Structure-preserving tangential-interpolation based model reduction of port-Hamiltonian systems

by

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Abstract

Port-Hamiltonian systems result from port-based network modeling of physical systems and are an important example of passive state-space systems. In this paper, we develop the framework for model reduction of large-scale multi-input/multi-output port-Hamiltonian systems via tangential rational interpolation. The resulting reduced-order model not only is a rational tangential interpolant but also retains the port-Hamiltonian structure; hence is passive. This reduction methodology is described in both energy and co-energy system coordinates. We also introduce an $H_2$-inspired algorithm for effectively choosing the interpolation points and tangential directions. The algorithm leads a reduced port-Hamiltonian model that satisfies a subset of $H_2$-optimality conditions. We present several numerical examples that illustrate the effectiveness of the proposed method showing that it outperforms other existing techniques in both quality and numerical efficiency.

Key words: Model Reduction, Interpolation, Port-Hamiltonian Systems, Structure Preservation, $H_2$ Approximation

1 Introduction

The modeling of complex physical systems often involves systems of coupled partial differential equations, which upon spatial discretization, lead to dynamical models with very large state-space dimension. This creates a need for model reduction methods that can produce (comparatively) low dimensional surrogate models that nonetheless are able to closely mimic the original system’s input/output map. Port-based network modeling of the original system will lead directly to a representation as a port-Hamiltonian system as well, a representation which encodes structural properties related to the manner in which energy is distributed and flows through the system. When the related Hamiltonian function is non-negative, port-Hamiltonian systems constitute an important class of passive state-space systems.

Structure preserving reduction of port-Hamiltonian systems using Gramian (balancing)-based methods was considered in [24,23,25]; we briefly review one such approach in Section 3.1. For the special case of single-input/single-output (SISO) systems, the use of (rational) Krylov methods is employed in [21,13,22,33] where [21,22,33] deal with interpolation at a single point only, as opposed to multi-point interpolation rational tangential interpolation of multi-input/multi-output (MIMO) systems we consider in this paper. Preservation of the port-Hamiltonian structure by reducing the underlying full order Dirac structure is presented in [30]. A perturbation approach is considered in [16,15]. See [20] for an overview of recent port-Hamiltonian model reduction methods.
For general multi-input/multi-output (MIMO) dynamical systems, interpolatory model reduction methods construct reduced-order models whose transfer functions interpolate the original system transfer function at selected points in the complex (frequency) plane along selected input and output directions. The main implementation cost involves solving (typically sparse) linear systems, which gives a significant advantage in large-scale settings over competing Gramian (balancing)-based methods (such as balanced truncation) that must contend with a variety of large-scale dense matrix operations. Until recently, there were no systematic strategies for selecting interpolation points and directions, but this has largely been resolved by Gugercin et al. [10,11,12] where an interpolation point/tangent direction selection strategy leading to $H_2$-optimal reduced order models has been proposed. See [31,6,18] for related work and [2] for a recent survey.

The goal of this work is to demonstrate that interpolatory model reduction techniques for linear state-space systems can be applied to MIMO port-Hamiltonian systems in such a way as to preserve the port-Hamiltonian structure in the reduced models, and preserve, as a consequence, passivity as well. Moreover, we introduce a numerically efficient $H_2$-based algorithm for structure-preserving model reduction of port-Hamiltonian systems that produces high quality reduced order models in the general MIMO case. Numerical examples are presented to illustrate the effectiveness of the proposed method.

In Section 2, we review the solution to the rational tangential interpolation problem for general linear multi-input/multi-output systems. Brief theory on port-Hamiltonian systems is given in Section 3. Structure preserving interpolatory model reduction of port-Hamiltonian systems in different coordinates is considered in Section 4 where we show theoretical equivalence of interpolatory reduction methods using different coordinate representations of port-Hamiltonian systems and discuss which is most robust and numerically effective. $H_2$-based model reduction for port-Hamiltonian systems together with the proposed algorithm is presented in Section 5 followed by numerical examples in Section 6.

**2 Interpolatory Model Reduction**

Let $G(s)$ be a dynamical system with a state-space realization given as

$$ G(s) : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1} $$

where $A, E \in \mathbb{R}^{n \times n}$, $E$ is nonsingular, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$. In (1), for each $t$ we have: $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ denoting, respectively, the state, the input, and the output of $G(s)$. By taking the Laplace transform of (1), we obtain the associated transfer function $G(s) = C(sE-A)^{-1}B$. Following the usual abuse of notation, the underlying dynamical system and its transfer function will both be denoted by $G(s)$.

We wish to produce a surrogate dynamical system $G_r(s)$ of much smaller order with a similar state-space form

$$ G_r(s) : \begin{cases} \dot{x}_r(t) = A_rx_r(t) + B_ru(t) \\ y_r(t) = C_rx_r(t), \end{cases} \tag{2} $$

where $A_r, E_r \in \mathbb{R}^{r \times r}$, $B_r \in \mathbb{R}^{r \times m}$, and $C_r \in \mathbb{R}^{p \times r}$ with $r \ll n$ such that the reduced-order system output $y_r(t)$ approximates the original system output, $y(t)$, with high fidelity, as measured in an appropriate sense. The appropriate sense here will be with respect to the $H_2$ norm and will be discussed in §5.

We construct reduced order models $G_r(s)$ via Petrov-Galerkin projections. We choose two $r$-dimensional subspaces $\mathcal{V}_r$ and $\mathcal{W}_r$ associated with $\mathcal{V}_r = \text{Range}(V_r)$ and $\mathcal{W}_r = \text{Range}(W_r)$). System dynamics are approximated as follows: we approximate the full-order state $x(t)$ as $x(t) \approx V_rx_r(t)$ and force a Petrov-Galerkin condition as

$$ W_r^T(EV_r \dot{x}_r - AV_rx_r - Bu) = 0, \quad y_r = CV_r x_r, \tag{3} $$

leading to a reduced-order model as in (2) with

$$ E_r = W_r^T EV_r, \quad B_r = W_r^T B \quad A_r = W_r^T AV_r, \quad C_r = CV_r. \tag{4} $$

The quality of the reduced model depends solely on the selection of the two subspaces $\mathcal{V}_r$ and $\mathcal{W}_r$, (or equivalently on the choices of $V_r$ and $W_r$). We choose $V_r$ and $W_r$ to force interpolation.

**2.1 Interpolatory projections**

The construction of reduced order models with interpolatory projections was initially introduced by Skelton et. al. in [7,35,36] and later put into a robust numerical framework by Grimm [9]. This problem framework was recently adapted to MIMO dynamical systems of the form (1) by Gallivan et al. [8] and then extended to a much class of transfer functions, namely those having a coprime factorization of the form $H(s) = C(sK(s)^{-1}B)(s)$ with $B(s)$, $C(s)$, and $K(s)$ given as meromorphic matrix-valued functions by Beattie and Gugercin [4].

