Transfer function of acoustically perturbed Bunsen flames. Theoretical investigation

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Abstract

The kinematic models for perturbed flames from [7, 6] are extended to account for flames with arbitrary cone angles and a burning velocity with non constant direction. This extension improves the description of the front close to the boundary and, consequently, the behaviour of the flame response to velocity perturbations.

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1 Introduction

Understanding the interaction between premixed Bunsen flames and acoustic perturbations of the gas velocity would be important for understanding combustion noise in small-scale combustion devices [4, 12].

The response of a Bunsen flame to velocity perturbations is evaluated in terms of the flame transfer function (TF), which is the ratio of the heat release rate perturbation to velocity perturbation in the frequency domain. The main features of the flame kinematics can be described by the G-equation model [13] in which the flame is treated as a surface (flame front) that separates the burnt from the unburnt gas. In this model the movement of the flame front under the action of the perturbed flow velocity \( \mathbf{v} \) and of the laminar burning velocity \( \mathbf{s}_L \) is given by the solution of the G-equation. The perturbation of the heat release rate due to flame front movement can be considered proportional with the area of the flame [7, 6]. Therefore, the evaluation of the flame TF involves computing the area of the flame which consequently requires the solution of the

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G-equation. The G-equation can be solved provided that models for the gas velocity and for the laminar burning velocity are given. For example the laminar burning speed $S_L$ can be assumed constant except possibly near the tip and the base of the flame [10]. The unperturbed flow velocity at the tube exit can be modeled as a Poiseuille flow [7] or as a constant uniform flow [6, 2, 5]. Since from a practical point of view it is important to study the response of the flame to small amplitude velocity perturbations the solution of the G-equation can be approximated with an asymptotic expansion. Collecting the terms of the same order of the expansion, leads to a system of linear partial differential equations (PDE) that is very difficult to solve analytically. A possible way to overcome these difficulties is to add a set of constraints to the model that would lead to a linear form of the G-equation hence, to a simple system of equations that allows for an analytical solution. For example, linear G-equations were obtained by assuming very long flames parallel with the stream lines [7], or that the laminar burning velocity has a constant direction, normal to a stationary position of the flame [6, 2]. These linear models ([7, 6]) predict the correct behaviour of the magnitude of the flame TF (low pass filter), but fail in describing the phase of the TF. Unlike the phase of the measured TF which increases linearly with the increase of the frequency, the phase of the theoretical TF saturates to a level of $\pi/2$ [6]. To understand the origin of the discrepancy between experiments and theory in describing the behaviour of the phase TF, here we extend the kinematic models proposed previously in [7, 6]. Here we consider flames subject to a perturbed Poisseuile flow that have an arbitrary cone angle and a burning velocity with a normal direction relative to the instantaneous perturbed flame front and consequently, non-constant relative to the stationary front position. Thus, we employ the nonlinear form of the G-equation instead of a simplified linear one.

Computing the solution of the G-equation or of the corresponding system requires boundary conditions. To apply boundary conditions the assumption that the flame is attached at the burner rim was employed [7, 6, 2, 11] ignoring the motion of the flame edge above the burner rim. This assumption is possible because the quenching distance is very small compared with the flame length [10]. The use of a linear form of the G-equation allows the flame attachment independently of the models for the flow velocity and for the laminar burning speed. For the case of the nonlinear G-equation the flame attachment is possible only if the slope at the flame front is defined at the burner rim [5]. To obtain a well defined slope the assumption of a perturbation amplitude lower than a critical value was used for the case of a mean velocity profile [11]. For the case of a Poisseuile flow considered here it is not possible to find a critical value for the perturbation amplitude that would allow for a well defined slope. Thus, in order to impose boundary conditions and to compute the flame front location we restrict the computational domain to a physical interval where the slope is defined. To compute the flame front location the solution of the nonlinear G-equation is approximated with an asymptotic expansion from which the leading, the first and the second order terms are retained, resulting in a system of linear advection equations. Without employing the constrains from [7, 6, 2] the leading and the first order solution are expressed analytically in terms of elliptic integrals.

