Degenerate two-phase flow model in porous media including dynamics effects in the capillary pressure: existence of a weak solution

by

X. Cao, I.S. Pop

Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507
Degenerate two-phase flow model in porous media including dynamic effects in the capillary pressure: existence of a weak solution

X. Cao\textsuperscript{a,*}, I.S. Pop\textsuperscript{a,b}

\textsuperscript{a}CASA, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
\textsuperscript{b}Institute of Mathematics, Johannes Brunns GT. 12, University of Bergen, Bergen, Norway

Abstract
In this paper, we analyze a degenerate elliptic-parabolic system which describes the flow of two incompressible, immiscible fluids in porous media including dynamic effects in the phase-pressure difference. First, for a regularized diffusion coefficient, the existence and uniqueness of the weak solution in the non-degenerate case is obtained. Then we let the regularization parameter go to zero to show the existence of weak solutions under degenerate case.

Keywords: Degeneracy, dynamic capillary pressure, two-phase flow, weak solution, uniqueness, pseudo-parabolic system

1. Introduction
In this paper, we analyze the existence and, where appropriate, uniqueness of a weak solution to the elliptic-parabolic system:
\begin{equation}
\frac{\partial u}{\partial t} + \nabla \cdot (k(u)\nabla p) - \Delta \theta(u) = 0, \tag{1.1}
\end{equation}
\begin{equation}
\nabla \cdot (k(u)\nabla p) + \nabla \cdot (k_n(u)\nabla (\tau(u)\partial_t u)) = 0, \tag{1.2}
\end{equation}
complemented with initial and boundary conditions. The equations hold in $Q := (0, T_M) \times \Omega$. Here $\Omega$ is a bounded domain in $\mathbb{R}^d$, having Lipschitz continuous boundary, and $T_M > 0$ is a given maximal time. The unknowns are $u$ and $p$. The work is motivated by two-phase flow in porous media (e.g. oil and water).

1.1. Two phase flow model in porous media under non-equilibrium condition
The system (1.1) - (1.2) models two phase flow in porous media, with dynamic effects in the phase pressure difference. It is obtained by including Darcy’s law for both phases in the mass conservation laws. With $w, n$ being indices for the wetting, respectively, non-wetting phase, the mass conservation equations are (see [4, 21]):
\begin{equation}
\phi \frac{\partial s_{\alpha}}{\partial t} + \nabla \cdot q_{\alpha} = 0 \quad (\alpha = w, n). \tag{1.3}
\end{equation}
The coefficient $\phi$ represents the porosity of the porous medium, while $s_{\alpha}$ and $q_{\alpha}$ denote the saturation and the volumetric velocity of the $\alpha$ phase. The volumetric velocity $q_{\alpha}$ is deduced from the Darcy law as
\begin{equation}
q_{\alpha} = -\frac{k}{\mu_{\alpha}} k_{\alpha}(s_{\alpha}) \nabla p_{\alpha} \quad (\alpha = w, n), \tag{1.4}
\end{equation}

*Corresponding author
Email addresses: x.cao@tue.nl (X. Cao), i.pop@tue.nl (I.S. Pop)
URL: http://www.win.tue.nl/casa/people/tempstaff/168.html (X. Cao)
where  $\bar{k}$ is the absolute permeability of the porous medium, $p$ the pressure, $\mu$ the viscosity and $k_{r\alpha}$ the relative permeability of the $\alpha$ phase. The specific function of $k_{r\alpha}$ is assumed known. Substituting (1.4) in (1.3) gives

$$
\phi \frac{\partial s_\alpha}{\partial t} - \nabla \cdot \left( \frac{\bar{k}k_{r\alpha}}{\mu_\alpha} \nabla p_\alpha \right) = 0 \quad (\alpha = w, n). \tag{1.5}
$$

We assume that only two phases are present,

$$
s_w + s_n = 1. \tag{1.6}
$$

To complete the model, one commonly assumes a relationship between the phase pressure difference, and $s_w$. Under equilibrium assumption, this is

$$
p_n - p_w = p_c(s_w),
$$

with a given function $p_c = p_c(\cdot)$. Experimental results [15, 20] have, however, proved the limitation of this assumption. Alternatively, in [20] the following relation is proposed:

$$
p_n - p_w = p_c(s_w) - \tau(s_w) \frac{\partial s_w}{\partial t}, \tag{1.7}
$$

The damping function $\tau$ as well as the function $p_c$, which represents the capillary pressure under equilibrium condition, are assumed to be known. Summing the two equations from (1.5) and making use of (1.6) give:

$$
\nabla \cdot F = 0, \tag{1.8}
$$

where $F = \frac{\bar{k}k_w}{\mu_w} \nabla p_w + \frac{\bar{k}k_n}{\mu_n} \nabla p_n$ denotes the total flow.

