Waterhammer with column separation, fluid-structure interaction and unsteady friction in a viscoelastic pipe

by

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Abstract

There are four popular items which may affect classical waterhammer results: unsteady friction (UF), column separation (CS), fluid-structure interaction (FSI) and viscoelasticity (VE), each of which has been investigated separately in numerous researches. In this study, waterhammer in a reservoir-pipe-valve system is simulated including – for the first time – all four items CS, FSI, UF and VE. The relative importance of each of the four items is assessed. Conclusions are drawn by looking at the numerical simulations corresponding to all possible combinations of the mentioned items.

Keywords: waterhammer; fluid transient; pressure surge; fluid-structure interaction; pipe vibration; column separation; plastic pipe; viscoelasticity; unsteady friction

1 INTRODUCTION

The four items UF, CS, FSI and VE are separately well-known and there is a rich literature for each one. Considering this, the scope of the current research is in developing a thorough mathematical model for waterhammer which is capable of simulating any combination of these items.

With the inclusion of one or more of these items in the analysis, fifteen possibilities are offered from which some combinations already have been studied and some have not. The combinations of VE and UF (1-5), CS and UF (6-8), FSI and UF (9), FSI and CS (10-12), VE and CS (13-15) and most recently FSI and VE (16-17) have already been investigated. Combinations of three including CS, UF and FSI (18) and CS, UF and VE (19-20) have also been modelled. For two selected case studies, this paper provides results for all fifteen combinations. The results are examined and provisional conclusions are drawn.

The model describing FSI herein consists of the extended waterhammer equations combined with the axial pipe vibration equations. UF and VE introduce additional terms in the form of convolution integrals in both the hydraulic and structural equations. The integrals are approximated using calculated heads (for VE) and velocities (for UF) at the present and previous time steps. The hydraulic and structural equations are solved simultaneously using the established method of characteristics (MOC). CS is taken into account by the discrete vapour cavity model (DVCM).
2 MATHEMATICAL MODEL

2.1 Viscoelasticity (VE)

The surrounding pressure, weight of material and steady state flow introduce a static loading on the pipe material. Hydraulic transients account for a dynamic loading on the pipe system. Assuming linear viscoelasticity, the static and dynamic strain can be superposed. As a result, the static and dynamic deformations are separated and the mathematical formulation of viscoelasticity to account for the waterhammer responses is described in terms of the dynamic quantities. The dynamic description satisfies the zero-value condition for \( t < 0 \), which is necessary to conveniently use convolution integrals.

In this article, dynamic variables which satisfy the aforementioned zero-value condition before the transient flow inception are defined. These quantities correspond to the dynamic behaviour and are indicated by tilde, such as \( \tilde{\sigma} := \sigma - \sigma_0 \) and \( \tilde{H} := H - H_0 \), in which the zero subscript stands for values at \( t < 0 \) (static or steady state).

The viscoelastic behaviour of the pipe wall introduces retardation in the transient pressure responses. The material viscosity causes the hoop and axial strains to lag behind the applied stresses. Various mechanical models to simulate material viscoelasticity exist, among which the generalized Kelvin-Voigt model is employed herein. It consists of \( N_{CV} \) pairs of spring and dashpot (in parallel) connected to a leading spring. Each pair is called a Kelvin-Voigt element and it is linked with the next one in series. For the generalized Kelvin-Voigt model, the following creep compliance function is obtained:

\[
J(t) := J_0 + \sum_{k=1}^{N_{CV}} J_k \left( 1 - e^{-t/\tau_k} \right),
\]

where \( J(0) := J_0 := 1/E_0 \) accounts for the immediate response of the material; \( J_k \) defined by \( J_k := 1/E_k \) is the creep compliance of the \( k \)-th spring which has modulus of elasticity \( E_k \), and \( \tau_k := \mu_k/E_k \) is the retardation time of the \( k \)-th dashpot in which \( \mu_k \) is its viscosity.

The function \( J(t) \) is used to describe the stress-strain relation which for uniaxial viscoelastic media is

\[
\tilde{\varepsilon}(t) = \tilde{\sigma}(t)J(0) + \int_0^t \tilde{\sigma}(t-s) \frac{dJ}{ds} ds = \tilde{\sigma}(t)J(0) + \left( \tilde{\sigma} * \frac{dJ}{dt} \right)(t) = (\tilde{\sigma} * dJ)(t),
\]

where \( \ast \) is the Stieltjes convolution operator. The first term represents the immediate response and the second term describes the retarded response. Assuming a constant Poisson ratio, relation (2) for the three-directional case of strain in the \( z \)-direction of a cylindrical coordinate system extends to

\[
\tilde{\varepsilon}_z = \tilde{\sigma}_z * dJ - \nu (\tilde{\sigma}_t * dJ - \tilde{\sigma} * dJ).
\]

The retardation in the axial strain is calculated by the convolution integral \( I_{z,\tilde{\varepsilon}} \)

\[
I_{z,\tilde{\varepsilon}} := \int_b^a \tilde{\sigma}_z (t-s) \frac{dJ}{ds}(s) ds = \sum_{k=1}^{N_{CV}} \left[ \frac{J_k}{\tau_k} \tilde{\sigma}_z (t-s) e^{-t/\tau_k} \right] ds = \sum_{k=1}^{N_{CV}} I_{z,\tilde{\varepsilon}_k},
\]

and the retarded hoop strain (up to a constant factor) is defined by
The convolution integrals (4-5) are employed to include the viscoelastic behaviour of the pipe wall in the hydraulic and structural governing equations.