The first order term in the asymptotic expansion of the location of the flame front computed here by employing the nonlinear form of the G-equation is com-
pared with the solution derived by using a linear form in [7]. A detailed analysis reveals that the use of the nonlinear G-equation brings an improvement to the solution from [7] close to the burner rim. The source term of the second order equation depends on the slope of the first order equation. The analysis of the first order solution reveals that the slope is very steep close to the burner rim implying a similar behaviour for the second order solution. Thus the second order solution is computed numerically by employing a novel nondissipative box scheme adapted for a nonuniform grid. By combining the leading, the first and the second order solution the time dependent flame front location is obtained. The flame front location is employed to compute the time dependent area of the flame for different perturbation frequencies. The response of the flame area to velocity perturbation is computed in terms of a flame transfer function. In agreement with the experiments the magnitude of the transfer function has a low pass filter behaviour. The phase of the transfer function computed here is an improvement of the phase presented in [7, 6]. The improvement is due to a better flame representation close to the burner rim.

2 Flame model

By assuming that the flame is nowhere vertical the flame front is described by the following so-called G-equation, i.e., \( G(r, z, t) = z - \zeta(r, t) = 0 \) where \( G \) is some combustion variable, \( \zeta \) is the location of the flame front, \( r \) and \( z \) are the axial and the radial coordinates respectively and \( t \) is time. The flame front moves under the action of the perturbed gas velocity \( v \) and of the laminar burning velocity \( S_L \) whose direction is normal to the instantaneous flame front. The movement of the flame front is given by the following nonlinear kinematic relation,

\[
\frac{\partial \zeta(r, t)}{\partial t} + u \frac{\partial \zeta(r, t)}{\partial r} - v + S_L \sqrt{1 + \left( \frac{\partial \zeta(r, t)}{\partial r} \right)^2} = 0,
\]

(1)

where \( u(r, z, t) \) and \( v(r, z, t) \) are the radial and the axial components of the perturbed gas velocity, respectively. For the mean, unperturbed gas velocity we assume as in [7] a Poiseuille profile, i.e.,

\[
\tau(r) = 0, \quad \tau(r) = v_0 \left( 1 - \left( \frac{r}{R} \right)^2 \right),
\]

(2)

where \( v_0 > 0 \) and \( R \) are the maximum velocity at the centerline and the radius of the duct, respectively. The perturbed gas velocity is the sum of the mean velocity with a perturbation modeled by

\[
u'(r, t) = 0, \quad \nu'(r, t) = \varepsilon v_0 \sin(\omega t),
\]

(3)

where \( \omega \) and \( \varepsilon \) are the angular frequency and the amplitude of the velocity oscillation, respectively. Here we consider as in [6, 2, 7] that the amplitude \( \varepsilon \) is small and that the laminar burning speed \( S_L \) is constant. To find the response of the flame to velocity perturbations, equation (1) needs to be solved. Equation (1) cannot be solved analytically for \( \varepsilon \neq 0 \). Instead, \( \zeta(r, t) \) is approximated by the following asymptotic expansion

\[
\zeta(r, t) = \zeta_0(r, t) + \varepsilon \zeta_1(r, t) + \varepsilon^2 \zeta_2(r, t) + \cdots
\]

(4)
By substituting expression (4) into equation (1) and collecting terms of the same order a system of linear advection equation is obtained. The small amplitude assumption allows us to take into consideration only the leading, the first, and the second order equations. After introducing the scaling, \( r^* := r/R, \ t^* := t/\tau, \ z^* := z/R, \ \omega^* = \omega R/S_L \), the system in dimensionless variables (omitting the *) reads

\[
\begin{align*}
\frac{\partial \zeta_0}{\partial t} & - \hat{v}(1 - r^2) + \sqrt{\left( \frac{\partial \zeta_0}{\partial r} \right)^2 + 1} = 0, \\
\frac{\partial \zeta_1}{\partial t} & = \frac{\sqrt{\hat{v}^2(1 - r^2)^2 - 1} \ \partial \zeta_1}{\hat{v}(1 - r^2)} = \hat{v} \sin(\omega t), \\
\frac{\partial \zeta_2}{\partial t} & = \frac{\sqrt{\hat{v}^2(1 - r^2)^2 - 1} \ \partial \zeta_2}{\hat{v}(1 - r^2)} = -\frac{1}{2\hat{v}(1 - r^2)^2} \left( \frac{\partial \zeta_1}{\partial r} \right)^2,
\end{align*}
\]

where \( \tau := R/S_L \) and \( \hat{v} := v_0/\omega S_L \). Typical values for \( \hat{v} \) are \( \hat{v} \in [2, 10] \).