Introducing the normalized relative permeabilities

$$
k_\alpha := \frac{k_{r\alpha}}{\mu_\alpha} \quad (\alpha = w, n),
$$

and with $k = k_w + k_n$, we also define the fractional flow function

$$
f_w(s_w) := \frac{k_w}{k}.
$$

Then, we follow [2, 13], and define the global pressure $p$

$$
p = p_n - \int_{C_D} f_w(z)p'_w(z)dz, \tag{1.9}
$$

which leads to the following expression of the water pressure

$$
p_w = p + \int_{C_D} f_w(z)p'_w(z)dz - p_c(s_w) + \tau(s_w)\partial_t s_w. \tag{1.10}
$$

Here $C_D \in (0, 1)$ is a constant that will be used as the boundary value of water saturation.

Furthermore, define the complementary pressure $\theta$ as the integral (Kirchhoff) transformation

$$
\theta(s_w) = -\int_{C_D} \frac{k_w}{k} \frac{k_n}{k} (z)p'_w(z)dz. \tag{1.11}
$$

Then, from (1.5) for the wetting phase, by using (1.6) and (1.9) gives

$$
\phi \partial_t s_w + \nabla \cdot (\bar{k}k_w \nabla p_w) - \nabla \cdot (\bar{k} \nabla \theta(s_w)) = 0. \tag{1.12}
$$

Finally, (1.8) becomes

$$
\nabla \cdot (\bar{k}k_w \nabla p_w) + \nabla \cdot (\bar{k}k_n \nabla (\tau(s_w))) = 0. \tag{1.13}
$$
The system (1.12)-(1.13) is in dimensional form. Taking \( L_r, T_r, \) and \( P_r \) as characteristic values for the length, time, global pressure, respectively. Scaling the space variable \( x \) with \( L_r \), the time \( t \) with \( T_r \) and the pressure \( p \) with \( P_r \), and assuming

\[
\frac{T_r}{L_r^2} = \frac{\phi}{kP_r},
\]

we obtain the following system:

\[
\begin{align*}
\partial_t u + \nabla \cdot (k_n \nabla p) - \Delta \theta(u) &= 0, \\
\nabla \cdot (k \nabla p) + \nabla \cdot (k_u \nabla (\tau \partial_t u)) &= 0,
\end{align*}
\]

with \( u = s_u \).

### 1.2. Assumptions and known results

For the non-linearities appearing in the model, we refer to [4, 21]. The special assumptions are given below. Here we mention the typical choices of permeability in the literature ([7, 8])

\[
k_u(u) = u^\alpha, \quad k_n(u) = (1 - u)^\beta, \quad \text{with} \quad \alpha, \beta > 1,
\]

and

\[
-p'_c(u) = u^{-\lambda}, \quad \lambda > 1.
\]

These are defined for \( u \in [0, 1] \) (the physical range).

The model (1.15) - (1.16) is completed by the initial condition

\[
u(0, \cdot) = u_0 \quad \text{in} \quad \Omega,
\]

and the boundary conditions \( u = C_D, \quad p = 0 \) at \( \partial \Omega, \) for all \( t > 0, \)

where \( u_0 \) is a given function and the constant \( C_D \) satisfies \( 0 < C_D < 1. \)

**Remark 1.** To avoid an excess of technicalities, the pressure boundary condition in (1.18) is assumed constant. Other types of boundary conditions, as Naumann can be considered. Similarly, the boundary values of \( u \) may be non-constant, but should be bounded away from 0 and 1.

In this paper we assume

**A1:** The functions \( k_u, k_n \) \( [0, 1] \rightarrow [0, 1] \) are \( C^1, \) \( k_u \) is an increasing function, and \( k_n \) is decreasing. \( p_c \in C^1((0, 1], \mathbb{R}^+) \)

is a decreasing function, \( \tau \in C^1((0, 1], \mathbb{R}^+) \) is an increasing function satisfying \( \tau \geq \tau_0 \), for some \( \tau_0 > 0 \), and for all \( u \in [0, 1] \). Furthermore, we assume \( -p'_c k_u \leq C < +\infty \).

These functions (see also [4]) are defined on \([0, 1]\), which is the physically relevant range. For the analysis, we extend them to \( \mathbb{R} \).

a: \( k_u(u) = k_u(0), \) if \( u \leq 0, \) \( k_u(u) = k_u(1), \) if \( u \geq 1. \)

b: \( k_n(u) = k_n(0), \) if \( u \leq 0, \) \( k_n(u) = k_n(1), \) if \( u \geq 1. \)

c: \( \frac{1}{p_u(u)} = \frac{1}{p_u(0)}, \) if \( u \leq 0, \) \( \frac{1}{p_u(u)} = \frac{1}{p_u(1)}, \) if \( u \geq 1. \)

d: \( 1/\tau(u) = 1/\tau(0), \) if \( u \leq 0, \) \( 1/\tau(u) = 1/\tau(1), \) if \( u \geq 1. \)