2.2 Unsteady friction (UF)

The unsteady part of the friction term in the momentum equation comprises the convolution of a weighting function \( W \) with past temporal fluid accelerations \( \partial (V - \tilde{u}_i) / \partial t \) relative to the pipe wall:

\[
h_{m}(t) = \frac{16 \nu}{gD^2} \int W(t-s) \frac{\partial (V - \tilde{u}_i)}{\partial s}ds := \frac{16 \nu}{gD^2} \tilde{Y}(t),
\]

where \( \tilde{u}_i \) is the axial velocity of the pipe wall, \( \nu \) is the kinematic fluid viscosity, and \( D \) is the pipe inner diameter. In this equation, the use of \( V - \tilde{u}_i \) as the velocity models FSI friction coupling. Zielke (21) provided analytical solutions for the weighting function for a laminar flow in a static pipe. The provided weighting function was split into two parts because of slow convergence of the fitted exponential series for dimensionless times \( \tau \) less than 0.02 (22). Hence, the approximate weighting function was defined as (22)

\[
W(\tau) = \begin{cases} 
\sum_{j=1}^{6} \frac{m_j}{\pi} e^{\tau^{j-1}} & \tau \leq 0.02 \\
\sum_{j=1}^{4} e^{-m_j \tau} & \tau > 0.02, \quad \tau \approx \frac{4\nu t}{D^2}
\end{cases}
\]

where \( m_j \) and \( n_j \) are fitted constants to analytical expressions (23). Vardy and Brown (24) used the frozen viscosity assumption to derive a weighting function for smooth-pipe turbulent flow. The assumption is a constant kinematic viscosity profile across the pipe cross-section that remains unchanged during the transient event (22). Their approximate weighting function is

\[
W(\tau) = \frac{A e^{-A \tau}}{\sqrt{\tau}},
\]

where \( A' \) and \( B' \) depend on the Reynolds number \( \text{Re}_0 = V_0 D / \nu \) of the pre-transient flow, the dimensionless time \( \tau \) and the relative roughness of the pipe. Vardy and Brown (24) derived the coefficients \( A' \) and \( B' \) for smooth-pipe turbulent flow as

\[
A' = \frac{1}{2 \sqrt{\pi \nu}} \quad \text{and} \quad B' = \frac{\text{Re}_0^{\kappa}}{12.86} \quad \text{with} \quad \kappa = \log_{10}(15.29 \text{Re}_0^{0.058}).
\]

These coefficients are accurate in the range \( 2000 < \text{Re}_0 < 10^8 \) (23). For fully rough turbulent flow another set of coefficients was also provided by Vardy and Brown (25).

The frozen viscosity assumption is satisfactory for flow situations with a large Ghidaoui et al. (26) ratio \( P \) of the diffusion time scale to the wave time scale expressed by the following relation \((P \gg 1, \text{Ghidaoui et al. (26)})\)

\[
P = \frac{2D}{L} \left( \frac{V_0}{c_f} \right) \left( \frac{L}{c_f} \right)
\]

where \( L \) is the pipe length, \( f \) is the Darcy-Weisbach friction coefficient and \( c_f \) is the pressure wave speed.
The weighting function (8) is less convenient to calculate the convolution integral $Y(t)$ in equation (6). A computationally efficient form of the weighting function approximates it by a linear summation of exponential terms as follows:

$$W(\tau) \approx \sum_{m=1}^{N_{UF}} m_i e^{-\frac{\mu^m}{\tau}} \approx \sum_{m=1}^{N_{UF}} m_i e^{-\frac{\mu^m}{\tau}}$$

(11)

in which $m_i$, $n_i$ and $N_{UF}$ are evaluated according to $A^*$, $B^*$ and $\tau$ (22). The influence of axial and radial pipe motion on the analytical expressions (from which the weighting functions are derived) is ignored herein.

2.3 Fluid-structure interaction (FSI)

In a liquid-filled pipe system, any sudden excitation of the pipe wall or the contained fluid, results in a transient phenomenon. This means that stress waves in the pipe wall and pressure waves in the fluid will propagate along the pipeline. These waves may interact with each other through the three mechanisms: Poisson, friction and junction coupling (27), (28). Friction and Poisson coupling occur along the pipeline. Junction coupling only takes place at discrete locations like free bends as a result of unbalanced pressure forces (29). Friction coupling is due to fluid shear stress on the inner side of the axially moving pipe wall (28). The reason for Poisson coupling is the Poisson contraction coefficient which makes the hoop stress cause pipe strain in the axial direction and the axial stress cause hoop strain.

2.4 Column separation (CS)

In a transient flow, when the liquid pressure drops to vapour pressure, vapour bubbles will form. In a cavitation bubble zone, the pressure can be assumed equal to vapour pressure for a certain period of time (order of seconds for water at room temperature) before the release of dissolved gas increases the pressure. The challenge in one-dimensional simulations of this phenomenon is to locate the vapour cavities. A simple scheme which has been successfully checked against experimental results for column separation in the literature is the DVCM (30), (31). Introduced in MOC context, the cavities do not move and they occur only at computational sections. Their volumes are governed by the fluid velocities at the upstream ($V_u$) and downstream ($V_d$) ends of the cavity, thereby satisfying continuity (assuming the same axial pipe wall velocity at both sides of the cavity) according to

$$\frac{\partial V}{\partial t} = A(V_u - V_d).$$

(12)

2.5 Governing equations of pipe-flow transients with VE, UF, FSI and CS

The equations governing waterhammer with FSI and viscoelasticity have recently been provided in (16). They are now extended to include friction (quasi-steady and unsteady). Column separation is modelled using DVCM, Eq. (12), for $H = H_v = \text{vapour head}$.