Given a full-order system (1) and interpolation points $\{s_k\}_{k=1}^r \in \mathbb{C}$ with corresponding (right) tangent directions $\{b_k\}_{k=1}^r \in \mathbb{C}^m$, the tangential interpolation prob...
lem seeks a reduced-order system $\mathbf{G}_r(s)$ that interpolates $\mathbf{G}(s)$ at the selected points along the selected directions; i.e.,

$$\mathbf{G}(s_i)\mathbf{b}_i = \mathbf{G}_r(s_i)\mathbf{b}_i, \quad \text{for} \quad i = 1, \ldots, r,$$

(5)

Analogous left tangential interpolation conditions may be considered, however (5) will suffice for our purposes. Conditions forcing (5) to be satisfied by a reduced system of the form (4) are provided by the following theorem (see [8]).

**Theorem 1** Suppose $\mathbf{G}(s) = \mathcal{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$. Given a set of distinct interpolation points $\{s_k\}_{k=1}^r$ and right tangent directions $\{\mathbf{b}_k\}_{k=1}^r$, define $\mathbf{V}_r \in \mathbb{C}^{n \times r}$ as

$$\mathbf{V}_r = [(s_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{b}_1, \ldots, (s_r\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{b}_r]$$

(6)

Then for any $\mathbf{W}_r \in \mathbb{C}^{n \times r}$, the reduced order system $\mathbf{G}_r(s) = \mathcal{C}_r(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r$ defined as in (4) satisfies (5) provided that $s_i\mathbf{E} - \mathbf{A}$ and $s_i\mathbf{E}_r - \mathbf{A}_r$ are invertible for $i = 1, \ldots, r$.

**Remark 2** Attempting interpolation of the full transfer function matrix (in all input and output directions) generally inflates the reduced system order by a factor of $m$ (the input dimension). However, this is neither necessary nor desirable. Effective reduced order models, indeed, $H_2$-optimal reduced order models, can be generated by forcing interpolation along selected tangent directions.

**Remark 3** The result of Theorem 1 generalizes to higher-order interpolation (analogous to generalized Hermite interpolation) as follows: For a point $s \in \mathbb{C}$ and tangent direction $\mathbf{b}$, suppose

$$((s\mathbf{E} - \mathbf{A})^{-1}\mathbf{E})^{k-1}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{b} \in \text{Range}(\mathbf{V}_r)$$

(7)

Then, for any $\mathbf{W}_r \in \mathbb{C}^{n \times r}$, the reduced order system $\mathbf{G}_r(s)$ defined as in (4) satisfies

$$\mathbf{G}^{(\ell)}(s)\hat{\mathbf{b}} = \mathbf{G}_r^{(\ell)}(s)\hat{\mathbf{b}}, \quad \text{for} \quad \ell = 0, \ldots, N - 1,$$

(8)

provided that $s\mathbf{E} - \mathbf{A}$ and $s\mathbf{E}_r - \mathbf{A}_r$ are invertible where $\mathbf{G}^{(\ell)}(s)$ denotes the $\ell$th derivative of $\mathbf{G}(s)$. By combining this result with Theorem 1, one can match different interpolation orders for different interpolation points (analogous to generalized Hermite interpolation). For details; see, for example, [8, 2].

**Remark 4** Notice that in Theorem 1 and Remark 3, what guarantees the interpolation is the range of the matrix $\mathbf{V}_r$, not the specific choice of basis in (6). Hence, one may substitute $\mathbf{V}_r$ with any matrix $\hat{\mathbf{V}}_r$ having the same range. In other words, for any $\hat{\mathbf{V}}_r$ satisfying $\hat{\mathbf{V}}_r = \mathbf{V}_r\mathbf{L}$ where $\mathbf{L} \in \mathbb{R}^{r \times r}$ invertible, the interpolation property still holds true. This is a simple consequence of the fact that the basis change $\mathbf{L}$ from $\mathbf{V}_r$ to $\hat{\mathbf{V}}_r$ amounts to a state-space transformation in the reduced model. We will make use of this property in discussing the effect of different coordinate representations for port-Hamiltonian systems. When the interpolation points occur in complex conjugate pairs (as always occurs in practice), then the columns of $\mathbf{V}_r$ also occur in complex conjugate pairs and may be replaced with a real basis, instead. We write $\mathbf{V}_r = [\mathbf{v}_1, \ldots, \mathbf{v}_r]$ to represent that a real basis for $\mathbf{V}_r$ is chosen. Notice then that $\hat{\mathbf{V}}_r$ will also be a real matrix which then leads to real reduced order quantities as well.

### 3 Linear port-Hamiltonian Systems

In the absence of algebraic constraints, linear port-Hamiltonian systems take the following form ([29, 28, 23])

$$\dot{x} = (\mathbf{J} - \mathbf{R})\mathbf{Q}x + \mathbf{B}u,$$

$$y = \mathbf{B}^T\mathbf{Q}x,$$

(9)

where $\mathbf{J} = -\mathbf{J}^T$, $\mathbf{Q} = \mathbf{Q}^T$, and $\mathbf{R} = \mathbf{R}^T \geq 0$. $\mathbf{Q}$ is the energy matrix; $\mathbf{R}$ is the dissipation matrix; $\mathbf{J}$ and $\mathbf{B}$ determine the interconnection structure of the system. $H(x) = \frac{1}{2}x^T\mathbf{Q}x$ defines the total energy (the Hamiltonian). For all cases of interest, $H(x) \geq 0$, hence we assume in all that follows that $\mathbf{Q}$ is positive semidefinite. Note that the system (9) has the form (1) with $\mathbf{E} = \mathbf{I}$, $\mathbf{A} = (\mathbf{J} - \mathbf{R})\mathbf{Q}$, and $\mathbf{C} = \mathbf{B}^T\mathbf{Q}$.

The state variables $x \in \mathbb{R}^n$ are called energy variables, since the total energy $H(x)$ is expressed as a function of these variables. $u, y \in \mathbb{R}^m$ are called power variables, since the scalar product $u^Ty$ equals the power supplied to the system. Note that

$$\frac{d}{dt} \frac{1}{2}x^T\mathbf{Q}x = u^Ty - x^T\mathbf{QR}Qx \leq u^Ty$$

—that is, port-Hamiltonian systems are passive and the change in total system energy is bounded by the work done on the system.