**Remark 2.** A1 allows \( p'_c \), respectively \( \tau \) to become \( \infty \). Then, an extension is not meaningful. However, this extension makes sense for the reciprocals \( 1/p'_c, 1/\tau \). Also, the monotonicity of \( \tau \), as well as the assumption on a strictly positive lower bound can be compensated by additional technicalities. We omit this here.
In the proofs below, we will use the functions (see [10]) $G, \Gamma : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$
\[
G(u) = \int_{\mathcal{C}_0} \tau_k \mathcal{K}_k(u) \, dz, \quad \text{and} \quad \Gamma(u) = \int_{\mathcal{C}_0} G(z) \, dz, \quad (1.19)
\]
and the function $T(u)$:
\[
T(u) = \int_0^u \tau(z) \, dz + C_T, \quad (1.20)
\]
here $C_T$ is a constant specified below. Clearly, $\Gamma$ is a convex function satisfying $\Gamma(0) = G(0) = 0$, implying that
\[
\Gamma(u) \geq 0, \quad \text{for all } u \in \mathbb{R}. \quad (1.21)
\]

**A2:** Second, we assume that the initial condition $u_0$ satisfies $u_0 \in W^{1,2}_0(\Omega) + C_D$, $\int_\Omega \Gamma(u_0) < +\infty$, $T(u_0) \in W^{1,2}(\Omega)$.

The problem considered here is an extension of scalar, two-phase flow models studied in [10], while only one pseudo-parabolic equation is considered in the unknown $u$. The model in [10] can be derived from the present one assuming that the total flow is known (see [2, 18]). After non-dimensionalization and some algebraic manipulation one gets
\[
\partial_t u + \nabla \cdot \mathbf{F} - \Delta \theta(u) - \tau \nabla \cdot \left( \frac{k_n k}{k} \nabla \partial_t u \right) = 0. \quad (2.22)
\]

The existence and uniqueness of the weak solution has been shown in [18] in the linear case of the third order derivative term. In [27], the author has considered one phase flow model, and regularization is employed in to prove existence of a weak solution for degenerate case. Here we build on similar ideas to prove existence for the two phase flow model. For uniqueness but in the non-degenerate case, we refer to [5, 11].

Two-phase flow models are mainly analyzed under equilibrium assumptions, i.e. for $\tau = 0$. In particular, we refer to [3, 14, 24]. For non-equilibrium, two-phase flow models are analyzed in [23], where existence is obtained for non-degenerate cases, but including hysteresis effects, and in [12], where uniqueness is proved again, in non-degenerate cases.

In what follows, we analyze first the regularized model, and prove the existence and uniqueness of a solution, as well as a-priori energy estimates. This is achieved by Rothe’s method [22]. Finally, we pass the regularization parameter to 0 to obtain the existence of weak solutions in the degenerate case. Note that uniqueness remains open.

### 2. Existence and uniqueness of weak solutions in the regularized case

#### 2.1. The weak solution concept

We use the standard spaces $L^2(\Omega)$, $W^{1,2}(\Omega)$ and $W^{1,2}_0(\Omega)$ in the theory of partial differential equation, and $(\cdot, \cdot)$ and $\| \cdot \|$ denote the scalar product and the corresponding norm in $L^2(\Omega)$, or where needed, in $(L^2(\Omega))^d$. Furthermore, $L^2(0,T_M;X)$ denotes the Bochner space of $X$-valued functions.

Let us now define the set
\[
V := W^{1,2}_0(\Omega) + C_D.
\]

Inspired by [27], we define a weak solution to the model (1.15)-(1.18) solving

**Problem P:** Find $u \in L^2(0,T_M;V)$, such that $\partial_t u \in L^2(Q)$, and $p \in L^2(0,T_M;W^{1,r}_0(\Omega))$ (for some particular chosen $r^* \in (1,2)$), such that $u(0, \cdot) = u_0$, $\sqrt{n} \nabla p \in L^2(0,T_M;L^2(\Omega)^d)$, $\sqrt{\mu}(\nabla p + \nabla \theta T(u)) \in L^2(0,T_M;L^2(\Omega)^d)$, and
\[
(\partial_t u, \phi) - (k_n \nabla p, \nabla \phi) + (\nabla \theta(u), \nabla \phi) = 0, \quad (2.1)
\]
\[
(k \nabla p, \nabla \phi) + (k_n \nabla \partial_t u, \nabla \psi) = 0, \quad (2.2)
\]
for any $\phi, \psi \in L^2(0,T_M;W^{1,2}_0(\Omega))$.