The momentum equation of flow in the pipe including the friction term is

$$\frac{\partial V}{\partial t} + \frac{\partial H}{\partial z} = -gh_f \quad \text{with} \quad h_f = \frac{f}{2gD} \left| V' - \tilde{u} \right| V' - \tilde{u} + \frac{16\nu}{gD^2} V'$$

(13)

in which $Y(t)$ is defined in equation (6). This equation requires to be written in terms of dynamic head $\dot{H}$. Equation (13) for steady state flow in a static pipe is
\[ \frac{\partial H}{\partial z} = -h, \quad \text{with } h = \frac{f V^2}{2gD}, \]  
where subscript 0 indicates the initial steady state. The definition of the dynamic head tells

\[ \frac{\partial H}{\partial z} + \frac{\partial H_0}{\partial z} = \frac{\partial H_0}{\partial z} - h, \]  

If Eq. (15) is substituted in Eq. (13), the desired equation for modelling VE and UF is obtained:

\[ \frac{\partial V}{\partial t} + \frac{g}{c_f} \frac{\partial H}{\partial z} + 2 \frac{\partial \bar{u}_z}{\partial z} = \rho_f \frac{\partial}{\partial \bar{c}_f} \left( V^2 - 1 \right) \frac{D \partial \bar{I}_{\bar{H}}}{\partial \bar{c}_f}, \]  

where \( V_0 \) is the steady state velocity.

The continuity equation is, Ref. (16):

\[ \frac{\partial V}{\partial z} + \frac{g}{c_f} \frac{\partial H}{\partial z} - 2 \frac{\partial \bar{u}_z}{\partial z} = \rho_f \frac{\partial}{\partial \bar{c}_f} \left( V^2 - 1 \right) \frac{D \partial \bar{I}_{\bar{H}}}{\partial \bar{c}_f}, \]  

with

\[ c_f := \left( \rho_f \left( \frac{1}{K} + (1 - \nu^2) \frac{D}{eE_f} \right) \right)^{1/2}, \]

where \( c_f \) is a pressure wave speed, \( \nu \) is Poisson’s ratio, \( K \) is the bulk modulus and \( \rho \) is the density of the fluid. Note that \( \partial H/\partial t = \partial H/\partial t \) and the axial velocity of the pipe wall in steady state \( \bar{u}_z = \bar{u}_z \) other way around. The convolution \( I_\bar{H} \) is defined by (5). The unknown \( \bar{u}_z \) is governed by two equations for the axial vibration of the pipe wall. The first one is obtained by taking the time derivative of equation (3) for a thin-walled pipe:

\[ \frac{\partial \bar{u}_z}{\partial z} = \frac{\partial}{\partial \bar{c}_f} \left( \bar{\sigma}_z * dI \right) - \nu \frac{\partial}{\partial \bar{c}_f} \left( \bar{\sigma}_z * dJ \right), \]

If the Stieltjes convolution operators are written in full, then:

\[ \frac{\partial \bar{u}_z}{\partial z} = \frac{1}{\rho_c} \frac{\partial \bar{\sigma}_z}{\partial \bar{c}_f} + \rho_f g \frac{V D}{2eE_f} \frac{\partial H}{\partial \bar{c}_f} \frac{\partial \bar{I}_{\bar{H}}}{\partial \bar{c}_f} = \frac{\partial \bar{u}_z}{\partial \bar{c}_f} - \rho_f \frac{\partial \bar{I}_{\bar{H}}}{\partial \bar{c}_f}, \]

where \( \rho_0 \) is the wave speed in unrestrained solid bars, \( \rho_c \) is the density of the pipe material, and \( I_\bar{H} \) and \( I_{\bar{H}} \) are given by relations (4) and (5), respectively. The second pipe equation is due to conservation of momentum (32)

\[ \frac{\partial \bar{u}_z}{\partial \bar{c}_f} = \frac{1}{\rho_c} \bar{\sigma}_z = \frac{1}{\rho_c} \bar{\tau}, \]

in which \( \bar{\tau} \) is the shear stress between pipe wall and fluid. Subtracting

\[ \frac{\partial \bar{\sigma}_z}{\partial \bar{c}_f} = \frac{1}{\rho_f} \bar{\tau}, \]

from equation (22) gives
\[ \frac{\partial \hat{u}}{\partial t} - \frac{1}{\rho_1} \frac{\partial \hat{\sigma}_z}{\partial z} = \frac{1}{\rho_1} \hat{\tau}, \quad (24) \]

noting that \( \hat{\sigma}_z = \sigma_z - \sigma_m \) and \( \hat{\tau} = \tau - \tau_v \). If the shear stress \( \hat{\tau} \) is written in terms of fluid velocity, equation (24) becomes
\[ \frac{\partial \hat{u}}{\partial t} - \frac{1}{\rho_1} \frac{\partial \hat{\sigma}_z}{\partial z} = \left( \frac{\rho_1}{2} \dot{f} \frac{1}{8} \rho_1 \dot{e} (V - \dot{u}_s) \right) \left[ V - \dot{u}_s \right] = \frac{\rho_1}{\rho_1} \dot{f} \frac{1}{8} \rho_1 \dot{e} \left( V + \frac{4 \dot{f}}{\rho_1} \dot{e} Y' \right). \quad (25) \]

The derived mathematical model takes all three FSI coupling mechanisms into account.

### 2.6 Initial and boundary conditions

The hydraulic initial conditions are determined from the steady state energy equation considering the Darcy-Weisbach or Hazen-Williams friction losses in combination with the continuity equation. The structural initial conditions correspond to the static state of the pipeline, but the dynamic variables are zero by definition.

The boundary conditions for a reservoir-pipe-valve system are a zero dynamic head and zero dynamic structural velocity at the reservoir, knowing that vapour cavities will not form at the reservoir. For the downstream boundary in the analysis with junction coupling, assuming a fully free-to-move valve of zero mass that closes instantaneously, the applied relations
\[ V = \dot{u}_s \quad (26) \]
\[ \hat{\sigma}, A_r = \rho_1 g A_r \hat{H} \quad (27) \]
where \( A_r \) and \( A_i \) are the cross-sectional areas of flow and pipe wall, respectively. When column separation occurs at the closed valve, the pressure head is constant at \( H_v \) (vapour head) and equation (12) with the downstream fluid velocity \( (V_d) \) replaced by the valve velocity \( (\dot{u}_v) \), is used instead of (26).