**Example 1** Consider the Ladder Network illustrated in Fig. 1, with $C_1, C_2, L_1, L_2, R$ being the capacitances, inductances, and resistances of idealized linear circuit elements described in the figure. The port-Hamiltonian rep-
representation of this physical system has the form (9) with
\[
Q = \text{diag}(C_1^{-1}, L_1^{-1}, C_2^{-1}, L_2^{-1}),
\]
\[
J = \text{tridiag} \left(\begin{array}{cccc}
-1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0
\end{array} \right),
\]
\[
R = \text{diag}(0, 0, 0, R), \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \end{bmatrix}.
\]

The A matrix is
\[
A = \begin{bmatrix}
0 & -L_1^{-1} & 0 & 0 \\
C_1^{-1} & 0 & -C_2^{-1} & 0 \\
0 & L_1^{-1} & 0 & L_2^{-1} \\
0 & 0 & -C_2^{-1} & -RL_2^{-1}
\end{bmatrix}
\]
and the state-space vector x is given as
\[
x^T = \begin{bmatrix} q_1 & \phi_1 & q_2 & \phi_2 \end{bmatrix}
\]
with \(q_1, q_2\) being the charges on the capacitors \(C_1, C_2\) and \(\phi_1, \phi_2\) being the fluxes over the inductors \(L_1, L_2\) correspondingly. The inputs of the system, \(\{u_1, u_2\}\) are given by the current \(I\) on the left side and the voltage \(U\) on the right side of the Ladder Network. The port-Hamiltonian outputs \(\{y_1, y_2\}\) are the voltage over the first capacitor \(U_{C_1}\) and the current through the second inductor \(I_{L_2}\).

The system matrices A, B, C, and E of (1) follow directly from writing the linear input-state differential equation for this system. \(Q\) may be derived from the Hamiltonian \(H(x) = \frac{1}{2}x^TQx\). Once A and Q are known, it is easy to derive the dissipation matrix, R, and the structure matrix, J, corresponding to the Kirchhoff laws, such that \(A = (J - R)Q\). The port-Hamiltonian output matrix C is given as
\[
C = B^TQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{L_2} \end{bmatrix}.
\]

Suppose now that \(Q\) is positive definite. Recall from [28,20,23] that the port-Hamiltonian system (9) may be represented in co-energy coordinates as
\[
\begin{cases}
\dot{e} = Q(J - R)e + QBu, \\
y = B^Te.
\end{cases}
\]

The coordinate transformation [28,20] between energy coordinates, \(x\), and co-energy coordinates, \(e\), is given by
\[
e = Qx.
\]

Example 2 (continued). The co-energy state vector, \(e\), for the Ladder Network from Example 1 is given as
\[
e^T = \begin{bmatrix} U_{C_1} & I_{L_1} & U_{C_2} & I_{L_2} \end{bmatrix}
\]
with \(U_{C_1}, U_{C_2}\) being the voltages on the capacitors \(C_1, C_2\) and \(I_{L_1}, I_{L_2}\) being the currents through the inductors \(L_1, L_2\), respectively.

3.1 Balancing port-Hamiltonian systems

To make the presentation self-contained, we review a recent structure-preserving, balancing-based model reduction method for port-Hamiltonian systems, the effort-constraint method ([20], [25], [23]), which will be used for comparisons in our numerical examples.

Consider a full order port-Hamiltonian system realized as in (9) with respect to energy coordinates. Consider the associated balancing transformation, \(T_b\), defined in the usual way (see [1] for a complete treatment): \(T_b\) simultaneously diagonalizes the observability Gramian, \(\mathcal{G}_o\), and the controllability Gramian, \(\mathcal{G}_c\), so that
\[
T_b^{-1}\mathcal{G}_o T_b^{-T} = T_b^T \mathcal{G}_c T_b = \text{diag} (\sigma_1, \sigma_2, \ldots, \sigma_n).
\]

where \(\sigma_1, \sigma_2, \ldots, \sigma_n\) are the Hankel singular values of the system \(\mathcal{G}(s)\). This balancing transformation is the same that is used in the context of regular balanced truncation.

The state-space transformation associated with the balancing transformation will preserve the port-Hamiltonian structure of the system. Indeed, if we define balanced coordinates, \(x_b\), through \(x_b = T_bx\), then we have
\[
\begin{cases}
\dot{x}_b = (J_b - R_b)Q_b x_b + B_b u, \\
y = (B_b)^TQ_b x_b,
\end{cases}
\]

Fig. 1. Ladder Network
where \( T_b R_b^T = R_b = (R_b)^T \geq 0 \) is the dissipation matrix, \( T_b J_b^T = J_b = -(J_b)^T \) is the structure matrix and \( T_b^{-1} Q_b^{-1} = Q_b = (Q_b)^T \geq 0 \) is the energy matrix in the balanced coordinates \( x_b \). In this case \( B_b = T_b B \).

Now split the state-space vector, \( x_b \), into dominant and codominant components:

\[
x_b = \begin{bmatrix} x_{b1} \\ x_{b2} \end{bmatrix}
\]

Regular balanced truncation proceeds by truncating the codominant state-space components, in effect forcing the system to evolve under the constraint \( x_{b2} = 0 \). This will destroy port-Hamiltonian structure in the smaller system that remains determining the evolution of \( x_{b1} \).

The **effort-constraint method** proceeds first by partitioning the balanced co-energy coordinates, \( e_b = Q_b x_b \), so as to conform with the dominant/codominant partitioning of \( x_b \):

\[
e_b = \begin{pmatrix} e_{b1} \\ e_{b2} \end{pmatrix} = \begin{bmatrix} Q_{11}^b & Q_{12}^b \\ Q_{21}^b & Q_{22}^b \end{bmatrix} \begin{bmatrix} x_{b1} \\ x_{b2} \end{bmatrix}
\]

Partition the balanced matrices \( J_b, R_b, \) and \( B_b \) similarly:

\[
J_b = \begin{bmatrix} J_{11}^b & J_{12}^b \\ J_{21}^b & J_{22}^b \end{bmatrix}, \quad R_b = \begin{bmatrix} R_{11}^b & R_{12}^b \\ R_{21}^b & R_{22}^b \end{bmatrix}, \quad B_b = \begin{bmatrix} B_1^b \\ B_2^b \end{bmatrix}
\]

Now, in order to effect model reduction, we force the system to evolve under the constraint \( e_{b2} = 0 \). This implies that \( x_{b2} = -Q_{22}^{-1} Q_{21} x_{b1} \) and the evolution of \( x_{b1} \) is determined by a smaller port-Hamiltonian system given by

\[
\begin{align*}
x_{b1} &= (J_{11}^b - R_{11}^b) S_b x_{b1} + B_1^b u, \\
y_r &= (B_1^b)^T S_b x_{b1}^r,
\end{align*}
\]

where \( S_b = Q_{11}^b - Q_{12}^b (Q_{22}^b)^{-1} Q_{21}^b \) is the Schur complement of the energy matrix \( Q_b \) in balanced coordinates. The relationship to the reduction of the full order underlying Dirac structure is discussed in [20]. A more direct approach is given in [23]; another approach using scattering coordinates can be found in [25].

