Folding a cusp into a swallowtail

by

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Abstract

The Hamiltonian Hopf bifurcation is described by part of a swallowtail surface. In this short note it is shown that this swallowtail is actually a non-transversally unfolded cusp singularity, and that the swallowtail surface is obtained by folding a cusp along a fold line that is tangent to the cusp and moving along the cusp.

Key Words: Hamiltonian Hopf bifurcation, relative equilibria, cusp singularity, swallowtail singularity.

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1 Introduction

In this short note an example is considered of a phenomena which appears when studying bifurcations of vector fields. This phenomena is that the bifurcation curve or surface seems to fit into the classification scheme of elementary catastrophes in a way which differs from the classification of the corresponding bifurcation equation. In this note we deal with the example of the Hamiltonian Hopf bifurcation.

The Hamiltonian Hopf bifurcation is the bifurcation of relative equilibria at a stationary point observed in a Hamiltonian system of two degrees of freedom, when the eigenvalues of the linearized system pass, under the influence of a parameter, from two pairs of purely imaginary eigenvalues \( \pm i \alpha, \pm i \beta \), through two equal pairs of purely imaginary eigenvalues to four eigenvalues in the complex plane. An additional non-degeneracy condition on the fourth order terms of the local normal form at the stationary point is required. This bifurcation is described in detail in [4]. There it is shown that the bifurcation is described

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by part of a swallowtail surface. This description is obtained by reducing the general system to a standard form, which is symmetric, and describes the full local bifurcation of relative equilibria of the original system up to diffeomorphism. On $\mathbb{R}^4$ with coordinates $(x, y) = (x_1, x_2, y_1, y_2)$ and standard symplectic form this standard Hamiltonian system is given by the parameter dependent Hamiltonian function

$$H_\nu(x, y) = X + \nu Y + aY^2,$$

with $X = \frac{1}{2}(x_1^2 + x_2^2)$, $Y = \frac{1}{2}(y_1^2 + y_2^2)$. Note that this system is symmetric, having integral $S = x_2y_1 - x_1y_2$. Furthermore in this symmetric context the linearized system has a quadruple of zero eigenvalues at the bifurcation $\nu = 0$. In the non-symmetric context the normalized linear system will have Hamiltonian $S + X + \nu Y$.

Because of the symmetry this system can be reduced to a one-degree-of-freedom system living on the semi-algebraic variety in $(X, Y, Z)$-space given by $4XY = Z^2 + s^2$, $X \geq 0$, $Y \geq 0$, where the symplectic form is induced by the Poisson structure on $\mathbb{R}^3$ given by $\{X, Y\} = Z$, $\{X, Z\} = -2X$, $\{Y, Z\} = 2Y$, where $\{, \}$ is the standard Poisson bracket on $\mathbb{R}^4$. Here $s$ is introduced by restricting to $S(x, y) = s$, and $Z = x_1y_1 + x_2y_2$.

The relative equilibria are now given by the stationary points of the reduced system. One obtains a family of relative equilibria depending on the parameters $s$, $\nu$ and the energy $h$. These relative equilibria are also the points where the energy surface of the reduced energy $H_\nu(X, Y, Z) = h$ is tangent to the reduced phase space given by $4XY = Z^2 + s^2$, $X \geq 0$, $Y \geq 0$. It is easily seen that for these points $Z = 0$. Substituting $X = h - \nu Y - aY^2$ in the equation $4XY = Z^2 + s^2$, and setting $Z = 0$, one obtains the bifurcation equation

$$4aY^3 + 4\nu Y^2 - 4hY + s^2 = 0,$$

(1)

together with the inequalities $Y \geq 0$, $h - \nu Y - aY^2 \geq 0$. Let $\Delta$ denote the discriminant of (1) considered as an equation in $Y$. Then the bifurcation is given by the discriminant locus $\Delta = 0$.

In [4] it is shown that this discriminant locus is a swallowtail surface by showing that it is the same as the discriminant locus of the fourth degree equation

$$Y^4 - \frac{\nu}{2|a|} Y^2 + \frac{s}{\sqrt{|a|}} Y + \frac{g}{4a} + \frac{\nu^2}{16a^2} = 0.$$

It is well known that the discriminant locus or singularity of such a fourth degree equation is a swallowtail. However the generic singularity for a third degree equation as in (1) is a cusp. In the next section it will be shown how this cusp becomes a swallowtail.

2 Non-transversal unfoldings

In this section we will consider the above problem in a simpler context of folds and cusps using concepts from catastrophe theory and singularity theory (see [2], [3], [1]).
Consider the potentials

\[ V_2(X; a, b) = X^3 + 6\sqrt{3}bX^2 - \frac{32}{3}a^3X \, , \quad V_3(X; a, b) = X^4 + aX^2 + bX \, . \]

According to \[2\] a transversal unfolding of \( X^3 \) will give rise to a fold and a transversal unfolding of \( X^4 \) will give rise to a cusp. Actually \( V_3 \) is the standard form for the cusp catastrophe. Furthermore \( V_2 \) is a non-transversal unfolding of \( X^3 \) because \( \frac{\partial V_2}{\partial b}(0; 0, 0) = 0 \), and therefore does not give a fold singularity at the origin. A straightforward calculation will show that both potentials describe the same singularity. More precisely, the potential \( V_2 \) describes the fold line as it is embedded in the cusp singularity given by \( V_3 \).