### 3 NUMERICAL SOLUTION

#### 3.1 Treatment of convolution integrals

There are two types of convolution integrals to be evaluated numerically. They are used in the governing equations to model unsteady friction (the term \( Y(t) \) in Eq. (6)) and viscoelasticity (the expressions \( I_{n_x} \) and \( I_{n_y} \) given by definitions (4) and (5)). Consider the weighting function given by Eq. (11). The UF convolution is then
\[ Y(t) = \int_{\frac{n_x}{2}}^{\frac{n_x}{2}} \left( \sum_{i=1}^{n_x} m_i \int_{\frac{n_y}{2}}^{\frac{n_y}{2}} \frac{\partial (V - \dot{u}_m)}{\partial s} \right) \left( V - \dot{u}_m \right) ds \]
\[ = \sum_{i=1}^{n_x} m_i \int_{\frac{n_y}{2}}^{\frac{n_y}{2}} \frac{\partial (V - \dot{u}_m)}{\partial s} \left( V - \dot{u}_m \right) ds = \sum_{i=1}^{n_x} Y_i(t). \quad (28) \]
The use of exponential weighting functions permits a recursive formula to evaluate the convolution integrals. To this aim, a numerical time step $\Delta t$ and $Y(t - \Delta t)$ are introduced such that

$$Y(t - \Delta t) = \int_0^{\infty} m(e^{\sigma t} - e^{(t-\Delta t)\sigma}) \hat{e}(V - \dot{u}_s) s \, ds$$

$$= e^{\sigma \Delta t} m(e^{\sigma t} - e^{(t-\Delta t)\sigma}) \int_0^{\infty} e^{\sigma s} \frac{\hat{e}(V - \dot{u}_s)}{\hat{e}s} s \, ds.$$ (29)

Accordingly, $Y(t)$ can be approximated as follows

$$Y(t) = \sum_{i=1}^{N} m_i(e^{\sigma t} - e^{(t-\Delta t)\sigma}) \int_0^{\infty} e^{\sigma s} \frac{\hat{e}(V - \dot{u}_s)}{\hat{e}s} s \, ds$$

$$= \sum_{i=1}^{N} m_i e^{\sigma \Delta t} Y_i(t - \Delta t) + \sum_{i=1}^{N} m_i(e^{\sigma t} - e^{(t-\Delta t)\sigma}) \int_0^{\infty} e^{\sigma s} \frac{\hat{e}(V - \dot{u}_s)}{\hat{e}s} s \, ds$$

$$= \left(\sum_{i=1}^{N} m_i \left[V(t) - \dot{u}_s(t)\right]\right) + \sum_{i=1}^{N} e^{\sigma \Delta t} Y_i(t - \Delta t) - m_i \left[V(t - \Delta t) - \dot{u}_s(t - \Delta t)\right]$$

$$= a_i^* \left[V(t) - \dot{u}_s(t)\right] + a_i^*.$$ (30)

The VE convolution integrals denoted by $I_{\sigma}$ and $I_H$ have the following general form ($h$ is either $\sigma$ or $H$):

$$I_h(t) = \int_{-\infty}^{\infty} I_{h}(t) = \frac{J}{\tau_s} \int_{0}^{\infty} h(t - s) e^{-\frac{s}{\tau_s}} \, ds.$$ (31)

$I_h(t)$ can be approximated as follows, Ref. (16):

$$I_h(t) = J \int_{0}^{\infty} h(t - s) e^{-\frac{s}{\tau_s}} \, ds$$ (32)

$$= h(t) \left[J_s \frac{J}{\tau_s} \left(1 - e^{-\frac{t}{\tau_s}}\right)\right] + h(t - \Delta t) \left[J_s \frac{J}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_s}}\right) - J_s \frac{J}{\tau_s} \right] + e^{-\frac{t}{\tau_s}} I_h(t - \Delta t).$$

If $-\Delta t / \tau_s$ is small (say $\Delta t < 0.1 \min (\tau_s)$, $k = 1..N_{kv}$), then the exponential term $e^{-\frac{t}{\tau_s}}$ can be approximated by $1 - \Delta t / \tau_s$, simplifying Eq. (32) to

$$I_h(t) = h(t - \Delta t) \left[J_s \frac{J}{\tau_s} \right] + \left(1 - \frac{\Delta t}{\tau_s}\right) I_h(t - \Delta t).$$ (33)

In this equation, the amplification factor is smaller than 1, which means that the recursion is numerically stable. The time derivative of $I_h(t)$ is evaluated after some algebraic manipulation and making use of Eq. (33),

$$\frac{dI_h(t)}{dt} = \frac{J}{\tau_s} \left[\frac{J}{\tau_s} + \frac{J}{\Delta t}\right] h(t) \left[1 - \frac{\Delta t}{\tau_s}\right] I_h(t - \Delta t).$$ (34)

Now the integrals given by Eqs. (4) or (5) are approximated using the following formula
\[
\frac{dl}{dt}(t) = \left( \sum_{i=1}^{N_c} \frac{J_i}{\tau_i} \right) \hat{h}(t) - \sum_{i=1}^{N_c} \left[ \frac{J_i}{\tau_i} (t - \Delta t) \right] \hat{h}(t - \Delta t) + \frac{1}{\tau_i} l_i (t - \Delta t) = a_i h(t) + a_e. \tag{35}
\]