**Remark 5** The effort-constraint method can be formulated as a Petrov-Galerkin projection method [30]. Even though a balancing transformation is used in developing the effort-constraint method as expressed in (15), the method is not equivalent to the usual balanced truncation method, which will not preserve port-Hamiltonian structure. For more details on this issue see [20].

### 4 Interpolatory Model Reduction of port-Hamiltonian systems

The message of Theorem 1 is clear. Given interpolation points, \( \{s_i\} \), and the tangent directions, \( \{b_i\} \), construct \( V_r \) as shown in the theorem, then for any choice of \( W_r \), the reduced-model satisfies the interpolation conditions. If the only goal were to produce an interpolatory reduced-order model, this would be all one needs. However, the set-up here is much more complicated due to the structure of the original problem: the full-order model has the port-Hamiltonian structure which guarantees stability and passivity. Hence, the goal here is not only to satisfy the interpolation conditions but also to retain the structure, i.e., produce an interpolatory reduced-order model that has the exact port-Hamiltonian structure as the original one; and hence is both stable and passive. In this section, we show how to achieve this goal.

Assume that Theorem 1 is applied to a port-Hamiltonian system with an arbitrary choice of \( W_r \). As a result, \( A_r \) will have the form \( W_r^T (J_r - R_r) Q_r V_r \), which can be no longer represented as \( (J_r - R_r) Q_r \) with a skew-symmetric \( J_r \), and symmetric positive semi-definite \( R_r \) and \( Q_r \) matrices. Below, we will develop several approaches that will choose \( W_r \) carefully to allow the structure preservation.

#### 4.1 Interpolatory projection with respect to energy coordinates

We first show how to achieve interpolation and preservation of port-Hamiltonian structure simultaneously in the original energy coordinate representation. The choice of \( W_r \) plays a crucial role.

**Theorem 6** Suppose \( G(s) \) is a linear port-Hamiltonian system, as in (9). Let \( \{s_i\}_{i=1}^r \subset \mathbb{C} \) be a set of \( r \) distinct interpolation points with corresponding tangent directions \( \{b_i\}_{i=1}^r \subset \mathbb{C}^m \), both sets being closed under conjugation. Construct \( V_r \) as in (6) using \( A = (J_r - R_r) Q \) and \( E = I \), so that

\[
V_r = [(s_i I - (J_r - R_r) Q)^{-1} B_1, \ldots, (s_r I - (J_r - R_r) Q)^{-1} B_r].
\]

Let \( M \in \mathbb{C}^{r \times r} \) be any nonsingular matrix such that \( \tilde{V}_r = V_r M \) is real.

Define \( \tilde{W}_r = Q \tilde{V}_r (\tilde{V}_r^T \tilde{Q}_r \tilde{V}_r)^{-1} \) and

\[
\begin{align*}
J_r &= \tilde{W}_r^T J \tilde{W}_r, \\
Q_r &= \tilde{V}_r^T \tilde{Q}_r \tilde{V}_r, \\
R_r &= \tilde{W}_r^T R \tilde{W}_r, \quad \text{and} \quad B_r &= \tilde{W}_r^T B.
\end{align*}
\]

Then, the reduced-order model

\[
G_r(s) : \begin{cases}
\dot{x}_r = (J_r - R_r) Q_r x_r + B_r u \\
y_r = B_r^T Q_r x_r
\end{cases}
\]
is port-Hamiltonian, passive, and
\[ G(s_i)b_i = G_r(s_i)b_i, \quad \text{for } i = 1, \ldots, r. \]

That is, \( G_r(s) \) interpolates \( G(s) \) at \( \{s_i\}_{i=1}^r \) along the tangent directions \( \{b_i\}_{i=1}^r \).

**Proof:** In light of Remark 4, there is at least one (and hence, an infinite number) of possible \( M \in \mathbb{C}^{r \times r} \) satisfying the conditions of the Theorem and so, \( \tilde{V}_r \in \mathbb{R}^{n \times r} \). Trivially, one may observe that the reduced-order system (17) is real and retains port-Hamiltonian structure with a positive definite \( Q_r \), and so must be passive. To show that \( G_r(s) \) interpolates \( G(s) \) note, as before, that the port-Hamiltonian system (9) has the standard form (1) with \( E = I, A = (J - R)Q \) and \( C = B^TQ \). An interpolatory reduced model may be defined as in (2) using \( V_r \circ \tilde{V}_r = \tilde{V}_r \) and \( W_r \circ \tilde{W}_r = \tilde{W}_r \). We then have \( E_r = \tilde{W}_r^T \tilde{V}_r = I \) and \( A_r = \tilde{W}_r^T A \tilde{V}_r = \tilde{W}_r^T (J - R)Q \tilde{V}_r \). From Theorem 1, this model interpolates \( G(s) \) at \( \{s_i\}_{i=1}^r \) as required but it is not obvious that this is the same reduced system \( G_r(s) \) that we have defined above.

Note that \( \tilde{W}_r \tilde{V}_r^T \) is a (skew) projection onto \( \text{Range}(Q \tilde{V}_r) \) and so \( Q \tilde{V}_r = \tilde{W}_r \tilde{V}_r^T Q \tilde{V}_r \). Thus, we have
\[
A_r = \tilde{W}_r^T (J - R)Q \tilde{V}_r \\
= \tilde{W}_r^T (J - R) \tilde{W}_r \tilde{V}_r^T Q \tilde{V}_r \\
= (\tilde{W}_r^T J \tilde{W}_r - \tilde{W}_r^T R \tilde{W}_r) \tilde{V}_r^T Q \tilde{V}_r \\
= (J_r - R_r)Q_r
\]
and
\[
C_r = C \tilde{V}_r = B^T \tilde{Q} \tilde{V}_r \\
= B^T \tilde{W}_r \tilde{V}_r^T Q \tilde{V}_r = B^T Q_r,
\]
which verifies that the port-Hamiltonian reduced system, \( G_r(s) \), interpolates \( G(s) \) at \( \{s_i\}_{i=1}^r \) along \( \{b_i\}_{i=1}^r \) as required. \( \Box \).

**Remark 7** Theorem 6 can be generalized to include higher-order (generalized Hermite) interpolation following ideas described in Remark 3. Indeed, given an interpolation point \( s \) and tangent direction \( b \), augment \( V_r \) so that (7) holds. The remaining part of the construction of Theorem 6 proceeds unchanged and the resulting reduced-order model will retain port-Hamiltonian structure (hence be both stable and passive) and will interpolate higher-order derivatives of \( G(s) \) as in (8).

