Heat and moisture transport in a paper sheet moving over a hot print surface

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Abstract

The generation of moisture-induced deformations during the final use of paper is of concern for the printing industry. Knowing the moisture profiles during printing is a primary step in order to understand such deformations. In this paper we derive a mathematical model to describe the heat and moisture transport in paper, and use this model to study the moisture gradients induced on a paper sheet by a printer from Océ Technologies, which contains a hot print surface. The model regards paper as a porous medium saturated with air, and it considers heat and moisture transport along the length and thickness directions of the sheet. Several physical parameters are determined experimentally; others are taken from literature. The system of coupled differential equations is solved iteratively, and the numerical results are also used to identify the relevant driving forces in the temperature-induced moisture changes in paper.

Keywords: Porous medium; Paper; Heat transport; Moisture transport; Printer

1. Introduction

The dimensional stability of paper, i.e., the property to maintain its original dimensions while being used, is of importance for the printing industry. During printing, heating and adding ink/toner can change the moisture content of the paper sheets, thereby creating moisture gradients. These gradients are undesirable since they induce local expansion or compression of the fibers, generating mechanical instabilities like curls and cockles [1, 2]. Such instabilities affect the quality of the prints, and since they can cause paper jams, they also affect the productivity of the printers.

To understand the moisture-induced deformation of paper, the heat and moisture transport processes need to be understood first. Several authors have derived moisture transport models for the final use of paper. Dano and Bourque [3] modeled the moisture diffusion through the thickness of a paper sheet, using their model to predict the formation of curls. In their work, paper was modeled as a homogeneous medium, but as noted in [4], better results are obtained by modeling paper as a porous medium. More elaborate models considering paper as a composite of pores and fibers were presented in [5], [6],
Although these models showed good agreement with experiments, they were restricted to isothermal conditions, which do not always hold true for the printing practice. Few non-isothermal models for moisture transport at the end use of paper have been developed. Foss et al., [9] modeled the heat and moisture transport in a paper sheet in response to variations of the relative humidity in the surroundings of the sheet. They studied the temperature changes induced by water sorption, but the impact of temperature on the evolution of the moisture content was not clearly established. In order to study the moisture changes at the final use of printing paper, a model that accounts for moisture transport in response to changes in temperature still needs to be developed.

In this paper we derive a model to describe the heat and moisture transport in a paper sheet in response to temperature variations. By choosing appropriate boundary conditions, the model is used to describe the heat and moisture transport in a paper sheet inside a printer of Océ Technologies, which contains a hot print surface. In such a printer, moisture-induced mechanical instabilities can be observed during printing at the higher range of relative humidity. In order to gain insight in the cause of these instabilities, the model is used to estimate the temperature and moisture gradients induced by the printer, and to study how variations in physical and printing parameters affect those gradients. The remainder of the paper is organized as follows: Section 2 presents a detailed derivation of the governing equations for heat and moisture transport in paper. Based on these equations, Section 3 introduces a two-dimensional model of the target printing environment. Section 4 presents the experimental determination of some of the model parameters. Section 5 describes the numerical solution of the model, and the numerical results are presented and discussed in Section 6. Finally, Section 7 presents a summary and conclusions.

2. Governing equations for heat and moisture transport in paper

Paper is a porous medium composed of cellulose fibers and often fillers and additives that are embedded in a complex solid matrix. A framework to model transport in porous media was derived by Bear and Bachmat [10] with the method of Representative Elementary Volumes (REV). Following their approach, paper can be idealized at the macroscopic level as consisting of two overlapping continua: A solid matrix, mainly composed of cellulose fibers, and a pore space saturated with air. The pore space can contain water vapor, and the solid matrix can contain liquid water in an adsorbed state, bonded by physicochemical forces to cellulose fibers. At relative humidities (RH) above 80% water might also condensate as bulk liquid [11], but we neglect this effect since the operation range for the target printer is restricted to RH < 80%. In this section we derive mass balances of water and heat balances for the solid paper matrix and the pore space.

2.1. Mass balances

If we denote the pore space by the subscript p and the solid matrix by the subscript f (fibers), neglecting possible convection, and assuming the volumetric fraction of pores and fibers to be constant in time and space, the balances of water in paper are given by

\[
\eta \frac{\partial C_p}{\partial t} = \eta \nabla \cdot (D_{eff}^p \nabla C_p) + \eta \Gamma_p - S_{pf},
\]

\[
(1 - \eta) \frac{\partial C_f}{\partial t} = (1 - \eta) \nabla \cdot (D_{eff}^f \nabla C_f) + (1 - \eta) \Gamma_f + S_{pf},
\]

where \( C_p \) [kg m\(^{-3}\)] is the concentration of water vapor in the pores, \( C_f \) [kg m\(^{-3}\)] is the concentration of bonded water in the fibers, \( 0 < \eta < 1 \) [-] is the volumetric fraction of pore space, also known as the porosity of the paper sheet, \( (1 - \eta) \) [-] is the volumetric fraction of fibers, the diagonal tensors \( D_{eff}^p \) and
$D^\text{eff}_{f} \, [m^2 \, s^{-1}]$ are the effective diffusivities of water vapor in the pores and the fibers, respectively, $\Gamma_p$ is the rate of production of water in the pores per unit volume of pores, $\Gamma_f$ is the rate of production of water in the fibers per unit volume of fibers, and $S_{pf} \, [kg \, m^{-3} \, s^{-1}]$ is the rate of mass exchange from pores to fibers per unit volume of porous medium, in this case by sorption or desorption.