Consider the equations

\[ \frac{\partial V_2}{\partial X}(X; a, b) = 3X^2 + 12\sqrt{3}bX - \frac{32}{3}a^3 = 0 \, , \tag{2} \]

and

\[ \frac{\partial V_3}{\partial X}(X; a, b) = 4X^3 + 2aX + b = 0 \, , \tag{3} \]

describing the singular points of \( V_2 \) and \( V_3 \). Although the surfaces defined by these two equations are not the same, the set where also the second derivative vanishes is the same for both potentials. This set is given by the discriminant locus of the equation. The discriminant of (2) is \( \Delta_2 = 16(8a^3 + 27b^2) \) and the discriminant of (3) is \( \Delta_3 = -16(8a^3 + 27b^2) \).

This can also be obtained by putting \( V_2 \) into the standard form for the fold which is \( Y^3 + cY \). This can be done by the linear transformation \( X = Y - 2\sqrt{3}b \), which turns \( V_2 \) into \( Y^3 - (36b^2 - \frac{32}{3}a^3)Y + f(a, b) \). As a consequence the fold line \( c = 0 \) is embedded into \( (a, b) \)-space as the cusp-line \( 27b^2 + 8a^3 = 0 \). This also reflects the fact that \( V_2 \) is not a transversal unfolding of \( X^3 \).

Yet another way of representing both singularities is by projecting both surfaces given by equation (2) and (3) onto the \((a, b)\)-plane. For (2) this induces a map \( \mathbb{R}^2 \to \mathbb{R}^2; (X, b) \to (a, b) \) given by \( a = \left( \frac{3}{32}(3X^2 + 12\sqrt{3}bX) \right)^\frac{2}{3} \). For (3) one gets a map \( \mathbb{R}^2 \to \mathbb{R}^2; (a, X) \to (a, b) \) given by \( b = -4X^3 - 2aX \). The set of singular values of these maps is in both cases the cusp given by \( 27b^2 + 8a^3 = 0 \). This latter point of view goes back to the original approach of Whitney [5] to singularities of maps from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \).

Note that \( V_2 \) and \( V_3 \) are definitely not equivalent as singularities. It is only shown that at the appropriate level they have the same singularity. It might be confusing that it is the singular set at this level that gives the singularity or elementary catastrophe its name.
Consider the bifurcation equation for the Hamiltonian Hopf bifurcation (1). This equation describes the critical points of the potential

\[ V(Y; h, s, \nu) = aY^4 + \frac{4}{3}\nu Y^3 - 2hY^2 + s^2Y , \]

which is a non-transversally unfolded cusp singularity.

Although this is not a transversal unfolding of \( Y^4 \) and therefore a non-generic phenomenon when considered as the singularity of a function, it is generic as a bifurcation of relative equilibria, because, within the context of Hamiltonian systems, it is actually related to the singularity of an energy-momentum mapping (see [4]). Another explanation is that the non-transversality is a consequence of the term \( s^2 \), the presence of which is generic for families with a Hamiltonian Hopf bifurcation, because it is introduced by the Casimir of the \( sl(2,\mathbb{R}) \) Poisson structure defining the reduced system.

Now \( V(Y, h, s, \nu) \) can be put into the normal form \( X^4 + cX^2 + dX \) by the transformation \( Y = X - \frac{\nu}{4a} \). This gives

\[ c = -\frac{3
\nu^2}{8a^2} - 2\frac{h}{a} , \]

and

\[ d = \frac{1}{8a^3} + \frac{h\nu}{a^2} + \frac{s^2}{a} . \]

The cusp line \( 27d^2 + 8c^3 = 0 \) is now embedded in \((h, s, \nu)\)-space as the surface given by

\[ \frac{64h^3}{a^3} + \frac{9h^2\nu^2}{a^4} - \frac{54h\nu s^2}{a^3} - \frac{27\nu^3 s^2}{4a^4} - \frac{27s^4}{a^2} = 0 . \]

Replacing \( s^2 \) by \( t \) one gets
\[ \frac{64h^3}{a^3} + \frac{9h^2\nu^2}{a^4} - \frac{54h\nu t}{a^3} - \frac{27\nu^3 t}{4a^4} - \frac{27t^2}{a^2} = 0 , \]

which for \( \nu = \text{constant} \) gives a cusp curve in the \((t, h)\)-plane. For \( \nu = 0 \) the cusp point lies at the origin. For \( \nu \neq 0 \) the cusp point lies away from the origin. In all cases the cusp curve is at the origin tangent to the \( h \)-axis. See fig. 1. By replacing \( t \) by \( s^2 \) a fold is introduced in the mapping of parameter spaces and the part of the cusp in the \( t \geq 0 \) halfplane is folded into a swallowtail. See fig. 2. In fig. 1. and fig. 2. the situation is drawn for \( a < 0 \) and \( \nu > 0 \). Due to the presence of the inequalities in the case of the Hamiltonian Hopf bifurcation only the \( h \geq 0 \) part of the swallowtail curve is considered in this case. Other cases are drawn in figs. 3, 4, 5, 6.

Finally one can also obtain the relevant part of the swallowtail by projecting the surface given by the bifurcation equation (1) onto the \((s, h)\)-plane taking \( \nu = \text{constant} \). Varying \( \nu \) then gives the bifurcation. This comes down to considering for each \( \nu \) the singularity of the map \( \mathbb{R}^2 \to \mathbb{R}^2, (Y, s) \to (h, s) \) given by

\[ h = aY^2 + \nu Y + \frac{s^2}{4Y} . \]

This is illustrated in fig. 7. for the case \( a < 0 \) and \( \nu > 0 \).
Figure 7: Relative equilibria for $a < 0, \nu > 0$

References


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