### 3.2 MOC solution

The coupled equations (16), (17), (25) and (20) are written in matrix form as

\[
A \frac{\partial y}{\partial t} + B \frac{\partial y}{\partial z} = r, \tag{36}
\]

in which \(y\) and \(r\) are the vector of unknowns and the right-hand-side, respectively, and \(A\) and \(B\) are matrices of constant coefficients:

\[
y = \begin{bmatrix} \hat{V} \\ \hat{H} \\ \hat{u}_t \\ \hat{v}_t \\ \hat{u}_s \\ \hat{v}_s \\ \hat{u}_c \\ \hat{v}_c \end{bmatrix}, \quad r = \begin{bmatrix} \frac{f}{2D} (\hat{V} - \hat{u}_t) \quad (\hat{V} - \hat{u}_t) \quad \frac{V^2}{2D} + 16\nu \hat{V} \\ -\rho_g (1 - \nu^2) \frac{\partial}{\partial t} - \rho_g (1 - \nu^2) \frac{V^2}{D^2} \\ \frac{1}{\rho_e} \left( \frac{\rho_g f}{8} (\hat{V} - \hat{u}_t) \right) - \rho_g V^2 \left( 1 + \frac{4\rho_g}{8} \right) \end{bmatrix},
\]

\[
A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{g}{c_f} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \rho_c g \frac{\mu D}{2e E_0} & 0 & -1 & \rho_c c_f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & g & 0 & 0 \\ 1 & 0 & -2\nu & 0 \\ 0 & 0 & 0 & -1 \rho_c \\ 0 & 0 & 1 & 0 \end{bmatrix}.
\]

MOC transformation of these equations described in Refs. (16), (32) and (33) and then integration along the relevant characteristic lines connecting \(A_1\) and \(A_2\) with \(P\) corresponding to the eigenvalues (wave speeds) \(\hat{c}_f\) and \(\hat{c}_f\), and \(A_3\) and \(A_4\) with \(P\) corresponding to the eigenvalues \(\hat{c}_e\) and \(\hat{c}_e\) (see Fig. 1), yields the following compatibility equations (Ref. (16)):

\[
(TA)_{A_1} (V_n - V_{A_1}) + (TA)_{A_2} (\hat{H}_r - \hat{H}_{A_2}) + (TA)_{A_3} (\hat{u}_r - \hat{u}_{A_3}) + (TA)_{A_4} (\sigma_r - \sigma_{A_4}) = \Delta t \left[ T_{A_1} f_{A_1} + T_{A_2} f_{A_2} + T_{A_3} f_{A_3} + T_{A_4} f_{A_4} \right],
\tag{38}
\]

\[
(TA)_{A_2} (V_n - V_{A_2}) - (TA)_{A_2} (\hat{H}_r - \hat{H}_{A_2}) + (TA)_{A_3} (\hat{u}_r - \hat{u}_{A_3}) - (TA)_{A_4} (\sigma_r - \sigma_{A_4}) = \Delta t \left[ T_{A_1} f_{A_1} + T_{A_2} f_{A_2} + T_{A_3} f_{A_3} + T_{A_4} f_{A_4} \right],
\tag{39}
\]

\[
(TA)_{A_3} (V_n - V_{A_3}) + (TA)_{A_2} (\hat{H}_r - \hat{H}_{A_2}) + (TA)_{A_3} (\hat{u}_r - \hat{u}_{A_3}) + (TA)_{A_4} (\sigma_r - \sigma_{A_4}) = \Delta t \left[ T_{A_1} f_{A_1} + T_{A_2} f_{A_2} + T_{A_3} f_{A_3} + T_{A_4} f_{A_4} \right],
\tag{40}
\]

\[
(TA)_{A_4} (V_n - V_{A_4}) - (TA)_{A_2} (\hat{H}_r - \hat{H}_{A_2}) + (TA)_{A_3} (\hat{u}_r - \hat{u}_{A_3}) - (TA)_{A_4} (\sigma_r - \sigma_{A_4}) = \Delta t \left[ T_{A_1} f_{A_1} + T_{A_2} f_{A_2} + T_{A_3} f_{A_3} + T_{A_4} f_{A_4} \right].
\]
\[
\begin{align*}
(\mathbf{T}_3)_{i4} &\left(V_{i4} - V_{A_4}\right) - (\mathbf{T}_3)_{i3} \left(\bar{\bar{H}}_{i3} - \bar{\bar{H}}_{A_3}\right) - (\mathbf{T}_3)_{i4} \left(\sigma_{i4} - \sigma_{A_4}\right) \\
&= \Delta \left[T_{i4} \sigma_{i4} + T_{i3} \sigma_{i3} + T_{i2} \sigma_{i2} + T_{i1} \sigma_{i1}\right].
\end{align*}
\]

Subscript \(P\) indicates the unknowns at the current time step and subscripts \(A_1, A_2, A_3\) and \(A_4\) indicate the calculated values at the previous time step, where each of the characteristic lines through \(P\) meets that earlier time line (Fig. 1).

The slope of each characteristic line is equal to the reciprocal of the corresponding wave speed. Herein the numerical time step corresponds to the pressure wave grid \(\Delta t = \Delta z / \bar{c}_j\). The elements of matrix \(T\) in the Eqs. (38)-(41) are

\[
\begin{align*}
T_{i1} &= \sigma_{i1} T_{i1} = 1, \\
T_{i2} &= -\bar{c}_j, \\
T_{i3} &= T_{i4} = 2\nu \left( \frac{\bar{c}_j}{c_j} \right)^2 \left( 1 - \left( \frac{\bar{c}_j}{c_j} \right)^2 \right)^{\frac{3}{2}}, \\
T_{i5} &= T_{i6} = -\rho_j \bar{V}_j \bar{V}_j \left( \frac{\bar{c}_j}{c_j} \right)^2 \left( 1 - \left( \frac{\bar{c}_j}{c_j} \right)^2 \right)^{\frac{3}{2}}, \\
T_{i7} &= -\rho_j \bar{V}_j \bar{V}_j \left( \frac{\bar{c}_j}{c_j} \right)^2 \left( 1 - \left( \frac{\bar{c}_j}{c_j} \right)^2 \right)^{\frac{3}{2}}, \\
T_{i8} &= T_{i9} = -\bar{c}_j.
\end{align*}
\]

The velocities at the upstream (subscript \(u\)) and downstream (subscript \(d\)) sides of each computational section have been introduced to take column separation into account. They are identical when the calculated pressure is above the vapour pressure. The numerical integration scheme for the cavity volume equation (12) on rectangular grids has the form (radial pipe velocity is ignored (which is assumed valid for steel pipes) in the calculation of volume):

\[
\nabla (t) = \nabla (t - \Delta t) + \Delta t \left[ \psi \left[ V_u(t) - V_i(t) \right] + (1 - \psi) \left[ V_d(t - \Delta t) - V_i(t) \right] \right],
\]

in which \(\psi\) is a weighting factor; \(\psi = 0.5\) used herein. The right-hand-side vector \(\mathbf{r}\) contains expressions of unknowns that are evaluated at the appropriate grid points at previous time steps as listed by relations (44). The terms containing convolution integrals can be evaluated using the approximations provided in Section 3.1 (see Eqs. (30) and (34) for the quantities \(a_1^l, a_2^l, a_1^r\) and \(a_2^r\)).