### 4.2 The influence of state-space transformations

Let \( T \in \mathbb{R}^{n \times n} \) be an arbitrary invertible matrix representing a state-space transformation \( \tilde{x} = Tx \). As one may recall from the discussion of Section 3.1, port-Hamiltonian structure will be maintained under state-space transformations:
\[
\dot{x} = (J - R)x + Bu, \quad \iff \quad \dot{\tilde{x}} = (J - \tilde{R})\tilde{x} + \tilde{B}u,
\]
with transformed quantities
\[
\tilde{J} = TJT^T, \quad \tilde{R} = TRT^T, \quad \tilde{Q} = T^{-T}QT^{-1}, \quad \text{and } \tilde{B} = TB.
\]

**Theorem 8** Suppose \( G(s) \) is a port-Hamiltonian system as in (9), with two different port-Hamiltonian realizations connected via a state-space transformation, \( T \), as above. Given a set of \( r \) distinct interpolation points \( \{s_i\}_{i=1}^r \in \mathbb{C} \) with corresponding tangent directions \( \{b_i\}_{i=1}^r \) (closed under conjugation), then interpolatory reduced-order port-Hamiltonian models produced from either realization via Theorem 6 will be identical to one another.

**Proof:** Define primitive interpolatory basis matrices for each realization as
\[
V_r = [(s_1I - (J - R)Q)^{-1}Bb_1, \ldots, (s_iI - (J - R)Q)^{-1}Bb_i] \quad \text{and} \quad \hat{V}_r = [\hat{s}_1I - (\hat{J} - \hat{R})\hat{Q}]^{-1}\hat{B}\hat{b}_i.
\]
Elementary manipulations show that \( \hat{V}_r = TV_r \). Note that any \( M \in \mathbb{C}^{r \times r} \) for which \( \hat{V}_r = V_rM \) is real makes \( \hat{V}_r = V_rM \) real as well (and vice versa). Then we define, following the construction of Theorem 6, \( \hat{W}_r = QV_r \) and \( \hat{W}_r = \hat{Q}V_r \). Observe that \( \hat{W}_r = T^{-T}\hat{W}_r \) and \( \hat{V}_r = TV_r \). So,
\[
\hat{J}_r = \hat{W}_r^T J \hat{W}_r = \hat{W}_r^T \hat{W}_r = J_r, \\
\hat{Q}_r = \hat{V}_r^T \hat{Q} \hat{V}_r = \hat{V}_r^T \hat{W}_r = \hat{Q}_r, \\
\hat{R}_r = \hat{W}_r^T R \hat{W}_r = \hat{W}_r^T \hat{W}_r = \hat{R}_r, \\
\text{and } B_r = \hat{W}_r^T B = \hat{W}_r^T \hat{W}_r = B_r. \quad \Box
\]

**Remark 9** One may conclude from Theorem 8 that prior to calculating a reduced-order interpolatory model, there may be little advantage in first applying a state-space transformation (e.g., a balancing transformation, \( T_b \), as described in Section 3.1 or transforming to coenergy coordinates with \( T = Q \) as in (13)). Indeed, choosing a realization that exhibits an advantageous sparsity pattern to facilitate the linear system solves that produce \( V_r \) will likely be the most effective choice. If we choose \( T \) to satisfy \( T^TT = Q \) (for example, if...
\( \mathbf{T} \) is a Cholesky factor for \( \mathbf{Q} \) then with respect to the new \( \mathbf{x} \)-coordinates (called “scaled energy coordinates”) we find that \( \tilde{\mathbf{Q}} = \mathbf{I} \) and so \( \tilde{\mathbf{W}}_r = \tilde{\mathbf{V}}_r \). Notice that in this case, scaled energy coordinates and scaled co-energy coordinates become identical. Notwithstanding these simplifications, unless the original \( \mathbf{Q} \) is diagonal (so that transformation to scaled energy coordinates is cheap), there appears to be little justification for global coordinate transformations preceding the construction of an interpolatory reduced order model.

5 \( \mathcal{H}_2 \)-based approximation of port-Hamiltonian systems

The selection of \( r \) interpolation points \( \{s_i\}_{i=1}^r \in \mathbb{C} \) and corresponding tangent directions \( \{b_i\}_{i=1}^r \in \mathbb{C}^m \) completely determines projecting subspaces and so are the sole factors determining the quality of a reduced-order model. Until recently only heuristics were available to guide this process and the lack of a systematic selection process for interpolation points and tangent directions had been a key drawback to interpolatory model reduction. However, Gugercin et al. [12] introduced a framework for determining optimal interpolation points and tangent directions to find optimal \( \mathcal{H}_2 \) reduced-order approximations, and also proposed an algorithm, the Iterative Rational Krylov Algorithm (IRKA) to compute the associated reduced-order models. In this section we briefly review the interpolatory optimal \( \mathcal{H}_2 \) problem and propose an algorithm informed by these methods that produces high quality reduced-order port-Hamiltonian models of port-Hamiltonian systems.

5.1 Interpolation for optimal \( \mathcal{H}_2 \) approximation

Let \( \mathcal{M}(r) \) denote the set of reduced order models having state-space dimension \( r \) as in (2). Given a full order port-Hamiltonian system

\[
\mathbf{G}(s) = \mathbf{B}^T \mathbf{Q}(s \mathbf{E} - \mathbf{A})^{-1} \mathbf{B},
\]

the optimal-\( \mathcal{H}_2 \) reduced-order approximation to \( \mathbf{G}(s) \) of order \( r \) is a solution to

\[
\| \mathbf{G} - \mathbf{G}_r \|_{\mathcal{H}_2} = \min_{\mathbf{G}_r \in \mathcal{M}(r)} \| \mathbf{G} - \hat{\mathbf{G}}_r \|_{\mathcal{H}_2},
\]

where

\[
\| \mathbf{G} \|_{\mathcal{H}_2} := \left( \frac{1}{2} \int_{-\infty}^{\infty} \| \mathbf{G}(i\omega) \|_2^2 \, d\omega \right)^{1/2}.
\]

Note that the solution to (18), in general, will not be port-Hamiltonian.

There are a variety of methods that have been developed to solve (18). They can be categorized as Lyapunov-based methods such as [34,27,14,17,32,37] and interpolation-based optimal \( \mathcal{H}_2 \) methods such as [19,12,11,6,10,18,3,5]. Although both frameworks are theoretically equivalent [12], interpolation-based optimal \( \mathcal{H}_2 \) methods carry important computational advantages. Our focus here will be on interpolation-based approaches.