### 3 Boundary conditions

To solve (5) the location of the flame front position at the burner rim \( (r = 1, \text{ in dimensionless variables}) \) needs to be prescribed. In the sequel we will derive boundary conditions for (5) by analysing the scaled equation for the flame front corresponding to a purely axial time dependent flow, i.e.,

\[
\frac{\partial \zeta}{\partial t} - v(r, t) + \sqrt{1 + \left( \frac{\partial \zeta}{\partial r} \right)^2} = 0,
\]

where \( v(r, t) \) is the axial velocity of the flow. The assumed attachment of the flame front at the burner rim implies \( \zeta(1, t) = 0 \). Thus from (6) we obtain the slope at the flame front at the point \( r = 1 \),

\[
\frac{\partial \zeta}{\partial r}(1, t) = -\sqrt{v(1, t)^2 - 1}.
\]

Then the boundary condition \( \zeta(1, t) = 0 \) can be imposed only if the condition

\[
v(1, t)^2 - 1 \geq 0,
\]

is fulfilled. Depending on the model for the axial gas velocity, condition (8) can limit the range of amplitude of the velocity perturbation. For example, in the case in which the axial velocity is assumed constant and the perturbation is uniform in space and harmonic in time condition (8) is satisfied when the magnitude of the velocity perturbation is lower than a critical value, see [11]. For our case (Poiseuille profile) condition (8) becomes

\[
(\hat{v}\varepsilon \sin(\omega t) - 1)(\hat{v}\varepsilon \sin(\omega t) + 1) \geq 0.
\]

Condition (9) cannot be satisfied for an \( \varepsilon \) independent of \( t \). Boundary conditions for the situation when \( v(1, t) < 1 \) were suggested in [5] for the case of a ducted flame. However, the use of the boundary condition from [5] would allow the flame to propagate into the tube. Here, to prevent flash back and to allow
for the computation of the flame TF, we employ an idea from [1] used to apply boundary conditions to (5a). By solving equation (5a) with the initial condition \( \zeta(r, 0) = 0 \) the evolution of a the Bunsen flame from an initial flat profile to its steady conical shape was investigated in [1]. To avoid non-physical solutions generated by the fact that the flow speed does not balance the laminar flame speed at the rim, the computation of the transient flame position was restricted to a physical interval on which the flow speed is larger than the laminar speed and thus condition (8) is fulfilled. Thus, we determine the physical interval \([0, \delta_{\infty}(t)]\) in which the condition (8) is fulfilled and we restrict our computational domain to this interval. Taking into account that (5a) is defined on \([0, \delta]\), 

\[
\delta_{\infty}(t) := \sqrt{1 - \frac{1}{\hat{v}}}, \quad \text{when the flow is perturbed } \delta_{\infty}(t) \text{ belongs to the interval}
\]

\[
\left(1 - \epsilon \sin(\omega t) - \frac{1}{\hat{v}} \right) \in \left[\sqrt{1 - \frac{1}{\hat{v}}}, \delta\right],
\]

provided that \( \hat{v} > 1 \) and \( 0 < \epsilon < 1 - 1/\hat{v} \). To avoid the difficulties of working on an interval that is time dependent we restrict further the domain to \([0, \delta_{\infty}]\) where

\[
\delta_{\infty} = \sqrt{1 - \frac{1}{\hat{v}}}. \tag{11}
\]

To find the flame front position \( \zeta(r, t) \) we will compute the solution of (5) subject to the following initial and boundary conditions

IC \quad \zeta_1(r, 0) = \zeta_2(r, 0) = 0, \tag{12a}

BC \quad \zeta_0(\delta_{\infty}, t) = \zeta_1(\delta_{\infty}, t) = \zeta_2(\delta_{\infty}, t) = 0. \tag{12b}