Before dealing with the degenerate case, we consider first the regularized case. To this aim, we define:

\begin{itemize}
  \item[a'] $k_{\alpha\beta}(u) = k_\alpha(u + \delta), \quad \text{if } 0 < u < 1 - \delta$, $k_{\alpha\beta}(u) = k_u(\delta)$, \quad \text{if } u < 0$, $k_{\alpha\beta}(u) = k_\alpha(1)$, \quad \text{if } u > 1 - \delta$.
  \item[b'] $k_{\alpha\beta}(u) = k_\alpha(u - \delta), \quad \text{if } \delta < u \leq 1$, $k_{\alpha\beta}(u) = k_\alpha(0)$, \quad \text{if } u < \delta$, $k_{\alpha\beta}(u) = k_\alpha(1 - \delta)$, \quad \text{if } u > 1$.
\end{itemize}
Below we analyze the following:

\[ c' \colon -p_{c,\delta}'(u) = -p_c(u + \delta), \text{ if } 0 \leq u \leq 1 - \delta, \quad -p_{c,\delta}'(u) = -p_c'(\delta), \text{ if } \delta < 0, \quad -p_{c,\delta}'(u) = -p_c'(1), \text{ if } u > 1 - \delta, \]

\[ d' \colon \tau_\delta(u) = \tau(u - \delta), \text{ if } \delta \leq u \leq 1, \quad \tau_\delta(u) = \tau(0), \text{ if } u < \delta, \quad \tau_\delta(u) = \tau(1 - \delta), \text{ if } u > 1. \]

This immediately follows:

\[ G_\delta(u) = \int_{C_0}^{u} \frac{\tau_\delta k_\delta}{k_{\text{mod}} k_\delta}(z) dz, \quad \Gamma_\delta(u) = \int_{C_0}^{u} G_\delta(z) dz, \quad (2.3) \]

\[ k_\delta(u) = k_{\text{mod}}(u) + k_{\text{mod}}(u), \quad f_{\text{mod}} = \frac{k_{\text{mod}}}{k_\delta}, \quad \phi_\delta(u) = -\int_{C_0}^{u} \frac{k_{\text{mod}} k_{\text{mod}}}{k_\delta}(z) p_{c,\delta}'(z) dz, \quad (2.4) \]

\[ T_\delta(u) = \int_{0}^{u} \tau_\delta(z) dz + C_T. \quad (2.5) \]

We now investigate the regularized model

\[ \partial_t u_\delta + \nabla \cdot (k_{\text{mod}} \nabla p_{\text{mod}}) - \Delta \phi_\delta(u_\delta) = 0, \quad (2.6) \]

\[ \nabla \cdot (k_\delta \nabla p_\delta) + \nabla \cdot (k_{\text{mod}} \nabla \theta_\delta (u_\delta)) = 0, \quad (2.7) \]

with the following initial and boundary conditions

\[ u_\delta(0, \cdot) = u^0 \quad \text{in} \quad \Omega, \quad (2.8) \]

\[ u_\delta = C_D, \quad p_\delta = 0 \quad \text{at} \quad \partial \Omega, \quad \text{for all} \quad t > 0. \quad (2.9) \]

Below we analyze the following:

**Problem P_\delta**: Find \( u_\delta \in L^2(0, T_M; V) \), such that \( \partial_t u_\delta \in L^2(Q) \), and \( p_\delta \in L^2(0, T_M; W_0^{1,2}(\Omega)) \), such that \( u_\delta(0, \cdot) = u^0 \), \( \nabla \theta_\delta (u_\delta) \in L^2(0, T_M; L^2(\Omega)^d) \) and

\[
\int_{0}^{T_M} \left( \partial_t u_\delta, \phi \right) dt - \int_{0}^{T_M} (k_{\text{mod}} \nabla p_{\text{mod}}, \nabla \phi) dt + \int_{0}^{T_M} (\nabla \phi_\delta(u_\delta), \nabla \phi) dt = 0, \quad (2.10) \\
\int_{0}^{T_M} (k_\delta \nabla p_\delta, \nabla \psi) dt + \int_{0}^{T_M} (k_{\text{mod}} \nabla \theta_\delta (u_\delta), \nabla \psi) dt = 0, \quad (2.11) 
\]

for any \( \phi, \psi \in L^2(0, T_M; W_0^{1,2}(\Omega)) \).

We show the existence of a solution to Problem P_\delta by the method of Rothe [22]. In doing so, we use the elementary inequality:

\[ ab \leq \frac{1}{2\sigma} a^2 + \frac{\sigma}{2} b^2, \quad \text{for any} \quad a, b \in \mathbb{R}, \sigma > 0. \quad (2.12) \]

### 2.2. The time discretization

With \( N \in \mathbb{N} \), let \( h = T_M/N \) and consider the Euler implicit discretization of (2.10)-(2.11).