Neglecting possible addition of ink leads to $\Gamma_p = \Gamma_f = 0$. We also assume the diffusion of bonded water along the fibers network to be negligible. This assumption is based on the diffusivity values along the thickness direction of a paper sheet reported by Bandyopadhyay [5] and Foss [9] (of the order of $10^{-14} \, [m^2 \, s^{-1}]$). For diffusivities of the order of $10^{-14}$ to $10^{-12} \, [m^2 \, s^{-1}]$ the characteristic diffusion time would be of the order of hours for the thickness direction, or higher for the in-plane directions. These diffusion time scales are much larger than the time scales of interest during printing (in the order of a minute).

The water sorption on paper fibers has been successfully modeled by means of a linear driving force [5, 6, 9, 12]. Following this approach, the term $S_{pf}$ can be expressed as

$$S_{pf} = (1 - \eta) K_{mi} (C_{f,\text{equ}} - C_f), \quad (3)$$

with $C_{f,\text{equ}} \, [kg \, m^{-3}]$ the water concentration in the fibers that would be in equilibrium with the local vapor concentration in the pores, and $K_{mi} \, [s^{-1}]$ an internal (pores to fibers) mass transfer coefficient. This coefficient accounts for the resistance to water vapor adsorption as well as the internal area available for contact between pores and fibers. An explicit relation between $C_{f,\text{equ}}$ and $C_p$ can be derived by using a sorption isotherm, i.e., a relation between the equilibrium moisture content in a material and the relative humidity at a given constant temperature. One of the isotherms that has been reported to fit data for paper well is the Guggenheim-Anderson-De Boer or GAB-isotherm [11, 13]. It is given by

$$X_{\text{equ}} = \frac{(X_m CK) \rho_{\text{RH}}}{(1 - K \rho_{\text{RH}})(1 - K \rho_{\text{RH}} + CK \rho_{\text{RH}})}, \quad (4)$$

where $\rho_{\text{RH}}$ [-] is the relative humidity; $X_{\text{equ}}$ [-] is the equilibrium moisture content on a dry basis, i.e., the mass of water per unit mass of dry paper; and $C$ [-], $K$ [-] and $X_m$ [-] are the GAB-parameters. These parameters are temperature dependent. For a given temperature they are determined experimentally by fitting (4) to equilibrium moisture data, measured at that temperature, for different relative humidities. Since for paper the adsorption isotherms can differ from the desorption isotherms [14], the GAB-parameters also depend on the history of the sorption process. However, for simplicity, hysteresis effects are not considered. The equilibrium moisture content $X_{\text{equ}}$ is related to the water concentration in pores and fibers by

$$X_{\text{equ}} = \frac{\eta C_p + (1 - \eta) C_{f,\text{equ}}}{\eta \rho_p + (1 - \eta) \rho_f} \approx \frac{C_{f,\text{equ}}}{\rho_f}, \quad (5)$$

where $\rho_p$ and $\rho_f$ [kg m$^{-3}$] are the density of air and dry fibers, respectively. Since $\eta C_p \sim 10^{-3}$, $(1 - \eta) C_{f,\text{equ}} \sim 10^{1}$, $\eta \rho_p \sim 10^{-1}$ and $(1 - \eta) \rho_f \sim 10^{2}$, the approximation in the right hand side is based on the fact that $\eta C_p \ll (1 - \eta) C_{f,\text{equ}}$ and $\eta \rho_p \ll (1 - \eta) \rho_f$.

The relative humidity $\rho_{\text{RH}}$ in (4) is defined as the ratio of the partial pressure of water vapor in air to the saturated vapor pressure of water at a given temperature. By using the ideal gas law, $\rho_{\text{RH}}$ can be expressed in terms of the water concentration in the pores as

$$\rho_{\text{RH}} = \frac{C_p}{C_p}, \quad (6)$$
with $C^s_p$ the saturated water vapor concentration in air. $C^a_p$ can be obtained as a function of the temperature from a straightforward substitution of the ideal gas law in the Arden-Buck’s relation [15] as follows

$$C^s_p = \frac{A_1}{T} \exp \left[ \frac{-A_2 + A_3 T - A_4 T^2}{T - A_5} \right],$$  \hspace{1cm} (7)$$

with $T$ the temperature in Kelvin, and the constants $A_1$ to $A_5$ given by $A_1 = 1.32 \text{[K]}, A_2 = 5.42 \times 10^3 \text{[K]}, A_3 = 21.01 \text{[]}, A_4 = 4.26 \times 10^{-3} \text{[K]}^{-1}$ and $A_5 = 16.01 \text{[K]}$.

After substituting (3) in (1) and (2), and taking into account that $\Gamma_p$, $\Gamma_f$ and $D_{f}^{\text{eff}}$ equal zero, the model for moisture transport in paper reads

$$\frac{\partial C_p}{\partial t} = \nabla \cdot (D_{p}^{\text{eff}} \nabla C_p) - \frac{(1-\eta)K_{mi}}{\eta} (C_{f,\text{equ}} - C_t),$$  \hspace{1cm} (8)$$

and

$$\frac{\partial C_t}{\partial t} = K_{mi} (C_{f,\text{equ}} - C_t)$$  \hspace{1cm} (9)$$

where $C_{f,\text{equ}} = C_{f,\text{equ}} (C_p,T)$ is the nonlinear function of $C_p$ and $T$ given by (4), (5) and (6), with $C^a_p = C^a_p(T)$ defined in (7) and $K$, $C$ and $X_m$ implicit functions of the temperature (see Table 1).