The solution of the system (5) on the physical interval \([0, \delta_{\infty}]\) can be used to compute the area of the flame by employing the formula

\[
A_{\infty}(t) = 2\pi \int_0^{\delta_{\infty}} r \sqrt{\left(\frac{\partial \zeta}{\partial r}\right)^2 + 1} dr. \tag{13}
\]

4 Solution of the leading and of the first order equations

To simplify the calculations we assume that perturbation in the gas velocity is introduced after the stabilisation of the flame above the burner rim, so that the leading order term in (4) can be replaced by the stationary position of flame front. The stationary position found in [1] recovers the steady solution of \( \zeta_0 \) whose expression in terms of elliptic integrals reads [1, 10],

\[
\zeta_0(r) = \frac{\sqrt{\hat{v} + 1}}{3\sqrt{\hat{v}}} \left( 2\hat{v}(E(\kappa) - E(\theta(r), \kappa)) - 2(K(\kappa) - F(\theta(r), \kappa)) - (\hat{v} - 1) \sin \theta(r) \cos \theta(r) \sqrt{1 - \lambda^2 \sin^2 \theta(r)} \right), \tag{14}
\]

where

\[
\theta(r) := \arcsin \left( \frac{r\sqrt{\hat{v}}}{\sqrt{\hat{v} - 1}} \right), \quad \kappa := \frac{\sqrt{\hat{v} - 1}}{\hat{v} + 1}. \tag{15}
\]
Here $F(\theta(r), \kappa)$ and $E(\theta(r), \kappa)$ are the incomplete elliptic integrals of first and second kind, respectively and $K(\kappa)$ and $E(\kappa)$ are the complete elliptic integrals of first and second kind, respectively (see [3]).

Equation (5b) can be written as

$$\frac{\partial \zeta_1}{\partial t} - \frac{d}{dr}\frac{\partial \zeta_1}{\partial r} = \hat{\upsilon} \sin \omega t, \quad (16)$$

where

$$\frac{d}{dr} = \frac{\hat{\upsilon}(1 - r^2)}{\sqrt{\hat{\upsilon}^2(1-r^2)^2 - 1}} \quad (17)$$

By using [3] $B$ can be expressed in terms of elliptic integrals,

$$B(r) = \frac{1}{\sqrt{\hat{\upsilon}^2 \hat{\upsilon} + 1}} ((\hat{\upsilon} + 1) E(\beta(r), \lambda) - F(\beta(r), \lambda)) + c, \quad (18)$$

where $c$ is constant. Equation (16) along with the boundary condition, $\zeta_1(\delta_r, t) = 0$, and the initial condition, $\zeta_1(r, 0) = 0, 0 \leq r \leq \delta_r$, is integrated using the Laplace transform in $t$ to yield

$$\zeta_1(r, t) = \frac{\hat{\upsilon}}{\omega} ((\cos(\omega(-B(\delta_r) + B(r) + t)) - \cos(\omega t)), \quad (19)$$

for $t > B(\delta_r)$. In order to derive the area of the flame, the derivative of $\zeta$ with respect with $r$ needs to be evaluated. From (19) follows that

$$\lim_{\varepsilon \to 0} \frac{\partial \zeta_1}{\partial r}(\delta_r) = \lim_{\varepsilon \to 0} -\hat{\upsilon} \frac{dB}{dr}(\delta_r) \sin(\omega t) = \left\{ \begin{array}{cl} -\infty, & \text{if } \sin(\omega t) > \varepsilon, \\ 0, & \text{if } |\sin(\omega t)| < \varepsilon, \\ +\infty, & \text{if } \sin(\omega t) < -\varepsilon. \end{array} \right. \quad (20)$$

The above analysis reveals that both $\zeta_1$ and the slope of $\zeta_1$ have a boundary layer at $\delta_r$ for except for $t = \pi k/\omega, k \in \mathbb{Z}$, see Figure 1. This analysis also proves that enforcing the correct boundary conditions (12) is crucial in the assessment of the response of the flame to velocity perturbations. For example if the computational domain would be reduced to $[0, \delta]$ instead of $[0, \delta_r]$ the slope of $\zeta_1$ would be divergent for any $t \neq \pi k/\omega, k \in \mathbb{Z}$ and consequently the area (13) cannot be computed.