**Problem P_\delta^h**: Given \( u_\delta^{n-1} \in V (n = 1, 2 \ldots N) \), find \( u_\delta^n \in V \) and \( p_\delta^n \in W_0^{1,2}(\Omega) \), such that

\[
\left( \frac{u_\delta^n - u_\delta^{n-1}}{h}, \phi \right) - (k_{\text{mod}}(u_\delta^n) \nabla p_\delta^n, \nabla \phi) + (\nabla \phi_\delta(u_\delta^n), \nabla \phi) = 0, \quad (2.13) \\
(k_\delta(u_\delta^n) \nabla p_\delta^n, \nabla \psi) + (k_{\text{mod}}(u_\delta^n) T_\delta(u_\delta^n) - T_\delta(u_\delta^{n-1}) \frac{h}{h}, \nabla \psi) = 0, \quad (2.14) 
\]

for any \( \phi, \psi \in W_0^{1,2}(\Omega) \). Here \((\cdot, \cdot)\) means \( L^2 \) inner product.

**Lemma 2.1**: Problem P_\delta^h has a solution.
Further, define 
\[ g \]
As above, for 
\[ A1 \]
By 
\[ M \]
\[ P \]
Proof. We start with a finite dimension approximation (Garlerin), for which we prove the existence of a solution to Problem P_0. To this aim, we use Lemma 1.4 (pp. 164 in [30]). Then we pass to the limit for the discrete solution, and use compactness to show the existence of a solution for Problem P_0.

Let \( \{v_m\}_{m=1}^{\infty} \) be the countable basis of the separable space \( W_1^0(\Omega) \). Set \( W_m = \text{Span}(v_m) \) for \( k = 1, ..., m \). Then, given \( \alpha_1, ..., \alpha_m \in \mathbb{R}, \beta_1, ..., \beta_m \in \mathbb{R} \), define \( \nu = \sum_{i=1}^{m} \alpha_i v_i \in W_m \), \( w = \sum_{i=1}^{m} \beta_i v_i \in W_m \), and set \( \varsigma = \nu + C_D \).

A finite dimensional solution \( (\varsigma, w) \in [W_m + C_D] \times [W_m] \) of (2.13)-(2.14) satisfies
\[
\begin{align*}
\left( \frac{\varsigma - \varsigma_0^{n-1}}{h}, \phi \right) - (k_{\alpha}(\varsigma) \nabla w, \nabla \phi) + (\nabla \theta_0(\varsigma), \nabla \phi) &= 0, \\
(k_{\delta}(\varsigma) \nabla w, \nabla \psi) + (k_{\alpha}(\varsigma) \nabla T_{\alpha}(\varsigma) - T_{\alpha}(\varsigma_0^{n-1}) \nabla \psi) &= 0,
\end{align*}
\]
for any \( \phi, \psi \in W_m \).

To avoid an excess of notations in the first part, we do not use any different indices for the solution pair \( (\varsigma, w) \) in the finite dimensional case.

For \( i = 1, ..., m \), define
\[
\dot{\beta}_i = (k_{\alpha}(\varsigma) \nabla w, \nabla v_i) + \frac{1}{h} (k_{\alpha}(\varsigma) \nabla(T_{\alpha}(\varsigma) - T_{\alpha}(\varsigma_0^{n-1})), \nabla v_i), \quad (i = 1, 2, ..., m).
\]

Note that if \( (\varsigma, w) \) is a solution pair to (2.15)-(2.16), one has \( \dot{\beta}_i = 0 \) for all \( i \).

Since \( w = \sum_{i=1}^{m} \beta_i v_i \), one clearly has
\[
\sum_{i=1}^{m} \dot{\beta}_i = (k_{\alpha}(\varsigma) \nabla w, \nabla w) + \frac{1}{h} (k_{\alpha}(\varsigma) \nabla(T_{\alpha}(\varsigma) - T_{\alpha}(\varsigma_0^{n-1})), \nabla w)
\]
\[
= \left\| \frac{\varsigma}{k_{\alpha}(\varsigma)} \nabla w \right\|^2 + \frac{1}{h} (T_{\alpha}(\varsigma)k_{\alpha}(\varsigma) \nabla w, \nabla w) - \frac{1}{h} (T_{\alpha}(\varsigma_0^{n-1})k_{\alpha}(\varsigma) \nabla w, \nabla w).
\]

Further, define \( g : \mathbb{R} \rightarrow \mathbb{R} \) by
\[
g(\varsigma) = \int_0^\varsigma \frac{T_{\alpha}(\varsigma)}{k_{\alpha}(\varsigma)} dz.
\]

By A1, there exist \( 0 < M_0 \leq M_1 < \infty \) possibly depending on \( \delta \), such that
\[
g(\varsigma) \geq M_0 \varsigma^2, \quad |g(\varsigma)| \leq M_1 |
\varsigma|.
\]

As above, for \( i = 1, ..., m \), we define
\[
l_i = \frac{(g(\varsigma), v_i)}{|v_i|^2}, \quad (i = 1, 2, ..., m),
\]
which immediately implies
\[
g(\varsigma) = \sum_{i=1}^{m} l_i v_i.
\]

Define
\[
\ddot{l}_i = \left( \frac{\varsigma - \varsigma_0^{n-1}}{h}, v_i \right) - \frac{1}{h} (k_{\alpha}(\varsigma) \nabla w, \nabla v_i) + \frac{1}{h} (\nabla \theta_0(\varsigma), \nabla v_i),
\]
as in above, if \( (\varsigma, w) \) is a solution pair, then \( \ddot{l}_i = 0 \) for all \( i \).