2.2. Heat balance

The heat balance of a porous medium saturated with one fluid (in this case air) and with a thermoelastic solid matrix was derived in [10]. If there is no convective flow through the porous medium and for relatively small solid grains, the fluid and the solid matrix can be assumed to be in thermal equilibrium. With this assumption, the heat balance on paper can be written as

$$\frac{\partial}{\partial t} \left( (\eta \rho_p c_v + (1-\eta)\rho_f c_t) T \right) = \nabla \cdot (\Lambda \nabla T) + \eta \Gamma_{h,p} + (1-\eta)\Gamma_{f,h} + (1-\eta)\nu \frac{\partial C_t}{\partial t} T,$$  \hspace{1cm} (10)$$

where $c_v$ [J kg$^{-1}$ K$^{-1}$] is the heat capacity at constant volume of the air inside the pores, $c_t$ [J kg$^{-1}$ K$^{-1}$] is the heat capacity of the fibers, $\rho_p$ and $\rho_f$ [kg m$^{-3}$] are the density of the air inside the pores and of the fibers, respectively, the diagonal tensor $\Lambda$ [J m$^{-1}$ s$^{-1}$ K$^{-1}$] is the thermal conductivity of paper, $\Gamma_{h,p}$ [J m$^{-3}$ s$^{-1}$] is the rate of heat production in the pores per unit volume of pores, $\Gamma_{h,f}$ [J m$^{-3}$ s$^{-1}$] is the rate of heat production in the fibers per unit volume of fibers, $\epsilon_f$ [K$^{-1}$] is the solid’s dilatation, $\nu$ [J m$^{-3}$] is the thermoelastic coefficient of the fibers network, and $T$ is the temperature of the paper sheet, which is assumed to be the same for pores and fibers.

The term $\eta \rho_p c_v + (1-\eta)\rho_f c_t$ represents the volumetric heat capacity of paper. In practice, it is obtained by multiplying the density of paper with its specific heat capacity, both of them determined experimentally. Thus, this term can be rewritten as

$$\eta \rho_p c_v + (1-\eta)\rho_f c_t = c \rho,$$ \hspace{1cm} (11)$$

with $\rho$ [kg m$^{-3}$] the density of paper and $c$ [J kg$^{-1}$ K$^{-1}$] the specific heat capacity of paper.

$\Gamma_{h,p}$ and $\Gamma_{h,f}$ are heat sources associated with the adsorption and desorption of water molecules on the fibers. Since we assumed the bonded water diffusivity to be negligible, the concentration in the fibers can only change due to water adsorption or desorption. Therefore, the heat sources can be expressed in terms of the rate of change of $C_t$ as follows

$$\eta \Gamma_{h,p} + (1-\eta)\Gamma_{h,f} = -(1-\eta)\Delta H_{\text{ads}} \frac{\partial C_t}{\partial t},$$ \hspace{1cm} (12)$$
with $\Delta H_{\text{ads}} < 0$ [J kg$^{-1}$] the specific enthalpy of adsorption of water vapor on the fibers. $\Delta H_{\text{ads}}$ is a function of $C_f$ [16]. At lower $C_f$-values the adsorbed water molecules are mainly bound to the solid’s molecules. However, as $C_f$ increases, water molecules also bind other water molecules creating a multi-layer. Since the binding energy for molecules bound to other water molecules is lower than for those directly in contact with the solid [17], the magnitude of $\Delta H_{\text{ads}}$ decreases with increasing $C_f$. A curve relating $\Delta H_{\text{ads}}$ and $C_f$ is presented in Section 4 together with other model parameters.

From (10) to (12), neglecting thermal expansion ($\nu = 0$), and assuming $\rho$ and $c$ to be constant in time, the heat balance of paper reads

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (\Lambda \nabla T) - (1 - \eta) \Delta H_{\text{ads}} \frac{\partial C_f}{\partial t},$$

with $\Delta H_{\text{ads}} = \Delta H_{\text{ads}}(C_f)$ (see Figure 2).

### 2.3. Dimensionless equations

The heat and mass balances are made non-dimensional by introducing the dimensionless variables $\hat{C}_p$, $\hat{C}_f$, $\hat{T}$, $\hat{x}$ and $\hat{t}$ as follows

$$\hat{C}_p = \frac{C_p}{C_{p,\text{ref}}}, \quad \hat{C}_f = \frac{C_f}{C_{f,\text{ref}}}, \quad \hat{T} = \frac{T}{T_{\text{ref}}}, \quad \hat{x} = \frac{x}{\ell}, \quad \hat{t} = \frac{D_{\text{ref}}}{\ell^2},$$

with $C_{p,\text{ref}}$, $C_{f,\text{ref}}$, $T_{\text{ref}}$ the highest possible values of $C_p$, $C_f$ and $T$, respectively, $\ell$ a characteristic length, and $D_{\text{ref}}$ a reference diffusivity. In terms of the dimensionless variables, equations (8), (9) and (13) read

$$\frac{\partial \hat{C}_p}{\partial \hat{t}} = \hat{\nabla} \cdot (\hat{D}_p \nabla \hat{C}_p) - N_p \left( \hat{C}_{f,\text{equ}} - \hat{C}_f \right),$$

$$\frac{\partial \hat{C}_f}{\partial \hat{t}} = N_f \left( \hat{C}_{f,\text{equ}} - \hat{C}_f \right),$$

$$\frac{\partial \hat{T}}{\partial \hat{t}} = \hat{\nabla} \cdot \left( \hat{L}_e \nabla \hat{T} \right) - \Delta \hat{H}_{\text{ads}} \frac{\partial \hat{C}_f}{\partial \hat{t}},$$

with $N_p$ and $N_f$ dimensionless constants, $\hat{D}_p$ the dimensionless water vapor diffusivity tensor, $\hat{L}_e$ the Lewis number tensor for thermal diffusion, and $\Delta \hat{H}_{\text{ads}}$ the dimensionless enthalpy of adsorption. They are given by

$$N_f = \frac{\ell^2 K_{\text{mi}}}{D_{\text{eff}}}, \quad N_p = \frac{(1 - \eta) C_{f,\text{ref}}}{\eta C_{p,\text{ref}}} N_f, \quad \hat{D}_p = \frac{D_{\text{eff}}}{D_{\text{ref}}}, \quad \hat{L}_e = \frac{\Lambda}{D_{\text{ref}} \rho c}, \quad \Delta \hat{H}_{\text{ads}} = \frac{(1 - \eta) C_{f,\text{ref}}}{\rho c T_{\text{ref}}} \Delta H_{\text{ads}}.$$

The remainder of this paper will refer to dimensionless variables unless stated otherwise. For ease of notation, the hat-signs will be omitted.