The optimization problem (18) is nonconvex, so obtaining a global minimizer is difficult in the best of circumstances. Typically local minimizers are sought instead and as a practical matter, the usual approach is to find reduced-order models satisfying the first-order necessary conditions of \( \mathcal{H}_2 \)-optimality.

Interpolation-based \( \mathcal{H}_2 \)-optimality conditions for SISO systems were introduced by Meier and Luenberger [19], Interpolatory \( \mathcal{H}_2 \)-optimality conditions for MIMO systems have recently been developed by [12,6,31].

**Theorem 10** Given a full-order system \( \mathbf{G}(s) = \mathbf{C}(s \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \), let \( \mathbf{G}_r(s) = \sum_{i=1}^r \frac{1}{s - \lambda_i} \mathbf{c}_i \mathbf{b}_i^T \) be the best \( r \)th order approximation of \( \mathbf{G} \) with respect to the \( \mathcal{H}_2 \) norm. Then

\[
\begin{align*}
(a) \; & \mathbf{G}(-\hat{\lambda}_k) \mathbf{b}_k = \mathbf{G}_r(-\hat{\lambda}_k) \mathbf{b}_k, \\
(b) \; & \mathbf{c}_k^T \mathbf{G}(-\hat{\lambda}_k) = \mathbf{c}_k^T \mathbf{G}_r(-\hat{\lambda}_k), \text{ and} \\
(c) \; & \mathbf{c}_k^T \mathbf{G}'(-\hat{\lambda}_k) \mathbf{b}_k = \mathbf{c}_k^T \mathbf{G}'_r(-\hat{\lambda}_k) \mathbf{b}_k
\end{align*}
\]

for \( k = 1, 2, \ldots, r \).

Theorem 10 asserts that any solution to the optimal-\( \mathcal{H}_2 \) problem (18) must be a bitangential Hermite interpolant to \( \mathbf{H}(s) \) at the mirror images of the reduced system poles. This would be a straightforward construction, if the reduced system poles and residues were known \textit{a priori}. However, they are not and the \textit{Iterative Rational Krylov Algorithm} IRKA [12] resolves the problem by iteratively correcting the interpolation points by reflecting the current reduced-order system poles across the imaginary axis to become the next set of interpolation points and the directions. The next tangent directions are residue directions taken from the current reduced model. Upon convergence, the necessary conditions of Theorem 10 are met. For details on IRKA, we refer the reader to [12].

5.2 \( \mathcal{H}_2 \)-based reduction for port-Hamiltonian systems

Even though the optimality conditions (19)-(21) can be satisfied for generic model reduction settings starting with a general system \( \mathbf{G}(s) = \mathbf{C}(s \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \) and reducing to \( \mathbf{G}_r(s) = \mathbf{C}_r(s \mathbf{E}_r - \mathbf{A}_r)^{-1} \mathbf{B}_r \), without any restriction on structure, it will generally not be possible to satisfy all optimality conditions while also retaining port-Hamiltonian structure; there are not enough degrees of freedom to achieve both goals simultaneously (outside of the trivial case where \( \mathbf{J} = \mathbf{0} \)).

We chose instead to force only a subset of first-order conditions and use the remaining degrees of freedom in pre-
Algorithm 1 An $\mathcal{H}_2$-based IRKA for MIMO port-Hamiltonian systems (IRKA-PH):

(1) Let $G(s) = B^T Q (sI - (J - R)Q)^{-1} B$ be as in (9).
(2) Choose initial interpolation points $s_1, \ldots, s_r$ and tangential directions $\{ b_1, \ldots, b_r \}$; both sets closed under conjugation.
(3) Construct $V_r = \left[ (s_1 I - (J - R)Q)^{-1} B b_1, \ldots, (s_r I - (J - R)Q)^{-1} B b_r \right]$. 
(4) Calculate $\tilde{V}_r = V_r L^{-1}$ where $V_r^T Q V_r = L^T L$.
(5) Calculate $\tilde{W}_r = Q \tilde{V}_r$.
(6) until convergence
(a) $J_r = \tilde{W}_r^T J \tilde{W}_r, R_r = \tilde{W}_r^T R \tilde{W}_r, Q_r = I_r$, and $B_r = \tilde{W}_r^T B$.
(b) $A_r = J_r - R_r$.
(c) Compute $A_r x_i = \lambda_i x_i, y_i^T A_r = \lambda_i y_i^*$ with $y_i^* x_j = \delta_{ij}$ for left and right eigenvectors $y_i^*$ and $x_i$ associated with $\lambda_i$.
(d) $s_i I - \lambda_i I$ and $b_i^T + y_i^* B_r$ for $i = 1, \ldots, r$.
(e) Compute $V_r = \left[ (s_1 I - (J - R)Q)^{-1} B b_1, \ldots, (s_r I - (J - R)Q)^{-1} B b_r \right]$.
(f) Compute $\tilde{V}_r = V_r L^{-1}$ where $V_r^T Q V_r = L^T L$.
(g) Compute $\tilde{W}_r = Q \tilde{V}_r$.
(7) The final reduced model is given by $J_r = \tilde{W}_r^T J \tilde{W}_r, R_r = \tilde{W}_r^T R \tilde{W}_r, B_r = \tilde{W}_r^T B, Q_r = I_r$, and $C_r = B_r Q_r = B^T \tilde{W}_r$.

Corollary 11 Let $G(s) = B^T Q (sI - (J - R)Q)^{-1} B$ be a port-Hamiltonian system as in (9). Suppose IRKA-PH as described in Algorithm 1 converges to a reduced model $G_r(s) = \sum_{\lambda_i} \frac{1}{\lambda_i - \lambda} c^T b_i^T$. Then $G_r(s)$ is port-Hamiltonian and both stable and passive. Moreover, $G_r(s)$ satisfies the necessary condition (19), for $\mathcal{H}_2$ optimality, i.e., $G_r(s)$ interpolates $G(s)$ at $-\lambda_i$ along the tangential directions $b_i$ for $i = 1, \ldots, r$.

Proof: The port-Hamiltonian structure, and consequently passivity, are direct consequences of the construction of $G_r(s)$ as shown in Theorem 6. The $\mathcal{H}_2$-based interpolation property results from the way the interpolation points $s_i$ and the tangential directions $b_i$ are computed in Steps 6-c) and 6-d) throughout the iteration. Hence upon convergence: $s_i = -\lambda_i$ and $b_i$ is obtained from the residue of $G_r(s)$ corresponding to $\lambda_i$ as desired. \Box

Convergence behavior of IRKA has been studied in detail in [12]; see, also [2]. In the vast majority of cases $IRKA$ converges rapidly to the desired first-order conditions regardless of initialization. Effective initialization strategies have been proposed in [12] although random initialization in many cases performs very well. We look at different initialization techniques for IRKA-PH in Section 6 illustrating the fast/robust convergence behavior of IRKA-PH.