Fig. 1. Computational grid based on pressure waves and characteristic lines for interior nodes and nodes adjacent to boundaries.
\[
\begin{align*}
\mathbf{r}_{i,t} &= -\frac{f}{2D} (V_{n} - \bar{u}_{i}) \left[ \psi_{i} - \bar{u}_{i} \right] + f \frac{V_{n}^{2}}{2D} - \frac{16\nu}{D} \left[ \bar{a}_{i}^{e} (V_{n} - \bar{u}_{i}) + \bar{a}_{n}^{e} \right] \quad \text{for } i = 1, 3, \\
\mathbf{r}_{i,t} &= -\frac{f}{2D} (V_{n} - \bar{u}_{i}) \left[ \psi_{i} - \bar{u}_{i} \right] + f \frac{V_{n}^{2}}{2D} - \frac{16\nu}{D} \left[ \bar{a}_{i}^{e} (V_{n} - \bar{u}_{i}) + \bar{a}_{n}^{e} \right] \quad \text{for } i = 2, 4, \\
\mathbf{r}_{i,4} &= -\rho_{t} \left( 1 - \nu \right) \frac{D}{\nu} \left( a_{i} \bar{H} - \bar{a}_{i}^{0} \right) \quad \text{for } i = 1, 2, 3, 4, \\
\mathbf{r}_{i,1} &= \left( \frac{\rho_{t} f}{8} (V_{n} - \bar{u}_{i}) \left[ \psi_{i} - \bar{u}_{i} \right] - \rho_{t} \frac{V_{n}^{2}}{8} + \frac{4\rho_{t} \nu}{D} \left[ \bar{a}_{i}^{e} (V_{n} - \bar{u}_{i}) + \bar{a}_{n}^{e} \right] \right) \quad \text{for } i = 1, 3, \\
\mathbf{r}_{i,2} &= \left( \frac{\rho_{t} f}{8} (V_{n} - \bar{u}_{i}) \left[ \psi_{i} - \bar{u}_{i} \right] - \rho_{t} \frac{V_{n}^{2}}{8} + \frac{4\rho_{t} \nu}{D} \left[ \bar{a}_{i}^{e} (V_{n} - \bar{u}_{i}) + \bar{a}_{n}^{e} \right] \right) \quad \text{for } i = 2, 4, \\
\mathbf{r}_{i,4} &= \left( a_{i} \bar{\sigma}_{n} + \bar{a}_{i}^{0} \right) - \frac{\rho_{t} \nu}{2\nu} \frac{D}{\nu} \left( a_{i} \bar{H} - \bar{a}_{i}^{0} \right) \quad \text{for } i = 1, 2, 3, 4.
\end{align*}
\]  

\( a_{i}^{e} \) is calculated using the velocities at the previous time step, which for points with vapour cavities are averaged between upstream and downstream velocities: \( \nu = 0.5 (V_{n} + V_{l}) \).

The four compatibility equations (38)-(41) (now with five unknowns) are solved simultaneously. If there is no column separation, the downstream and upstream velocities at each section are the same, and the remaining four unknowns are evaluated using the four compatibility equations. As soon as the calculated pressure gets to the liquid vapour pressure or below, the pressure remains at vapour pressure and two distinctive velocities at each section cause a cavity to open. They are calculated from the four equations (38)-(41) knowing that \( \bar{H} = H - H_{w} \) (\( H_{w} \) is the vapour pressure head), where the remaining four unknowns are \( V_{n}, V_{l}, \bar{u}_{i}, \text{ and } \bar{a}_{i}^{0} \). If the cavity volume \( \forall \) (determined by \( V_{n} \) and \( V_{l} \)) becomes zero or negative, the computation returns to the pure-liquid condition.

4 \hspace{1cm} \text{NUMERICAL RESULTS}

4.1 \hspace{1cm} \text{Case study description}

Two reservoir-pipeline-valve systems are used to investigate the four items CS, FSI, UF and VE, separately and simultaneously. One is known as the Imperial College experiment performed by Covas et al. (1-3) and the other was conducted by Güney at the Laboratory of Fluid Mechanics of I.N.S.A. in Lyon (34). Their specifications are given in Tables 1A and 1B, respectively. It is noted that Table 1A is different from Table 6 in our Ref. (16), because we have used hypothetical wave speeds there. For the analyses with VE, two PE materials with the creep behaviour specified in Tables 2A and 2B, corresponding to each experiment, were used. For the FSI cases, the pipeline was considered to be straight and axially free to move to enhance Poisson coupling effects. In addition, the massless valve was allowed to move to introduce junction coupling. Steady and unsteady friction together account for friction coupling. The valve was closed instantaneously at \( t = 0 \). The I.C. case study is indicated by A and the I.N.S.A. study by B.
Table 1A. Specification of the Imperial College experiment (1-3); the starred items are hypothetically assumed to generate FSI and CS.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>Length</td>
<td>277 m</td>
</tr>
<tr>
<td>Inner diameter</td>
<td>50.6 mm</td>
</tr>
<tr>
<td>Wall thickness</td>
<td>6.3 mm</td>
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<tr>
<td>Young’s modulus ($E_0$)</td>
<td>1.43 GPa</td>
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<tr>
<td>Poisson’s ratio</td>
<td>0.46</td>
</tr>
<tr>
<td>Kinematic viscosity</td>
<td>$10^{-6}$ m$^2$/s</td>
</tr>
<tr>
<td>Darcy-Weisbach coefficient</td>
<td>0.02</td>
</tr>
<tr>
<td>Vapour pressure head</td>
<td>$-$10 m</td>
</tr>
<tr>
<td>Steady velocity</td>
<td>1 m/s</td>
</tr>
<tr>
<td>Pressure wave speed $\hat{c}_p$</td>
<td>451.3 m/s</td>
</tr>
<tr>
<td>Stress wave speed $\hat{c}_s$</td>
<td>1091.1 m/s</td>
</tr>
<tr>
<td>Fluid density</td>
<td>1000 kg/m$^3$</td>
</tr>
<tr>
<td>Pipe’s material density</td>
<td>1200 kg/m$^3$</td>
</tr>
</tbody>
</table>