### 3. Mathematical model of the printing environment

A schematic representation of a paper sheet in the target printing environment is shown in Figure 1. When paper is printed, it is moved along two metallic surfaces. The first surface, defined as the print surface (ps), is where printing takes place. It has a constant homogeneous temperature $T_{ps}$ and length $L_{ps}$. The second surface, defined as the drying surface (ds), is used to keep the sheet straight while the
toner dries. It has the same temperature as the bulk environment \((T^\infty)\), and its length is denoted by \(L_{ds}\).

While being printed, the sheet is moved instantaneously over a distance \(Y_{step}\) after every time interval \(t_{step}\). This movement is modeled using the coordinates of the paper sheet as the reference frame, i.e., the point \(y = 0\) is always located at the left corner of the sheet, while the position of the print and the drying surfaces change in time. In this way, the \(y\)-coordinate of the left extreme of the print surface \(Y_{ps}(t)\) is given by

\[
Y_{ps}(t) = -Y_{step} \left\lfloor \frac{t}{t_{step}} \right\rfloor + Y_{ps}(0),
\]

with \(\lfloor \cdot \rfloor\) the floor function and \(Y_{ps}(0)\) the initial position of the print surface.

Since the thickness of the paper sheet is much smaller than its width and length, the area available for heat and mass exchange on the lateral surfaces of the sheet is negligible compared to the area on the top and bottom surfaces. Thus, we can assume the lateral surfaces to be isolated. Taking this into account, and assuming there are no other factors inducing gradients along the width of the sheet (\(x\)-direction), we can describe the heat and moisture transport in a sheet on the printing environment with a two-dimensional model in \(y\) and \(z\). In terms of the dimensionless variables, this model is given by the two-dimensional version of the system in (14), i.e.,

\[
\frac{\partial C_p}{\partial t} = D_{p,y} \frac{\partial^2 C_p}{\partial y^2} + D_{p,z} \frac{\partial^2 C_p}{\partial z^2} - N_p \left( C_{t,equ} - C_t \right),
\]

\[
\frac{\partial C_t}{\partial t} = N_t \left( C_{t,equ} - C_t \right),
\]

\[
\frac{\partial T}{\partial t} = L_{e,y} \frac{\partial^2 T}{\partial y^2} + L_{e,z} \frac{\partial^2 T}{\partial z^2} - \Delta H_{ads} \frac{\partial C_f}{\partial t},
\]

with \(D_{p,y}\) and \(D_{p,z}\) the in-plane and through-plane dimensionless diffusivities of water vapor in paper, respectively, and similarly for the Lewis numbers \(L_{e,y}\) and \(L_{e,z}\). The boundary conditions at \(y = 0\) and \(y = L_y\) are given by

\[
\frac{\partial C_p}{\partial y} = 0, \quad \frac{\partial T}{\partial y} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = L_y.
\]

Due to the relative movement of the print and drying surfaces, the bottom of the paper sheet can be locally exposed to three different temperature and moisture boundary conditions: Being over the print
surface, being over the drying surface, or being exposed to the environment. At the top surface of the sheet, water vapor and heat are transported by convection. The boundary conditions at the bottom and top of the sheet for the variable $C_p$ are given by

\[
\frac{\partial C_p}{\partial z} = \text{Bi}_m^{-}(C_p - C_p^\infty) \quad \text{at} \quad z = 0,
\]

\[
\frac{\partial C_p}{\partial z} = \text{Bi}_m^{+}(C_p^\infty - C_p) \quad \text{at} \quad z = L_z,
\]

where $C_p^\infty$ is the dimensionless water vapor concentration in the bulk environment, $L_z$ is the dimensionless thickness of the sheet, and $\text{Bi}_m^-$ and $\text{Bi}_m^+$ are the Biot numbers for mass transfer at the bottom and top of the sheet, respectively. They are given by

\[
\text{Bi}_m^- = \frac{L_zK_m^-}{D_{p,z}}, \quad \text{Bi}_m^+ = \frac{L_zK_m^+}{D_{p,z}},
\]

where $K_m^+$, with units [m s$^{-1}$], is a convective mass transfer coefficient at the top of the sheet, and $K_m^-$, with units [m s$^{-1}$], is the mass transfer coefficient at the bottom of the sheet. $K_m^- = 0$ wherever the sheet is in contact with the print or the drying surface, and we assume $K_m^- = K_m^+$ wherever the bottom of the sheet sheet is exposed to the environment, i.e.,

\[
K_m^-(t, y) = \begin{cases} 
K_m^+ & \text{if} \quad y < Y_{ps}(t), \\
0 & \text{if} \quad Y_{ps}(t) \leq y < Y_{ps}(t) + L_{ps} + L_{ds}, \\
K_m^+ & \text{if} \quad y \geq Y_{ps}(t) + L_{ps} + L_{ds}.
\end{cases}
\]

