On a pathological dependence of traffic flow fluxes on vehicle acceleration – a dimensional renormalization exercise

by

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Looking particularly at traffic flow scenarios, we show in this note that assuming the fluxes entering the conservation equation of cars to be dependent on the local density and acceleration is generally false. Our proof is based on a dimension renormalization argument.

1 Background

Traffic flows in urban areas became in the last years a very active research field; see, for instance, [4, 5, 8] and references cited therein. One of the main questions is the derivation of fundamental diagrams - explicit relations between numbers of cars (densities) and driving velocities. Note that fundamental diagram are very easily quantified and interpreted experimentally, provided the traffic is not too heavy. But why should one completely neglect in the fundamental diagrams the acceleration information? The acceleration mechanism seems to be responsible for producing stop-and-go-effects which typically cannot be captured by standard fundamental diagrams; compare e.g. [4, 7, 2]. How to include then acceleration in macroscopic models? Can we put acceleration information directly in the macroscopic fluxes, or should one derive from vehicle particle systems (posed at the microscale) Boltzmann-like equations/continuum conservation laws starting off from Newton/Langevin-like equations describing the discrete/stochastic flow?
In this framework, we use dimensional analysis methods (see, for instance, [6, 3, 1]) to show that standard constitutive laws for car traffic fluxes are incompatible with the acceleration information. By this, we emphasize the need of a different route to accounting for acceleration effects in continuum traffic models.

2 Basic conservation law

We consider the following standard traffic problem (conservation law for the number of cars)

\[ \partial_t \rho + \text{div}_x q = 0, \]  

(1)

posed on a one dimensional road, where we take 

\[ \rho := \text{the car density/unit length}, \]
\[ q := \text{the flux of cars}, \]
\[ \omega := \text{the acceleration of cars}. \] 

(2)

In (1), \( \partial_t \) is the first order derivative with respect to \( t \), while \( \text{div}_x \) is the divergence operator with respect to \( x \) [in 1D, this coincides with \( \partial_x \) – the first order derivatives with respect to \( x \)].

Assume \( q = Q(\rho, \omega) \) to be such that \( \rho \) and \( \omega \) are independent of each other.

Considering the system of dimensions \( \{ M, L, T \} \) (mass, length, time), we have

\[ [\rho] = ML^{-1} \]
\[ [q] = MT^{-1} \]
\[ [\omega] = LT^{-2}. \] 

(3)

Since we assume \( \omega \) to be independent on \( \rho \), we can rewrite (1) as follows:

\[ \partial_t \rho + \frac{dQ}{dq} \partial_x q = 0 \text{ in } \Omega, \]

(4)

where \( t \) and \( x \) are the time and space variables.

3 Main question

This note addresses the following question:

[Q] Is the dependence shown in (2), i.e. \( q = Q(\rho, \omega) \) really possible?

In other words, can we expect that the quantity \( \frac{dQ}{d\rho} \) arising in (4) depends on the car density and acceleration?
4 A dimensional renormalization approach

To address the question [Q], we use a technique presented in [3], pp. 8-10, which we refer to as dimensional renormalization. As we will see in this modeling exercise, the dimensional renormalization technique is able to discover pathologic parameter dependencies, but is not necessarily able to explain how to repair in the correct way the discovered dimensional pathology.

We proceed as follows: Denote $\varphi := \frac{dQ}{d\rho}$ and assume

$$\varphi = F(\rho_0, \omega_0) \quad (5)$$

for some given function $F$, where $\rho_0$ and $\omega_0$ represent the car density and acceleration in a fixed scale of units in the $\{M, L, T\}$ system. Note also that $[\varphi] = LT^{-1}$.

In the spirit of [3], let us now renormalize the time dimension in (5). We obtain

$$\xi_T^2 \omega_0 = 1 \quad (7)$$

Denoting $F(\rho_0, 1)$ by $G(\rho_0)$, and than using (7) leads to

$$\varphi = \sqrt{\omega_0} G(\rho_0) \quad (8)$$

Now, we renormalize the space dimension and obtain

$$\xi_L^{-1} \varphi = \xi_L^{-1/2} \sqrt{\omega_0} G(\xi_L \rho_0) \quad (9)$$

where $\xi_L$ is an arbitrary positive renormalization factor.

Choosing

$$\xi_L \rho_0 = 1 \quad (10)$$

gives

$$\varphi = \xi_L^{-1/2} \sqrt{\omega_0} G(1) \quad (11)$$

Given $[G(1)] = 1$, i.e. $G$ is dimensionless, we note that

$$LT^{-1} = [\varphi] \neq [\sqrt{\rho_0 \omega_0}] [G(1)] = M^{1/2} T^{-1} \quad (12)$$

which is inconsistent.

Based on (12), we can now conclude that (5) as well as (2) cannot hold in this context.
It is worth also noting that if the traffic problem (1) would be posed in a higher-dimensional space, then the above conclusion would remain unchanged. We have deliberately chosen to present a one-dimensional formulation of this pathologic example mainly because this fits well to traffic flows in road networks.

5 Acknowledgments

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REFERENCES

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