukowsky (1898) in a comprehensive study of pressure waves in water supply lines. The same equation had earlier been derived and validated, through experiments in water-filled rubber hoses, by Kries (1883) in a study of the pulse. Independently, Frizell (1898) and Allievi (1902) derived the "Joukowsky equation" in pure theoretical studies.

It is Rankine (1870) who had already found the equation in a more general context, thus preceding Kries and Joukowsky. Rankine (1870) opened his paper by writing that: “The object of the present investigation is to determine the relations which must exist between the laws of the elasticity of any substance, whether gaseous, *liquid or solid*, and those of the wave-like propagation of a finite longitudinal disturbance in that substance.” He was fully aware of the analogy between waves in fluids and solids, given that Rankine (1867) had reviewed and translated an impressive piece of work by Saint-Venant (1867) on the elastic collision of two solid bars. Saint-Venant (1867) derived three equations, which combine into the "Joukowsky equation" for solids, $\sigma = -\rho c v$.

It is typical for Young (1802, 1807, 1808) that he had found all the ingredients to arrive at the "Joukowsky equation" for fluids *and* solids, but that his achievements were not picked up by his contemporaries.

Acknowledgements

This work was supported by funding under the European Commission’s Fifth Framework ‘Growth’ Programme via Thematic Network “Surge-Net”, contract reference: G1RT-CT-2002-05069 (*www.surge-net.info*).

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Appendix: Young (1808) and the waterhammer wave speed

Young (1808) is of fundamental importance to the history of waterhammer, concerning "the propagation of an impulse through an elastic tube", in which Young derived for the first time the now standard formula for waves of an incompressible fluid in an elastic tube (which forms half of the waterhammer wave velocity expression). Unfortunately his analysis was obscure and the actual formula was not explicitly written in his paper so his achievement (like many others) passed unnoticed until it was rediscovered nearly half a century later by the brothers Weber (1850, 1866).

Young's argument proceeded as follows. "The same reasoning, that is employed for determining the velocity of an impulse, transmitted through an elastic *solid or fluid* body, is also applicable to the case of an incompressible fluid contained in an elastic pipe" (this suggests that Young had obtained the speed of sound in a solid bar). The problem is then to determine the apparent modulus of elasticity conferred on the incompressible fluid by the elasticity of pipe walls, or, in Young's terminology, to discover "the height of the modulus" to be substituted into Newton's basic formula (Young 1802)

$$c = \sqrt{gh} \qquad \text{(A1)}$$

for the speed of sound, this formula giving a velocity half as great as that of a body falling freely from a height $2h$ [$2h = gt^2/2$ gives $t = \sqrt{(4h/g)}$, and therefore $gt = 2\sqrt{(gh)}$]. Note that Young first introduced his modulus with the dimension of height rather than the modern dimension of stress (Todhunter and Pearson 1886, p. 82; Straub 1952, p. 155; Timoshenko 1953, p. 92).

Continuing the argument, if the pipe is such that the increase in tension force varies as the increase in circumference or diameter from the natural state (i.e., the pipe is elastic and obeys Hooke's law) up to the limit (at which the pressure in the fluid must balance the tension in the pipe by Newton's first law) where an infinite increase in diameter occurs (i.e., plastic deformation at elastic limit), then the height of a column of liquid equivalent to the pressure causing failure is designated "the modular column of the pipe". This is an application of the maximum stress theory that was favoured by English writers over the maximum strain theory, which was favoured on the Continent (Timoshenko 1953, p. 89).

The relationship is readily demonstrated since, from the stress/strain curve up to the elastic limit $f = \sigma\varepsilon/2 = \sigma^2/(2E)$ (for $\sigma = E\varepsilon$) or, replacing the stresses with their equivalent "heights", $2h = (2h)^2 \rho g/(2E)$, i.e., $h = E/(\rho g)$.

For the equivalent elasticity conferred on the incompressible fluid Young used the continuity principle. If a short length of pipe of diameter D and length x is compressed in length by a pressure pulsation to $(x - \delta x)$, then if the fluid is incompressible the diameter D must increase to preserve continuity so that $(2\delta D/D - \delta x/x)$ = 0. But the increase in hoop strain $(\partial D/D)$ = (σ/E) for a pipe in tension, and the hoop stress for an increase in pressure δP is given by $D\delta P/(2e)$, so that $D/(Ee)$ = $\delta P/(\delta x/x)$. The right hand side of this last relationship defines precisely an apparent compressibility for the liquid, which is therefore given conveniently by the expression on the left hand side. Young terminated his argument at this point but it is a trivial matter to make the substitution into Eq. (A1) to give explicitly:

$$c = \sqrt{\frac{Ee}{\rho D}} \qquad \text{(A2)}$$

Young was undoubtedly in a position to obtain the celerity of the waterhammer wave if he so desired. The continuity method he used can be extended to take account of compressible fluids (indeed it was the method used by Korteweg, Kries and Joukowsky, seventy, seventy-five and ninety years later, respectively). Nevertheless he did not, though he did go on to consider the reflection and collision of waves, to state that the particle velocity must be less than the wave velocity and to examine the effect of a contraction in a pipe.

Notation

A	cross-sectional area, m^2
c	sonic wave speed, m/s
D	internal tube diameter, m
E	Young modulus, Pa
e	tube wall thickness, m
F	force, N
f	elastic limit, Pa
g	gravitational acceleration, m/s^2
h	height, pressure head, m
K	fluid bulk modulus, Pa
K^*	effective fluid bulk modulus, Pa
p	fluid pressure, Pa
t	time, s
v	velocity, m/s
x	length, m
Δ	change, jump
ε	longitudinal strain
ρ	mass density, kg/m^3
σ	longitudinal stress, Pa