The temperature boundary conditions at the bottom and top of the sheet are given by

\[
\frac{\partial T}{\partial z} = \text{Bi}_h^-(T - T^-) \quad \text{at} \quad z = 0,
\]

\[
\frac{\partial T}{\partial z} = \text{Bi}_h^+(T^\infty - T) \quad \text{at} \quad z = L_z,
\]

where $T^\infty$ and $T^-$ are the dimensionless temperatures of the bulk environment and the material present at the bottom of the sheet, respectively. $T^- = T_{ps}$ wherever the paper sheet is in contact with the print surface, and $T^- = T^\infty$ otherwise, i.e.,

\[
T^-(t, y) = \begin{cases} 
T^\infty & \text{if} \quad y < Y_{ps}(t), \\
T_{ps} & \text{if} \quad Y_{ps}(t) \leq y < Y_{ps}(t) + L_{ps}, \\
T^\infty & \text{if} \quad y \geq Y_{ps}(t) + L_{ps} + L_{ds}.
\end{cases}
\]

$\text{Bi}_h^-$ and $\text{Bi}_h^+$ are the Biot numbers for heat transfer at the bottom and top of the sheet, respectively. They are given by

\[
\text{Bi}_h^- = \frac{L_zK_h^-}{A_z}, \quad \text{Bi}_h^+ = \frac{L_zK_h^+}{A_z},
\]

where $K_h^+$, with units [W m$^{-2}$ K$^{-1}$], is a convective heat transfer coefficient at the top of the sheet, and $K_h^-$, with units [W m$^{-2}$ K$^{-1}$], is the heat transfer coefficient at the bottom of the sheet. $K_h^- = h_c$ whenever the paper is in contact with the print or drying surfaces, with $h_c$ the contact thermal conductance between paper and these metallic surfaces. We assume $K_h^- = K_h^+$ wherever the paper is exposed on both sides to the environment, i.e.,

\[
K_h^-(t, y) = \begin{cases} 
K_h^+ & \text{if} \quad y < Y_{ps}(t), \\
h_c & \text{if} \quad Y_{ps}(t) \leq y < Y_{ps}(t) + L_{ps} + L_{ds}, \\
K_h^+ & \text{if} \quad y \geq Y_{ps}(t) + L_{ps} + L_{ds}.
\end{cases}
\]
The dimensionfull parameters $K_h^+$ and $K_m^+$ are obtained from heat and mass transfer correlations for convective flux. $K_h^+$ is calculated based on the following expression for the average Nusselt number ($\text{Nu}$) in a laminar boundary layer over a flat plate [18]:

$$\text{Nu} = \frac{K_h^+ L}{\Lambda_a} = 0.664 \text{Re}^{1/2} \text{Pr}^{1/3},$$

with the Reynolds number ($\text{Re}$), the Prandtl number ($\text{Pr}$), and the plate length $L$ given by

$$\text{Re} = \frac{\rho_a V^\infty L}{\mu_a}, \quad \text{Pr} = \frac{\mu_a c_a}{\Lambda_a}, \quad L = L_{ps} + L_{ds}.$$ 

$V^\infty$ [m/s], $\rho_a$ [kg m$^{-3}$], $\mu_a$ [kg m$^{-1}$ s$^{-1}$], $c_a$ [J kg$^{-1}$ K$^{-1}$], and $\Lambda_a$ [W m$^{-1}$ K$^{-1}$] are the (dimensionfull) bulk velocity, density, dynamic viscosity, heat capacity and thermal conductivity of the air over the print surface, respectively. The correlation in (18) is valid for incompressible flows with uniform heat flux or temperature, and satisfying $\text{Re} < 2 \times 10^5$ and $0.6 \leq \text{Pr} \leq 50$. Although neither the heat flux nor the temperature are uniform for this application, we use this correlation to obtain an estimate of $K_h^+$ since no experimental value is currently available for this parameter.

Using the analogy between heat and mass transfer, $K_m^+$ is obtained from (18) by replacing the Prandtl number by the Schmidt number ($\text{Sc}$), and the average Nusselt number by the average Sherwood number ($\text{Sh}$) [18]. The resulting correlation reads

$$\text{Sh} = \frac{K_m^+ L}{D_a} = 0.664 \text{Re}^{1/2} \text{Sc}^{1/3},$$

with $D_a$ [m$^2$ s$^{-1}$] the diffusivity of water vapor in air, and $\text{Sc}$ given by $\text{Sc} = \mu_a / (\rho_a D_a)$.

4. Model parameters and experiments

Different experiments are executed in order to determine some of the transport parameters involved in the model. All experiments are conducted with commercial printing paper provided by Océ Technologies. The specific heat ($c$) and the in-plane and through-plane thermal conductivities ($\Lambda_y$ and $\Lambda_z$) are determined simultaneously with the Transient Plane Source Hot-Disc method, following the ISO/DIS 22007-2.2 protocol [19]. The sorption isotherms are measured at $T = 20$, 25 and 35$^\circ$C by means of the well known gravimetric method. The corresponding GAB-parameters are found by fitting (4) to the isotherms with the Least Squares method. These parameters are presented in Table 1. The enthalpy of adsorption ($\Delta H_{ads} = \Delta H_{ads}(C_f)$) is found from the sorption isotherms at several temperatures as described in [20]. The resulting curve is presented in Figure 2. The through-plane effective diffusivity ($D_{eff, p,z}$) is found by means of a diffusion cup experiment consisting of a minor modification of the diffusion stack introduced in [21]. The in-plane water vapor diffusivity ($D_{eff, p,y}$) is estimated from $D_{eff, p,z}$ and the in-plane and through-plane tortuosities, $\tau_{prop}$ and $\tau_{prop}$, reported in [22] for n-propanol vapor in bleached kraft paper board as $D_{eff, p,y} = D_{eff, p,z} (\tau_{prop} / \tau_{prop})$. The remaining physical parameters are obtained from literature. Table 2 presents the values of the physical parameters involved in the model and the parameters describing the printing environment.