The first order solution is similar to the first order solution derived in [7]. In [7] the nonlinear equation (1) was transformed into a linear equation by approximating the square root term with $-\partial \zeta/\partial r$. Then, a first order analysis of the linearised equation gives the following first order perturbation $\zeta_1^*$ (in our notation)

$$\zeta_1^* = \frac{\hat{\upsilon}}{\omega} (\cos(\omega(t - (B^r(1) - B^f(r)))) - \cos(\omega t)), \quad (21)$$

where $B^r(r) = r$. Thus, an eventual difference in the TF between our model and the more simple models from [7, 6] would originate in the difference between the function $B$, (18) and the function $B^f$, see Figure 2 (left).

A simple analysis shows that

$$\max_{r \in [0, \delta_r]} |B(r) - B^f(r)| < |B(\delta_r) - \delta_r| < |B(\delta) - \delta|. \quad (22)$$
Figure 1: (a) First order solution $\zeta_1$ as function of $r$. (b) Slope of the first order solution as function of $r$. The following parameters were used: $\hat{v}=9$, $\varepsilon=0.01$, $\omega t = \pi/2$.

![Graph](image_url)

Figure 2: (a) The functions $B(r)$ and $B^f(r)$ as function of $r$, $\varepsilon$: $B(r)$, $-\varepsilon$: $B^f(r)$, $\varepsilon=0.01$, $\hat{v}=3$. (b) The difference $|B(\delta_2) - \delta_1|$ as function of $\varepsilon$, $\hat{v}=3$.

![Graph](image_url)

We notice that $B(\delta) = t_a$, where $t_a$ is the time needed for a Bunsen flame to evolve in a Poiseuille flow from a initially flat profile to its stationary conical position, [1]. Then by using Lemma 6.1 from [1] and taking into account that $\hat{v} \in [2, 10]$ it follows that $1 \leq B(\delta) \leq 1.0439$ thus $0.0517 \leq |B(\delta) - \delta| \leq 0.3368$. The same Lemma implies that the difference $|B(\delta) - \delta|$ decreases with increasing $\hat{v}$. The largest difference $|B(r) - B^f(r)|$ is at the boundary, see Figure 2(a). The difference at the boundary decreases with increasing $\varepsilon$, see Figure 2(b).

5 Solution of the second order equation

Although (5c) is similar to (5b) an analytic solution of this equation is difficult to obtain due its complicated right hand side. Because the right hand side of (5c) contains the square of the slope of $\zeta_1$, we expect that the slope of $\zeta_2$ is steeper than the slope of $\zeta_1$. Then, $\zeta_2$ needs a careful approximation in the boundary layer. In order to compute the solution of (5c) we will introduce a nonuniform, nondissipative box scheme for a general advection equation of the
type
\[
\frac{\partial u}{\partial t} + h(r) \frac{\partial u}{\partial r} = g(r, t). \quad (23)
\]
To derive the scheme we introduce the nonuniform grid containing the points
\((r_j, t^n)\) given by
\[
0 = r_0 < r_1 < r_2, \ldots, < r_M = \delta_r, \quad t^n := n\Delta t, \quad n = 0, 1, 2, \ldots, \quad (24)
\]
Following the ideas from [14] we combine the trapezoidal rule for the time integration with the central difference approximation for the space derivative at the intermediate points \(r_{j+1/2}\) given by
\[
\begin{align*}
    r_{j+1/2} & := \frac{r_{j+1} + r_j}{2} \quad (25).
\end{align*}
\]
The source term in (23) is approximated by using the Crank-Nicolson rule. Finally, we approximate the intermediate values \(u(r_{j+1/2}, t^n), (m = n, n + 1)\) by linear interpolation to obtain the scheme
\[
\begin{align*}
    u_{j+1}^{n+1} &= u_j^{n+1} + \Delta r_j \frac{\Delta th(r_{j+1/2})}{\Delta r_j - \Delta th(r_{j+1/2})} (u_{j+1}^n - u_{j+1}^{n+1}) + \\
    &\frac{\Delta r_j \Delta t}{\Delta r_j - \Delta th(r_{j+1/2})} (g(t^n, r_{j+1/2}) + g(t^{n+1}, r_{j+1/2})), \quad (26)
\end{align*}
\]
where
\[
\Delta r_j := r_{j+1} - r_j, \quad j = 0, 1, 2, \ldots, M - 1. \quad (27)
\]
By combining the second order behaviour of the trapezoidal rule, of the central difference space discretisation, of the Crank-Nicolson scheme and of the linear interpolation we obtain a second order accurate nonuniform scheme that retains the nondissipative behaviour of the uniform box scheme. Although this scheme is implicit the computational time is comparable to the one of an explicit scheme. Indeed, knowing the boundary value \(u_M^{n+1}\) the other values at time level \(t^{n+1}\) can be computed from (26) in decreasing order for \(j\). To investigate the behaviour of the scheme (26) we employ it to compute the numerical solution of the first order equation (5b). To demonstrate that the boundary layer is well resolved we consider the case in which \(\zeta_1\) has a steep slope which according to (20) corresponds to a small \(\varepsilon\). The numerical and the analytical solutions are depicted in Figure 3.