Since \( \nabla g(\varsigma) = \frac{T_{\alpha}(\varsigma)}{k_{\alpha}(\varsigma)} \nabla \varsigma \), we get
\[
\sum_{i=1}^{m} l_i \ddot{l}_i = \frac{1}{h^2} (\varsigma - \varsigma_0^{n-1}, g(\varsigma)) - \frac{1}{h} (k_{\alpha}(\varsigma) \nabla w, \frac{T_{\alpha}(\varsigma)}{k_{\alpha}(\varsigma)} \nabla \varsigma) + \frac{1}{h} (\nabla \theta_0(\varsigma), \frac{T_{\alpha}(\varsigma)}{k_{\alpha}(\varsigma)} \nabla \varsigma).
\]
Adding (2.18) and (2.24) yields
\[
\sum_{i=1}^{m} \beta_i \tilde{t}_i + \sum_{j=1}^{m} l_j \tilde{t}_j = \left\| \sqrt{k_{\delta}(\varsigma)} \nabla w \right\|^2 + \frac{1}{h^2} (\varsigma - u_{\delta}^{-1}, g(\varsigma)) \\
+ \frac{1}{h} (\nabla \theta_\delta(\varsigma), \frac{\tau_{\delta} k_{\delta}}{k_{\delta}} (\varsigma) \nabla \varsigma) - \frac{1}{h} (\tau_{\delta} (u_{\delta}^{-1}) k_{\delta}(\varsigma) \nabla u_{\delta}^{-1}, \nabla w). 
\] (2.25)

The existence of a solution is provided if the sum on the left in (2.25) becomes positive for \((l_1, \ldots, l_m)\) and \((\beta_1, \ldots, \beta_m)\) sufficiently large.

To prove this, denote the terms on the right by \(T_1, T_2, T_3, T_4\). Note that \(T_1\) is positive.

For \(T_2\), we use (2.20) to obtain,
\[
\frac{1}{h^2} (\varsigma - u_{\delta}^{-1}, g(\varsigma)) = \frac{1}{h^2} (\varsigma, g(\varsigma)) - \frac{1}{h^2} (u_{\delta}^{-1}, g(\varsigma)) \\
\geq \frac{M_0}{h^2} ||\varsigma||^2 - \frac{1}{2h^2} \frac{M_1}{M_0} ||u_{\delta}^{-1}||^2 - \frac{M_0}{2h^2} \cdot ||\varsigma||^2 \\
= \frac{M_0}{2h^2} \cdot ||\varsigma||^2 - \frac{1}{2h^2} \frac{M_1}{M_0} ||u_{\delta}^{-1}||^2. 
\] (2.26)

Recalling (2.4), \(T_3\) becomes
\[
\frac{1}{h} (\nabla \theta_\delta(\varsigma), \frac{\tau_{\delta} k_{\delta}(\varsigma) \nabla \varsigma}{k_{\delta}}) = \frac{1}{h} \left( -\frac{k_{\delta}}{k_{\delta}} \frac{k_{\delta}}{k_{\delta}} \frac{p_{\delta}(\varsigma) \nabla \varsigma}{k_{\delta}} \right) \tau_{\delta} k_{\delta} \frac{\nabla \varsigma}{k_{\delta}} = \frac{1}{h} \left( \frac{k_{\delta}}{\sqrt{k_{\delta}}} \sqrt{-\tau_{\delta} P_{\delta}(\varsigma) \nabla \varsigma} \right)^2. 
\] (2.27)

For \(T_4\), one gets
\[
-\frac{1}{h} (\tau_{\delta} (u_{\delta}^{-1}) k_{\delta}(\varsigma) \nabla u_{\delta}^{-1}, \nabla w) \geq -\frac{1}{2h^2} \left( \frac{\tau_{\delta} (u_{\delta}^{-1}) k_{\delta}(\varsigma) \nabla u_{\delta}^{-1}}{k_{\delta}(\varsigma)} \right)^2 - \frac{1}{2} \left( \frac{\nabla \theta_\delta(\varsigma) \nabla \varsigma}{k_{\delta}(\varsigma)} \right)^2. 
\] (2.28)