5. Numerical solution

The differential equations derived in Section 3 are discretized following the method of lines (MOL) approach, i.e., the spatial derivatives are discretized first, and the resulting ODE system is subsequently
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<th>$C$</th>
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</table>

Table 1: GAB sorption isotherm parameters for different temperatures.

In these equations, \( \Delta t \) is the time step used for time integration, \( \theta \) is an adjustable parameter varying between 0 and 1, \( I \) is the identity matrix of size \( N_y \times N_z \), and the operator * is the Hadamard or entrywise matrix product [25]. \( c_p \), \( c_f \) and \( T \) are vectors containing the values of \( C_p \), \( C_f \) and \( T \) at all grid points, respectively. The superscript \( n \) is used to indicate the value of a given vector at the \( n \)th time level, and similarly for \( n+1 \). Thus, the vectors \( c_p^{n+1} \), \( c_f^{n+1} \) and \( T^{n+1} \) are the unknowns of the system. The vector \( c_{\text{equ}} \) is used to denote the non-linear function \( C_{f,\text{equ}}(C_p,T) \) evaluated at all grid points, and similarly for \( \Delta h_{\text{ads}} \). The matrices \( A_p \) and \( A_T \) are block tri-diagonals resulting from the space discretization of equations (15) and (17), respectively. \( b_p \) and \( b_T \) are vectors originating from the boundary conditions.
Equations (20) to (23) are coupled. To solve this system, we eliminate $c_t^{n+1}$ from (21) and substitute it in (20). With this substitution, and after rearranging terms, we obtain the following discrete equation for $C_p^c$:

\[
(I - \theta \Delta t A_p) c_p^{n+1} + \Delta t \theta N_p (1 - N_3) c_{f,\text{equ}}^{n+1} = (I + (1 - \theta) \Delta t A_p) c_p^n + \Delta t N_p \left( (N_2 - 1 + \theta) c_{\text{equ}}^{n} + (N_1 + 1 - \theta) c_t^n \right) + b_p,
\]

which is decoupled from (21) and implicitly coupled to the temperature equation (23) by means of the parameters $C_p^c$, $K$, $C$ and $X_m$ involved in the definition of $c_{f,\text{equ}}^{n+1}$. The constants $N_1$ to $N_3$ in (23) are given by

\[
N_1 = \frac{1 - (1 - \theta) \Delta t N_t}{1 + \theta \Delta t N_t}, \quad N_2 = \frac{(1 - \theta) \Delta t N_t}{1 + \theta \Delta t N_t}, \quad N_3 = \frac{\theta \Delta t N_t}{1 + \theta \Delta t N_t}.
\]

For each time step $n+1$, equations (21) to (23) are solved iteratively as follows:

1. Take $T = T^n$ as initial guess for the temperature vector at time $n+1$.
2. Use $T$ to calculate the temperature-dependent parameter vectors $c_p^{\text{eq}}$, $k$, $c$, and $x_m$. These vectors contain the values of $C_p^c$ and the GAB-parameters $K$, $C$, and $X_m$, respectively, at all grid points. $c_p^{\text{eq}}$ is obtained by evaluating (7) at each element of the vector $T$. For a given element of $T$, the isotherm parameters are estimated by linearly interpolating the curves from the temperatures available in Table 1, followed by fitting relation (4) to the interpolated curve with the Least Squares method.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_x$</td>
<td>$1.1 \times 10^{-4}$ m</td>
</tr>
<tr>
<td>$L_y$</td>
<td>0.25 m</td>
</tr>
<tr>
<td>$\rho$</td>
<td>818 kg m$^{-3}$</td>
</tr>
<tr>
<td>$\rho_t$</td>
<td>1500 kg m$^{-3}$ [1]</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.47</td>
</tr>
<tr>
<td>$D_{\text{eff}}$</td>
<td>$3 \times 10^{-6}$ m$^2$ s$^{-1}$</td>
</tr>
<tr>
<td>$c$</td>
<td>1200 J kg$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>$\Lambda_p$</td>
<td>0.06 W m$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>$\Lambda_t$</td>
<td>0.51 W m$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>$K_{\text{mi}}$</td>
<td>0.0035 s$^{-1}$ [5]</td>
</tr>
<tr>
<td>$h_c$</td>
<td>1180 W m$^{-2}$ K$^{-1}$</td>
</tr>
<tr>
<td>$\rho_a$</td>
<td>1.205 kg m$^{-3}$</td>
</tr>
<tr>
<td>$\mu_a$</td>
<td>$1.82 \times 10^{-5}$ kg m$^{-1}$ s$^{-1}$</td>
</tr>
<tr>
<td>$c_a$</td>
<td>1005 J kg$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>$\Lambda_a$</td>
<td>0.0257 W m$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>$D_a$</td>
<td>$2.53 \times 10^{-5}$ m$^2$ s$^{-1}$</td>
</tr>
</tbody>
</table>

$^{a}$Calculated from $\rho$ and $\rho_t$.

$^{b}$Estimated from the data reported in [23] for uncoated samples at a pressure of 1 atm.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{\text{ps}}$</td>
<td>32 $^\circ$C</td>
</tr>
<tr>
<td>$L_{\text{ps}}$</td>
<td>0.15 m</td>
</tr>
<tr>
<td>$L_{ds}$</td>
<td>0.2 m</td>
</tr>
<tr>
<td>$y_{\text{step}}$</td>
<td>0.025 m</td>
</tr>
<tr>
<td>$t_{\text{step}}$</td>
<td>2.5 s</td>
</tr>
<tr>
<td>$T_{\infty}$</td>
<td>20 $^\circ$C</td>
</tr>
<tr>
<td>$C_p^\infty$</td>
<td>$0.333 \times 10^{-1}$ kg m$^{-3}$ (RH = 65%)</td>
</tr>
<tr>
<td>$V_{\infty}$</td>
<td>3 m s$^{-1}$</td>
</tr>
<tr>
<td>Re</td>
<td>$6.9 \times 10^{4}$</td>
</tr>
<tr>
<td>Pr</td>
<td>0.713</td>
</tr>
<tr>
<td>Sc</td>
<td>0.597</td>
</tr>
<tr>
<td>$K_h^+$</td>
<td>11.48 W m$^{-2}$ K$^{-1}$</td>
</tr>
<tr>
<td>$K_m^+$</td>
<td>0.011 m s$^{-1}$</td>
</tr>
</tbody>
</table>

Table 2: List of parameters used in the simulations.