Remark 12 Let $PH(r)$ denote the set of port-Hamiltonian systems with state-space dimension $r$ as in (17) and consider the problem:

$$\|G - G_r\|_{\mathcal{H}_2} = \min_{G_r \in PH(r)} \|G - G_r\|_{\mathcal{H}_2}$$

where $G(s)$ and $G_r(s)$ are both port-Hamiltonian. This is the problem that we would prefer to solve and it is a topic of current research. Note that although IRKA-PH generates a reduced-order port-Hamiltonian model, $G_r(s)$, satisfying a first-order necessary condition for $\mathcal{H}_2$ optimality, this condition is not a necessary condition for $G_r(s)$ to solve (22), that is, for $G_r(s)$ to be the optimal reduced-order port-Hamiltonian system approximation to the port-Hamiltonian system $G(s)$. Nonetheless, we find that reduced-order models produced by IRKA-PH have much superior $\mathcal{H}_2$ performance compared to other approaches, as we illustrate in §6.

6 Numerical Examples

In this section, we illustrate the theoretical discussion on three port-Hamiltonian systems. The first two systems are of modest dimension so that we can compute all the norms (such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ error norms) explicitly for several $r$ values for many different methods to compare them to the fullest extent. Then, to illustrate that the proposed method can be easily applied in the large-scale settings as intended, we use a large-scale model as the third example.

6.1 MIMO Mass-Spring-Damper system

The full-order model is the the Mass-Spring-Damper system shown in Fig. 2 with masses $m_i$, spring constants $k_i$ and damping constants $c_i \geq 0$, $i = 1, \ldots, n/2$. $q_i$ is the displacement of the mass $m_i$. The inputs $u_1, u_2$ are the external forces applied to the first two masses $m_1, m_2$. The port-Hamiltonian outputs $y_1, y_2$ are the velocities of masses $m_1, m_2$. The state variables are as follows: $x_1$ is the displacement $q_1$ of the first mass $m_1$, $x_2$ is the momentum $p_1$ of the first mass $m_1$, $x_3$ is the displacement $q_2$ of the second mass $m_2$, $x_4$ is the momentum $p_2$ of the second mass $m_2$, etc.

A minimal realization of this port-Hamiltonian system for order $n = 6$ corresponding to three masses, three
springs and three dampers is

\[
\mathbf{B} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
\mathbf{C} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
\mathbf{J} = \begin{bmatrix}
k_1 & 0 & -k_1 & 0 & 0 & 0 \\
0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\
-k_1 & 0 & k_1 + k_2 & 0 & -k_2 & 0 \\
0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \\
0 & 0 & -k_2 & 0 & k_2 + k_3 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{m_3} \\
\end{bmatrix},
\]

\[
\mathbf{R} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_2 & 0 \\
0 & 0 & 0 & 0 & 0 & c_3 \\
\end{bmatrix},
\]

\[
\mathbf{Q} = \begin{bmatrix}
0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\
-k_1 & 0 & k_1 + k_2 & 0 & -k_2 & 0 \\
0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \\
0 & 0 & -k_2 & 0 & k_2 + k_3 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{m_3} \\
\end{bmatrix}.
\]

Then the \( \mathbf{A} \) matrix can be obtained as

\[
\mathbf{A} = (\mathbf{J} - \mathbf{R})\mathbf{Q} =
\begin{bmatrix}
0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\
-k_1 & 0 & k_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{m_3} & 0 \\
0 & k_2 & 0 & -k_2 & 0 & 0 \\
0 & 0 & k_2 & 0 & -k_2 & 0 \\
\end{bmatrix}.
\]

We used a 100-dimensional Mass-Spring-Damper system with \( m_i = 4, \ k_i = 4, \) and \( c_i = 1. \) We compare three different methods: (1) The proposed method in Algorithm 1 denoted by \textit{IRKA-PH} (2) The effort-constraint method (15) of \cite{20, 25, 23} denoted by \textit{Eff. Bal.}, and (3) One step interpolatory model reduction without the \( \mathcal{H}_2 \) iteration, denoted by \textit{1-step-Intrpl}. The reason for including \textit{1-step-Intrpl} is as follows: We choose a set of interpolation points and tangential directions; then using the interpolatory projection in Theorem 6, we obtain a reduced model. Then, we use the same interpolation points and directions as an initialization technique for \textit{IRKA-PH}. Then, we are able to illustrate starting from the same interpolation data, how \textit{IRKA-PH} corrects the interpolation points and directions, and how much it improves the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) behavior through out this iterative process.

Using all three methods, we reduce the order from \( r = 2 \) to \( r = 20 \) with increments of two. For \textit{IRKA-PH}, initial interpolation points are chosen as logarithmically spaced points between \( 10^{-3} \) and \( 10^{-1} \); and the corresponding directions are the dominant left and right singular vectors of the transfer function at each interpolation point (a small \( 2 \times 2 \) singular value problem). These same points and directions are also used for \textit{1-step-Intrpl}. The resulting relative \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) error norms for each order \( r \) are illustrated in Figure 3. Several important conclusions are in order: First of all, with respect to the both \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norm, \textit{IRKA-PH} significantly outperforms the other methods. As the figure illustrates, for \textit{1-step-Intrpl}, the performance hardly improves as \( r \) increases unlike \textit{IRKA-PH} for which both the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) errors decay consistently. This shows the superiority of \textit{IRKA-PH} over \textit{1-step-Intrpl}. The initial interpolation point and direction selection does not yield a satisfactory interpolatory reduced-order model; however instead of searching some other interpolation data in an \textit{ad hoc} way, the proposed method automatically corrects the interpolation points through out the iteration and yields a significantly smaller error norm. To see this effect more clearly, we plot the initial interpolation point selection, denoted by \( s^{(0)} \), and the final/converged interpolation points, denoted by \( s^{(\text{final})} \), in Figure 4 together with the mirror images of the original system poles, denoted by \( -\lambda_i \). As this figure illustrates, even starting with this logarithmically placed points on the real line, \textit{IRKA-PH} iteratively corrects the points so that upon convergence they automatically align themselves around the mirror images of the original system poles; we note that the full-order poles are computed only to obtain this figure and are not needed by \textit{IRKA-PH}.

The second observation from Figure 3 is that in comparison to \textit{Eff. Bal.}, \textit{IRKA-PH} achieves the better performance with less computational effort; the main cost is sparse linear solves and no dense matrix operations are needed unlike the balancing-based approaches where Lyapunov equations need to be solved. We also note that
even though the proposed method is $\mathcal{H}_2$-based, it produces a satisfactory $\mathcal{H}_\infty$ performance as well. This is not surprising as [12] discusses IRKA usually leads to high fidelity $\mathcal{H}_\infty$ performance as well. Finally, we note that the effort-constraint method used here is applied in the regular balanced basis. For this example, it appears that the usual balanced coordinates may not be the best coordinates to perform the effort-constraint method.