To find the second order solution we solve (5c) using (26), subject to the initial and boundary conditions given by (12). The second order solution as function of \(r\) has a boundary layer at \(r = \delta_r\), see Figure 4(a). The slope of the second order solution as function of \(r\) is indeed very steep for small \(\varepsilon\), see Figure 4(b).

### 6 Flame response to velocity perturbations

By combining the analytical solutions of the leading and of the first order equation and the numerical solution of the second order equation we obtain an
Figure 3: Numerical solution of the first order equation (5b) computed with the box scheme given by (26). * numerical solution, − analytic solution given by (19). The following parameters were used $\varepsilon = 10^{-6}$, $\hat{\nu} = 8.23$, $\hat{\delta}_c = 0.939$, $\omega t = \pi/2$, $\Delta t = 1.25 \times 10^{-4}$, $\Delta r = 1.5 \times 10^{-4}$ on $[0, 0.9]$, $\Delta r = 2 \times 10^{-5}$ on $[0.9, \hat{\delta}_c]$.

Figure 4: (a) Second order solution $\zeta_2$ as function of radius. (b) Slope of the second order solution as function of radius. The following parameters were used: $\hat{\nu} = 9$, $\varepsilon = 0.01$, $\omega t = \pi/2$.

The analytical-numerical description of the flame front location. The flame front location oscillates around its stationary position under the action of the velocity perturbation, see Figure 5(a). The oscillation of the flame front induces a perturbation of the flame surface area (13), see Figure 5(b). To compute the area the slope at the flame front needs to be evaluated. Although the derivatives of the first and second order solution have a boundary layer the integral in (13) is proper. It is easy to see from (20) that $\lim_{\varepsilon \to 0} \varepsilon \partial \zeta_1 / \partial r(\hat{\delta}_c) = 0$. From the numerical approximation of the second order solution we can conclude that the contribution of the term $\varepsilon^2 \partial \zeta_2 / \partial r(\hat{\delta}_c)$ is small. The area is computed by evaluating numerically (13) with the composite trapezoid rule on the nonuniform grid used to compute the second order solution. The response of the flame to velocity perturbations is given in terms of the transfer function which is defined as the ratio of the relative heat release rate perturbation to the relative velocity perturbation in the frequency domain. Under the assumption that the heat
release rate is proportional to the area of the flame, the transfer function of the flame \( H(\omega) \) becomes

\[
H(\omega) = \frac{A'_z}{\overline{A_z}} \varpi(\omega),
\]

where \( \overline{A_z} \) and \( A'_z \) are the mean value of the flame area and its variation, respectively [7]. The transfer function is completely determined by its angle or phase and its magnitude. The magnitude of the transfer function \( |H(\omega)| \) is given by