$^{c}$Meant to represent the printing speed of a 0.84m-wide poster.
3. Solve the system in (23) using the parameter vectors calculated in step 2 to locally define \( c_{p,n+1} \).

This nonlinear system for \( c_{p,n+1} \) is solved using Newton’s iteration method [24]. The solution constitutes a predictor of \( c_{p,n+1} \), denoted by \( c_{p,1} \).

4. Find \( c_{t,n+1} \) from equation (21) using \( c_{p,1} \) and the parameter vectors obtained in step 2 to evaluate the vector \( c_{t,n+1} \). The solution constitutes a predictor of \( c_{t,n+1} \), denoted by \( c_{t,1} \).

5. Solve the system in (23) using \( c_{t,1} \) to evaluate the vector \( \Delta h_{ads,n+1} \). \( \Delta h_{ads} \) is found by linear interpolation of the curve presented in Figure 2. The solution obtained in this step constitutes a predictor of \( T_{n+1} \), which is denoted by \( T_{1} \).

6. Take \( T = T_{1} \) and repeat steps 2 to 5. The solutions obtained in this step are correctors of \( c_{p,1} \), \( c_{t,1} \) and \( T_{1} \), denoted by \( c_{p,2} \), \( c_{t,2} \) and \( T_{2} \), respectively.

7. Check if the following conditions are satisfied: \( \| c_{p}^{(k+1)} - c_{p}^{(k)} \|_2 < \text{tol}, \| c_{t}^{(k+1)} - c_{t}^{(k)} \|_2 < \text{tol} \) and \( \| T^{(k+1)} - T^{(k)} \|_2 < \text{tol} \), with \( k \) an iteration counter and \( \text{tol} \) a prescribed tolerance. If at least one of the conditions is not satisfied, take \( T = T^{(k+1)} \), and repeat steps 2 to 5. As soon as the three conditions are satisfied, the iteration stops and the solution of the system is given by \( c_{p,n+1} = c_{p}^{(k+1)}, c_{t,n+1} = c_{t}^{(k+1)}, T_{n+1} = T^{(k+1)} \).

6. Results and discussion

The effect of temperature changes is included in the model for moisture transport in paper by means of the saturated vapor concentration \( C_{p}^s \) and the GAB-parameters \( K, C, \) and \( X_m \). For a fixed \( C_p \)-value, \( C_{p}^s \) induces desorption by decreasing the relative humidity for increasing temperatures, thus displacing the equilibrium concentration \( C_{t,eq} \) to lower values. Simultaneously, the GAB-parameters affect the moisture equilibrium by displacing the isotherms to lower values of \( C_{t,eq} \) for higher temperatures. In order to identify the dominant driving force for temperature-induced moisture changes, the model derived in Section 2 is first used to study a simple set up in which a paper sheet is heated homogeneously. The sheet is initially in thermodynamic equilibrium with an environment of 20\(^\circ\)C and 65% relative humidity. After 1 minute it is placed on a water impermeable surface maintained at 32\(^\circ\)C. Figure 3 shows the results for the evolution of the integrated moisture content of the sheet when the variation with temperature of \( C_{p}^s \) and the GAB-parameters is included. This figure also shows the results when either \( C_{p}^s \) or the GAB-parameters are considered constant with temperature. From these results we can conclude that the variation of \( C_{p}^s \) constitutes the main driving force for the desorption process. Nevertheless, the variation with temperature of the GAB-parameters also has a significant effect and should be taken into account in order to model temperature-induced moisture changes in paper.

The results from the simulation of a paper sheet in the printing environment are shown in Figure 4 and 5. The numerical solution is obtained with a spatial resolution \( \Delta z = 5 \times 10^{-6} \) m in the z-direction and \( \Delta y = 2.5 \times 10^{-3} \) m in the y-direction. The resolution for time integration was \( \Delta t = 0.2 \) s. Instead of presenting the results in terms of the variables \( C_{p} \) and \( C_{t} \), we use the scaled variables relative humidity in the pores (\( \text{RH}_{p} = C_{p}/C_{p}^s(T) \)) and moisture content (\( X = C_{t}/\rho_{t} \)), respectively, since the latter allow for easier interpretation and comparison with results from other authors. Figure 4 shows the temperature and moisture profiles along the length of the sheet for \( z = L_z/2 \) and \( t = 23 \) s. All variables show step-like profiles resulting from the step-wise movement of the print surface. Due to the high value of the thermal contact conductance between paper and the metallic surfaces, the temperature of the sheet follows the temperature of the print and drying surfaces, which creates temperature differences of up to 12\(^\circ\)C along the length of the sheet. Although these temperature differences generate changes of about 25% in RH, changes of only 0.2% in moisture content are observed. This is a result of the short time
interval for which the paper stays at the print surface. Each y-coordinate of the sheet is in contact for at most 15 s with the hot print surface, a very small time compared to the characteristic adsorption time, i.e., $1/K_{mi} = 286$ s.