Using (2.26)-(2.28), the Poincaré and Cauchy-Swarzz inequalities lead to
\[
\sum_{i=1}^{m} \beta_i \tilde{t}_i + \sum_{j=1}^{m} l_j \tilde{t}_j \geq C(\Omega) \frac{M_2}{2} ||\nabla \varsigma||^2 + \frac{M_0}{2h^2} ||\varsigma||^2 + \frac{M_3}{h} ||\nabla \varsigma||^2 - \frac{1}{2h^2} \left( \frac{\tau_{\delta} (u_{\delta}^{-1}) k_{\delta}(\varsigma) \nabla u_{\delta}^{-1}}{k_{\delta}(\varsigma)} \right)^2 - \frac{1}{2h^2} \frac{M_1^2}{M_0} \left( ||u_{\delta}^{-1}||^2 \right)^2 \\
\geq C(\Omega) \frac{M_2}{2} ||\nabla \varsigma||^2 + \frac{M_0}{2h^2} ||\varsigma||^2 + \frac{M_3}{h} ||\nabla \varsigma||^2 - \frac{1}{2h^2} \left( \frac{\tau_{\delta} (u_{\delta}^{-1}) k_{\delta}(\varsigma) \nabla u_{\delta}^{-1}}{k_{\delta}(\varsigma)} \right)^2 - \frac{1}{2h^2} \frac{M_1^2}{M_0} \left( ||u_{\delta}^{-1}||^2 \right)^2. 
\] (2.29)

Note that \(M_2\) is independent of \(\delta\), and \(M_3\) may depend on \(\delta\), but they all not depend on the dimension of the space \(W_m\).

We know that (2.17) and (2.23) define a mapping \(\xi_m : \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}\) by \(\xi_m \left( \begin{array}{c} \frac{1}{\beta} \\ \frac{1}{\beta} \end{array} \right) = \left( \begin{array}{c} \frac{1}{\beta} \\ \frac{1}{\beta} \end{array} \right)\), which is continuous by (A1). As ||\(\beta\)|| + ||\(\beta\)|| (e.g. in the standard Euclidean norm) becomes large enough, \(\xi_m\) is strictly positive, and Lemma 1.4 in [30] (pp. 164) guarantees that \(\xi_m\) has a zero, that means there exists a solution to the discrete system.

We now pass to the limit as \(m \rightarrow \infty\). Denoting by \((u_m, p_m) \in [W_m + C_D] \times [W_m]\) the finite-dimensional solution \((\varsigma, w)\) obtained above, from (2.29), we get
\[
C(\Omega) \frac{M_2}{2} ||\nabla p_m||^2 + \frac{M_0}{2h^2} ||u_m||^2 + \frac{M_3}{h} ||\nabla u_m||^2 \leq \frac{1}{2h^2} \left( \frac{\tau_{\delta} (u_{\delta}^{-1}) k_{\delta}(u_m) \nabla u_{\delta}^{-1}}{k_{\delta}(u_m)} \right)^2 + \frac{1}{2h^2} \frac{M_1^2}{M_0} \left( ||u_{\delta}^{-1}||^2 \right)^2 \leq C. 
\] (3.0)

Here \(C\) may depend on \(\delta\), but not on the dimension \(m\).

This means that \(u_m\) is uniformly bounded in \(V\), \(p_m\) is uniformly bounded in \(W_0^{1,2}(\Omega)\), so we can find \(\tilde{u} \in V, \tilde{p} \in W_0^{1,2}(\Omega)\), such that, \(u_m \rightharpoonup \tilde{u}\) weakly in \(V\) and \(p_m \rightharpoonup \tilde{p}\) weakly in \(W_0^{1,2}(\Omega)\). The compact embedding of \(W_1^1(\Omega)\) into \(L^2(\Omega)\) gives
Therefore, for any \( \phi \)

\[
(\kappa_0(u_m) \nabla p_m, \nabla \phi) \rightarrow (\kappa_0(\tilde{u}) \nabla \tilde{p}, \nabla \phi), \quad \text{for any } \phi \in W^{1,2}_0(\Omega).
\]

To do so, we choose the test function \( \phi \in C_0^\infty(\Omega) \), clearly, one yields

\[
(\kappa_0(u_m) \nabla p_m, \nabla \phi) = (\nabla p_m, \kappa_0(\tilde{u}) \nabla \phi) + (\nabla p_m, (\kappa_0(u_m) - \kappa_0(\tilde{u})) \nabla \phi),
\]

and since \( \kappa_0(\tilde{u}) \in L^\infty(\Omega) \), one has

\[
(\nabla p_m, \kappa_0(\tilde{u}) \nabla \phi) \rightarrow (\nabla \tilde{p}, \kappa_0(\tilde{u}) \nabla \phi).
\]

We now show that the limit of \( (\nabla p_m, (\kappa_0(u_m) - \kappa_0(\tilde{u})) \nabla \phi) \) is 0.