Table 1B. Specification of the Lyon experiment (34); the starred items are hypothetically assumed to generate FSI and CS.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>Length</td>
<td>43.1 m</td>
</tr>
<tr>
<td>Inner diameter</td>
<td>50 mm</td>
</tr>
<tr>
<td>Wall thickness</td>
<td>4.2 mm</td>
</tr>
<tr>
<td>Young’s modulus ($E_0$)</td>
<td>0.648 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
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<tr>
<td>Kinematic viscosity</td>
<td>$10^{-6}$ m$^2$/s</td>
</tr>
<tr>
<td>Darcy-Weisbach coefficient</td>
<td>0.02</td>
</tr>
<tr>
<td>Vapour pressure head</td>
<td>$-$10 m</td>
</tr>
<tr>
<td>Steady velocity</td>
<td>0.55 m/s</td>
</tr>
<tr>
<td>Reservoir head</td>
<td>0.55 m</td>
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<tr>
<td>Pressure wave speed $\hat{c}_p$</td>
<td>263.9 m/s</td>
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<tr>
<td>Stress wave speed $\hat{c}_s$</td>
<td>735.1 m/s</td>
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<tr>
<td>Fluid density</td>
<td>930 kg/m$^3$</td>
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<td>Pipe’s material density</td>
<td>1200 kg/m$^3$</td>
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Table 2A. Creep coefficients $J_k$ used to describe the VE behaviour of the pipe wall in the Imperial College experiment (taken from Ref. (3)).

<table>
<thead>
<tr>
<th>Retardation times $\tau_k$ (s) and creep coefficients $J_k$ (10$^{-10}$ Pa$^{-1}$)</th>
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</thead>
<tbody>
<tr>
<td>$\tau_1 = 0.05$</td>
</tr>
<tr>
<td>$\tau_2 = 0.5$</td>
</tr>
<tr>
<td>$\tau_3 = 1.5$</td>
</tr>
<tr>
<td>$\tau_4 = 5$</td>
</tr>
<tr>
<td>$\tau_5 = 10$</td>
</tr>
<tr>
<td>$J_1 = 1.057$</td>
</tr>
<tr>
<td>$J_2 = 1.054$</td>
</tr>
<tr>
<td>$J_3 = 0.9051$</td>
</tr>
<tr>
<td>$J_4 = 0.2617$</td>
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<td>$J_5 = 0.7456$</td>
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Table 2B. Creep coefficients $J_k$ used to describe the VE behaviour of the pipe wall in the Lyon experiment (taken from Ref. (17)).

<table>
<thead>
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<th>Retardation times $\tau_k$ (s) and creep coefficients $J_k$ (10$^{-10}$ Pa$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_1 = 0.0000089$</td>
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<tr>
<td>$\tau_2 = 0.022$</td>
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<td>$\tau_3 = 1.864$</td>
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<tr>
<td>$J_1 = 7.54$</td>
</tr>
<tr>
<td>$J_2 = 10.46$</td>
</tr>
<tr>
<td>$J_3 = 12.37$</td>
</tr>
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4.2 Results and discussion

On the basis of various terms switched on or off in the governing equations, analysis with or without FSI, UF, CS and VE was carried out. All the aforementioned effects are analysed separately, as seen in Fig. 2A, to investigate how each one changes the classical waterhammer pressures at Location 1 at distance $x = 6$ m from the valve (Ref. (3)). In all figures $H = \hat{H} + H_c$ is displayed. Looking at Fig. 2A reveals that only the two items FSI and CS can produce pressures larger than the Joukowsky pressure and the other items UF and VE mostly bring about pressure reduction, though in different patterns. The initial UF and VE effects are opposite, as they have increasing (due to line pack) and decreasing
(due to wall retardation) pressure plateaus, respectively. The amount of pressure reduction due to the VE effect depends on the retardation times in comparison with the fundamental waterhammer period. As Eq. (1) indicates, the shorter the retardation time, the more the corresponding Kelvin-Voigt elements will contribute to the retardation. As a result, the retardation times should be compared with the period of the transients. In this case study, the first three times $\tau_k (0.05s, 0.5s, 1.5s)$ are all smaller than the waterhammer period $\hat{T} = 2.74s$. As seen in Fig. 2A(d), the creep behaviour is about fully developed in the first half cycle and the head graph becomes approximately horizontal at $t = 1.37s$. In the UF results (Fig. 2A(b)), the line pack depends on the initial pressure gradient (steady state) of the flow. CS is the only phenomenon that produces steep wave fronts (at cavity collapse).