Figure 5 shows the moisture and temperature profiles along the thickness of the sheet for $y = Ly/2$ and $t = 17.6$ s. This time corresponds to 1 second after the print surface has arrived to the middle of the sheet ($Ly/2$), when the moisture gradients at that point are the largest. Although the figure shows temperature and relative humidity gradients along the thickness, these gradients only appear shortly after the print surface reaches $y = Ly/2$, and they are smoothed out after 2 seconds. Since the characteristic time for the desorption process is much larger than 2 seconds, no gradients in the moisture content are observed along the thickness of the sheet. Small moisture and temperature gradients along the thickness direction were expected since the Biot numbers for heat ($Bi_h^+ = 0.02$) and mass transfer ($Bi_m^+ = 0.4$) are lower than 1.

The moisture profiles in Figure 4 and 5 suggest that the contact time with the hot print surface is too short to generate significant gradients that could cause dimensional instabilities in the printed sheet. According to experimental research performed by Océ Technologies (data not published), a paper sheet placed over two surfaces with different temperatures only develops dimensional instabilities when the induced difference in moisture content between those surfaces is of the order of 3% or higher. Also, when modeling cockling phenomena in paper, Lipponen et al., [2] found that local out-of-plane deformation only takes place when the difference in moisture content between the top and bottom of the sheet exceeds a value of about 2%. For the physical parameters listed in Table 2, differences of the order of 3% in moisture content along the length of the sheet only appear when each y-coordinate is in contact with the print surface for about 3 minutes (data not shown). This corresponds to a printing speed of the order of 1 mm s$^{-1}$, ten times slower than the real printing speed. Moreover, when the sheet is heated for longer times, moisture content differences of 2% along the thickness of the sheet only appear when the Biot number for mass transfer is significantly higher, i.e., the sheet is then much thicker and/or $K_m^+$ is higher.
Figure 4: Moisture and temperature profiles along the length of a paper sheet in the printing environment for \( z = L_z/2 \) and \( t = 23 \) s.

It is important to take into account that the predicted moisture gradients might be affected by inaccuracies in the estimation of the model parameters. The model is particularly sensitive to the value of the pores to fibers mass transfer coefficient \( K_{mi} \), indicating that the evolution of the moisture content is greatly determined by the speed of the sorption process. This coefficient was not measured for this application, but obtained from the value reported in [5] for bleached kraft board. Several processes might lead to variations in \( K_{mi} \) among different paper types. Differences in the manufacturing of the sheet, for example during bleaching or addition of sizings and additives, can change the affinity between fibers and water vapor, affecting the kinetics of the sorption process. Moreover, since \( K_{mi} \) also accounts for the surface area available for contact between pores and fibers, variations in the internal surface area of the sheet will also be reflected on the \( K_{mi} \)-value. Finally, the \( K_{mi} \)-value reported in [5] was measured under isothermal conditions, and no variations with temperature of this coefficient are included in our model.

In order to determine the exact magnitude of the moisture differences induced by the printing environment, the experimental determination of \( K_{mi} \) for the specific paper used in this application as well as the dependency of \( K_{mi} \) on the temperature are needed. However, further conclusions can be drawn
by studying which parameter values lead to significant moisture content differences along the length and thickness of the sheet. Figure 6 shows the moisture profiles obtained with different values of the pores to fibers mass transfer coefficient $K_{mi}$ and the convective heat and mass transfer coefficients at the top of the sheet, $K_{m}^{+}$ and $K_{h}^{+}$. The figure shows the results when $K_{mi}$ is 10 and 100 times higher than the value reported in Table 2. Variations in $K_{m}^{+}$ and $K_{h}^{+}$ are introduced indirectly by changing the bulk velocity of air $V_{\infty}$. Whenever the Reynolds number exceeds $2.5 \times 10^5$, $K_{m}^{+}$ and $K_{h}^{+}$ are calculated from the correlation reported in [18] for the average heat transfer coefficient for turbulent boundary layers over a flat plate.

Figure 6 shows that in order to achieve moisture content differences of the order of 3%, $K_{mi}$ and $K_{m}^{+}$ need to be significantly higher than the values reported in Table 2. Such differences are only achieved when $K_{mi}$ is 100 times higher and for air velocities above 15 m s$^{-1}$. These results support the idea that mechanical stabilities reported by Océ Technologies in the target printer are induced by other factors besides the movement over the hot print surface. We have found that air velocities above 15 m s$^{-1}$ are necessary to obtain large moisture content gradients, but this velocity is significantly higher than the one present in the target printer. These results also hold when the simulations are performed at higher relative humidities.

7. Summary and Conclusions

The goal of this paper was to model heat and moisture transport at the final use of paper, when paper is heated by, for example, a printer. Therefore, an elaborated two-phase non-isothermal model was derived. This model enabled us to identify the relevant driving forces behind the temperature-induced moisture changes in paper. It has been shown that the variation with temperature of the saturated vapor concentration constitutes the main driving force for moisture desorption. Although to a lesser extent, variation with temperature of the GAB-parameters also affects the evolution of the moisture content and should be included in the model.

The model was applied to describe the heat and moisture transport in a paper sheet moving over a hot print surface. The corresponding numerical simulations predicted changes in moisture content of only 0.2%. This suggests that the contact time with the hot print surface is too short to create moisture gradients able to induce dimensional instabilities on the printed sheet. However, the experimental determination of the pores to fibers mass transfer coefficient $K_{mi}$ as well as its variation with temperature need to be investigated in order to obtain more reliable predictions. Variation in other
model parameters like the printing speed and the bulk velocity of the air inside the printer might lead to larger moisture gradients. The model can therefore be helpful in the design phase of new printers. Furthermore, it can be used to calculate moisture profiles in any application that considers the final use of printing paper.

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