Before we conclude this example, we show, in Fig 5, how the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ errors evolve during IRKA-PH for $r = 20$, the largest reduced-order. The figure reveals that IRKA-PH converges after seven or eight steps. Moreover, the large initial relative errors are reduced drastically even after two steps of the iteration; hence illustrating the effectiveness of the proposed method.

Fig. 3. Relative $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for Mass-Spring-Damper System

Fig. 4. Initial and converged interpolation points

Fig. 5. Relative $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for Mass-Spring-Damper System

6.2 MIMO port-Hamiltonian Ladder Network

As a second MIMO port-Hamiltonian system, we consider an $n$-dimensional Ladder Network as shown in Fig. 6.

We take the current $I$ on the left side and the voltage $U$ on the right side of the Ladder Network as the inputs. The port-Hamiltonian outputs are the voltage over the first capacitor $U_{C_1}$ and the current through the last inductor $I_{L_{n/2}}$. The state variables are as follows: $x_1$ is the charge $q_1$ of $C_1$, $x_2$ is the flux $\phi_1$ of $L_1$, $x_3$ is the charge $q_2$ of $C_2$, $x_4$ is the flux $\phi_2$ of $L_2$, etc. The directions chosen for the internal currents of the Network are shown by plus- and minus-signs in Fig 6. A minimal realization of this port-Hamiltonian Ladder Network for order $n = 4$ is given in Example 1.

Adding another LC pair to the network, which would correspond to an increase of the dimension of the model by two, will modify the system matrices as follow: The subdiagonal of the matrix $A$ will contain additionally $L_{n/2-1}^{-1}$ with the plus-sign in the $(n/2 + 1, n/2)$ position. The superdiagonal of $A$ will contain $-C_{n/2}^{-1}, L_{n/2}^{-1}$ with the minus-sign in the $(n/2, n/2 + 1)$ position. Furthermore, the main diagonal of $A$ will have zeros in the $(n - 2, n - 2)$ and
$(n-1, n-1)$ positions and $-\frac{R}{R_{n/2}}$ in the $(n, n)$ position. $B$ and $C$ matrices will be of the similar structure as in (10), (11). The matrix $B$ will have ones in the (1, 1) and (n, 2) positions and zeros in the other positions. The only nonzero entries of $C$ are $\frac{1}{k}$ in the (1, 1) position and $\frac{1}{R_{n/2}}$ in the (2, n) position.

Here, we first consider the 100-dimensional full order port-Hamiltonian network with $C_1 = 5$ and $L_2 = 12$ and $R = 2$. For this model, we compare $IRKA-PH$ with regular balanced truncation (denoted by Reg. Bal). We would like to emphasize that balanced truncation does not preserve the port-Hamiltonian structure. We choose to compare with regular balanced truncation to better illustrate the effectiveness of our proposed method by showing that $IRKA-PH$ can outperform even regular balanced truncation which is known to yield high-fidelity $H_\infty$ and $H_2$ performance and is not constrained to preserve the port-Hamiltonian structure.

We reduce the order from $r = 2$ to $r = 30$ with increments of two. The resulting relative $H_2$ and $H_\infty$ error norms are depicted in Figure 7. First observation is that in terms of the $H_2$ performance, the proposed method drastically outperforms balanced truncation. Secondly, we observe that for most of the $r$ values, the proposed method outperforms balanced truncation in terms of the $H_\infty$ behavior as well. This is crucial since balanced truncation is a method tailored towards $H_\infty$ behavior. Moreover unlike $IRKA-PH$: it is not constrained to preserve the structure. Therefore, this is an important illustration of the quality of the proposed method.

Similar to the previous example, we illustrate the convergence behavior of $IRKA-PH$, in this case for $r = 30$, in Figure 8. In this case, the convergence is even faster than the previous example with the algorithm converging after four to five steps. For this model, we have initialized $IRKA-PH$ arbitrarily in the rectangular region with real parts of the interpolation points in the interval $[10^{-7}, 10^{-3}]$ and the imaginary parts in the interval $[10^{-2}, 10^{-1}]$. This roughly corresponds to the extreme eigenvalues of $A$ reflected across the imaginary axis suggested as an effective initialization strategy in [12]. We emphasize that this initialization does not require computing all the eigenvalues; it only needs the two extreme ones which are cheaply and effectively computable by methods such an Implicitly Restarted Arnoldi Algorithm of Sorensen [26].

6.3 Mass-Spring-Damper system with $n = 20000$

In the previous two examples, to have a thorough analysis of the models with as many system norm computations as possible, we have chosen a very modest system size of 100. In this example, to illustrate that we can effectively apply our method in large-scale settings, we modify the Mass-Spring-Damper model to have a full model of order $n = 20000$. Then, using $IRKA-PH$, we reduce the order to $r = 20$, $r = 30$ $r = 40$ and $r = 50$. In each case, the initial interpolation points are chosen as logarithmically spaced points between $10^{-3}$ and $10^{-1}$ with the corresponding tangential directions chosen as the leading singular vectors of the transfer function at these points. Before we present the results, we note that even at this scale, the method took less than one minute to converge with a rather straightforward implementation in Matlab. We have not tried to optimize the \begin{align*}
\|G(\omega)\|_2 \text{ vs. } \omega \text{ of the full-order model } G(s) \text{ and all four reduced models are plotted in Figure 9. Except the } r = 20 \text{ case, all the reduced models provide a high quality approximation to the full-order model of order } n = 2000. \text{ To illustrate the approximation quality further, we depict the sigma plots of the error.
\end{align*}
As $r$ increases, the quality of the approximation improves consistently; for $r = 50$, we obtain an approximate relative $\mathcal{H}_\infty$ error of $7.90 \times 10^{-4}$. Hence, with the proposed algorithm, we are able to reduce a port-Hamiltonian system of order $n = 20000$ in a numerically effective structure-preserving way using interpolation. Moreover, with the $\mathcal{H}_2$-inspired interpolation point and tangential direction selection, the reduced-model of order $r = 50$ is accurate to a relative error of $7.90 \times 10^{-4}$.

---

1 We use the term approximate since both $\|G\|_{\mathcal{H}_\infty}$ and $\|G - G_{50}\|_{\mathcal{H}_\infty}$ are computed by 500 frequency sampling points over the imaginary axis as opposed to an exact $\mathcal{H}_\infty$ norm computation. However, since the error plot is smooth, we expect this error number to be accurate enough.


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