\[
|H(\omega)| = \left( \frac{A^n_z}{\overline{A_z}} \right) / (v^n_z / \varpi),
\]

where \( A^n_z \) and \( v^n_z \) are the amplitude of the area oscillation and velocity fluctuation, respectively. The angle of the transfer function \( \angle H(\omega) \) is the phase difference between the relative velocity perturbation and the relative area perturbation in the centre of the duct. The transfer function \( H(\omega) \) is computed over the range of dimensionless frequencies \( \omega \in [1, 100] \). For each sample frequency \( \omega \) we evaluate the slope of the perturbed flame front to compute the normalised area of the flame (the ratio between the perturbed area and its mean, \( A'_z / \overline{A_z} \)). The time variation of the normalised area of the flame can be fitted by means of the least-squares approximation to a function \( A^n_z(t) \) of the following form

\[
A^n_z(t) = 1 + a^n_z \sin(\omega t + b^n_z).
\]

Thus, the angle and the phase difference corresponding to frequency \( \omega \) are given by

\[
\angle H(\omega) = b^n_z, \quad |H(\omega)| = a^n_z / \varepsilon.
\]

The TF derived with our model is depicted in Figure 6 along with the TF derived in [7]. In agreement with experiments, in both models the magnitude of the TF has a low pass filter behaviour. The phase of both models agrees well with the experiments up to the dimensionless frequency \( \omega = 6 \) [6], but the linear increase of the phase with the increase of the excitation frequency is not
captured. Nevertheless, due to the difference $|B(r) - B'(r)|$ at the boundary, our model gives a value for the saturation of the flame TF that is slightly larger than $\pi/2$. Compared to the previous models [7, 6], the model proposed here improves the description of the front close to the boundary which results in an improvement of the phase behaviour. These results concur with the experimental observations from [15] that indicate that for the case of an attached Bunsen flame the phase of the flame transfer function saturates at a level between $\pi/2$ and $\pi$. By enforcing a small amplitude axial movement of the flame anchoring point the same phase behaviour was obtained theoretically in [9]. Thus, the response of an attached flame obtained with our extended model is equivalent with the one of a flame with a moving base in the axial direction.

In [15, 8] the suggestion was made that in order to understand the behaviour of the flame transfer function the motion of the moving flame edge needs to be taken into account. Our results show that the G-equation model for attached flames is capable to predict the transfer function behaviour of non attached flames with a moving base in the axial direction. Thus, for a better understanding of the flame TF only the contribution of the edge radial movement needs to be quantified.

7 Conclusions

The response of a Bunsen flame to velocity perturbations was investigated by using an extended kinematic model that accounts for flames with arbitrary flame angles and burning velocity with non constant direction. Our extension requires the use of a nonlinear form of the G-equation model which for the case of a Poiseuille flow considered here, does not allow for the flame attachment at the burner rim. To apply boundary conditions the computational domain was restricted to a physical interval where the flow speed is larger than the laminar burning speed. On the physical interval a first order approximation of the solution of the G-equation is expressed in terms of elliptic integrals. A detailed analysis reveals that close to the burner rim our first order solution is an improvement of the solution computed with the linear form of the G-equation. The analysis

Figure 6: (a) The phase difference and (b) the magnitude of the transfer function as function of dimensionless frequency. * our model; — model from [7]. The following parameters were used: $\dot{v} = 9$, $\varepsilon=0.1$. 
also proves that the correct application of the boundary conditions is crucial in computing the flame TF. Since the analysis predicts a steep behaviour close to the boundary for the second order solution a novel nondissipative box scheme adapted for a nonuniform grid was used to compute the solution of the second order equation. The flame front position obtained by combining the leading, the first and the second order solutions was employed to compute the response of the flame to velocity perturbations. In agreement with the experiments the magnitude of the transfer function has a low pass filter behaviour. The phase of the transfer function computed here is an improvement of the phase presented in [7, 6]. The improvement is due to a better flame representation close to the burner rim. Moreover our TF results are comparable with the ones for the case of non attached flames with a moving base in the axial direction indicating that for a better understanding of the flame TF only the contribution of the edge in the radial direction needs to be quantified.

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