![Fig. 2A. Pressure head at Location 1 in the Imperial College experiment for the analysis with only one effect in each graph.](image)

The same analysis was carried out for the second case study and the results are depicted in figures with the same number but followed by “B”. The same patterns are observed when the two sets of figures A and B are compared. Accordingly, similar explanations hold for the influence of each phenomenon, so that the conclusions can be generalized to a certain extent.
The four items UF, VE, FSI and CS will make six different two-combinations, the results of which are depicted in Figs. 3A and 3B. UF and VE have dampening effects and reduce the pressure amplitudes. Conversely, FSI and CS bring about more pressure peaks compared to the classical results. Consequently, the result with the lowest pressure amplitudes is that of the combination of UF and VE given in Fig. 3A(a) and the one with the highest pressure amplitudes corresponds to the combination of FSI and CS as seen in Fig. 3A(f). The others, having one decreasing (like UF or VE) and one increasing (like FSI or CS) effect can be placed between these graphs. Meanwhile, the pressure rise due to only FSI (Figs. 2A(a)) is larger than that of only CS (Fig. 2A(c)). As a result, one can predict that the pressure peak in the analysis with FSI and VE (Fig. 3A(d)) (or with FSI and UF given in Fig. 3A(b)) is larger than that of analysis with CS and VE (Fig. 3A(e)) (or with CS and UF given in Fig. 3A(c)). Similar prognosis can be made for the decreasing effects, observing that VE causes more pressure reduction than UF in this case study (compare Figs. 2A(b) and 2A(d)). Accordingly, the pressure rise in the analysis with VE and FSI (Fig. 3A(d)) (or with VE and CS given in Fig. 3A(c)) is less than that of analysis with UF and FSI (Fig. 3A(b)) (or with UF and CS given in Fig. 3A(c)). Based on the results in Fig. 2A, the two-combinations in Fig. 3A can be sorted in terms of largest maximum pressure as: (f), (b), (c) or (d), (e) and (a). This prediction is in agreement with the numerical results shown in the figure (Fig. 3A). Similar interpretations can be made for Fig. 3B, noting that therein, CS leads to larger pressures than FSI does, as is already evident from Fig. 2B.
Fig. 3A. Pressure head at Location 1 in the Imperial College experiment for the analysis with all possible two-combinations of UF, VE, FSI and CS.
Fig. 3B. Pressure head at the valve in the Lyon experiment for the analysis with all possible two-combinations of UF, VE, FSI and CS.

The results for three-combinations of the four items (UF, VE, FSI, CS) are plotted in Figs. 4A and 4B. The conclusions made for Fig. 3A can be applied here to find out that the combination of UF, FSI and CS produces the largest pressures among the four graphs. This is because FSI and CS are the effects causing pressure peaks larger than classical results and UF is less dominant in pressure dampening than VE. Likewise, as stated before for the I.C. case (A), FSI is more dominant than CS in increasing pressures, so one can expect that the lowest possible pressures belong to analysis with UF, VE and CS as shown in Fig. 4A(d). In Fig. 4B(d) the dampening effect of VE and UF is so strong that CS does not occur. Be aware that all discussions are only valid for the two specific case studies and that for any other case it is possible that results with CS are much larger than those with FSI. However, from the sequential figures and corresponding explanations, one can figure out clues regarding the results to be expected for various combinations.
Fig. 4A. Pressure head at Location 1 in the Imperial College experiment for the analysis with all combinations of three out of UF, VE, FSI and CS.

Fig. 4B. Pressure head at the valve in the Lyon experiment for the analysis with all combinations of three out of UF, VE, FSI and CS.
Figures 5A and 5B correspond to the ultimate situation when all four items are included in the analysis. Although the two damping effects UF and VE exist in the I.C. simulation (A), the maximum pressure is still larger than the classical Joukowsky value (broken line). However, VE significantly damps out the pressure rises caused by FSI and CS. In the I.N.S.A. simulation (B) the damping is so strong that CS does not occur. Figure 5B is therefore the same as Fig. 4B(a) and not shown (again). Despite the complexity of the mathematical model and its computer implementation, numerical artefacts are kept to a minimum. A positive side effect is that VE, UF and FSI suppress the numerical spikes that normally show up in DVCM results.

![Graph](image)

Fig. 5A. Head at Location 1 in the Imperial College experiment for the analysis with UF, VE, CS and FSI.

Fig. 5B = Fig. 4B(a).

5 SUMMARY AND CONCLUSIONS

A mathematical model and a numerical method of solution are provided for waterhammer with FSI, CS, VE and UF. All three mechanisms of fluid-structure interaction, that is Poisson, junction and friction coupling, were included. The applied numerical method was MOC for both fluid and pipe. Column separation was modelled with DVCM such that vapour cavities were allowed to occur in all grid points. Convolutions of state variables and weight functions accounted for Kelvin-Voigt viscoelasticity and Vardy-Brown unsteady friction.

Two well-known laboratory experiments on polyethylene pipes were simulated for all possible combinations of the aforementioned four items. The resulting figures were compared in an attempt to draw conclusions regarding general tendencies and the significance of each effect.
Among the four items, FSI and CS were able to cause pressures larger than the Joukowsky pressure, so that CS with FSI is the most dangerous combination. In addition, CS is able to produce steep wave fronts. The two items VE and UF caused attenuation of the transient pressures, where the effect of VE was much larger than UF. The viscoelastic behaviour of the pipe wall is an important damping mechanism that not only reduces maximum pressures but also increases minimum pressures, thus preventing the dangerous phenomenon of column separation. The ultimate result herein, that is a simulation with FSI, CS, VE and UF, showed no resemblance with the classical waterhammer square-wave solution (for frictionless, motionless, high-pressure, elastic systems).

6 REFERENCES


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<th>Author(s)</th>
<th>Title</th>
<th>Month</th>
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<td>12-41</td>
<td>A.S. Tijsseling, A. Bergant</td>
<td>Exact computation of waterhammer in a three-reservoir system</td>
<td>Dec. ’12</td>
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<tr>
<td>12-43</td>
<td>A. Keramat, A.S. Tijsseling</td>
<td>Waterhammer with column separation, fluid-structure interaction and unsteady friction in a viscoelastic pipe</td>
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