Since \( \nabla p_m \) is bounded uniformly in \( (L^2(\Omega))^d \), one has

\[
[(\nabla p_m, (\kappa_0(u_m) - \kappa_0(\tilde{u})) \nabla \phi)] \leq \|\nabla p_m\| \cdot \|(\kappa_0(u_m) - \kappa_0(\tilde{u})) \nabla \phi\| \leq C \left( \int_\Omega (\kappa_0(u_m) - \kappa_0(\tilde{u}))^2 |\nabla \phi|^2 \, dx \right)^{\frac{1}{2}}.
\]

Because \( \kappa_0 \) is bounded, we get

\[
|(\kappa_0(u_m) - \kappa_0(\tilde{u})) \nabla \phi| \leq C|\nabla \phi| \quad \text{in } \Omega.
\]

Further, since \( u_m \rightarrow \tilde{u} \) strongly in \( L^2(\Omega) \), in the view of the continuity of \( \kappa_0 \), we have

\[
\kappa_0(u_m) \rightarrow \kappa_0(\tilde{u}) \quad \text{a.e.}
\]

Then, by the Dominated Convergence Theorem

\[
\left( \int_\Omega (\kappa_0(u_m) - \kappa_0(\tilde{u}))^2 |\nabla \phi|^2 \, dx \right)^{\frac{1}{2}} \rightarrow 0.
\]

Therefore, for any \( \phi \in C_0^\infty(\Omega) \), since \( \kappa_0(u_m) \nabla p_m \rightharpoonup \chi \) (weakly in \( W^{1,2}(\Omega) \)), due to the density of \( C_0^\infty(\Omega) \) in \( W^{1,2}(\Omega) \), this allows identifying \( \chi = \kappa_0(\tilde{u}) \nabla \tilde{p} \) for any \( \phi \in W^{1,2}_0(\Omega) \).

Similarly, we can also prove that

\[
(\kappa_0(u_m) \nabla p_m, \nabla \phi) \rightarrow (\kappa_0(\tilde{u}) \nabla \tilde{p}, \nabla \phi),
\]

\[
(\tau(\tilde{u}^{m-1}), k_0(u_m) \nabla u^{m-1}, \nabla \phi) \rightarrow (\tau(\tilde{u}), k_0(\tilde{u}) \nabla u^{m-1}, \nabla \phi),
\]

\[
(\tau(\tilde{u}, k_0(\tilde{u})) \nabla u_m, \nabla \phi) \rightarrow (\tau(\tilde{u}), k_0(\tilde{u}) \nabla u, \nabla \phi),
\]

\[
(\theta(t_m) u_m, \nabla \phi) \rightarrow (\theta(t) \tilde{u}, \nabla \phi),
\]

for any \( \phi \in W^{1,2}_0(\Omega) \).

Therefore a solution pair to Problem \( P_0 \) from now on is denoted by \( (\tilde{u}_0^k, \tilde{p}_0^k) \).

\[ \square \]

2.3. \textit{A priori estimates}

Having established the existence for the time discrete problems, we now prove the existence of a solution to Problem \( P_0 \). To achieve this, we use the elementary result.

\textbf{Lemma 2.2.} Let \( k \in 1, ..., N \), \( m \geq 1 \). Given two sets of real vectors \( a^k, b^k \in \mathbb{R}^m \) (\( k = 1, ..., N \)), one has

\[
\sum_{k=1}^N (a^k - a^{k-1}, a^k) = \frac{1}{2} (|a^N|^2 - |a^0|^2) + \sum_{k=1}^N (|a^k|^2 - |a^{k-1}|^2).
\]

We have the following:
Lemma 2.3. A \( C > 0 \), not depending on \( h \), exists such that, for any \( N^* \in 1, 2, \ldots, N \), one has

\[
\int_{\Omega} \nabla \theta(u^N_w)dx + \frac{1}{2} \left\| \nabla \Gamma_{a}(u^N_w) \right\|^2 + \sum_{n=1}^{N^*} \left\| \nabla (T_{\delta}(u^n_w) - T_{\delta}(u^{n-1}_w)) \right\|^2 + h \sum_{n=1}^{N^*} \left\| \nabla \phi \right\|^2 \leq C, \tag{2.35}
\]

\[
\left\| u^N_w \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \nabla u^N_w \right\|^2 + h \sum_{n=1}^{N^*} \left\| \nabla u^n_w \right\|^2 \leq C. \tag{2.36}
\]

**Proof.** Taking \( \phi = \int_{C_\delta}^{(\zeta)} \frac{\theta \delta k_\delta}{k_{w,\delta} k_{n,\delta}} d\zeta \) in (2.13) and \( \psi = \int_{C_\delta}^{(\zeta)} \frac{\tau \delta k_\delta}{k_{w,\delta} k_{n,\delta}} d\zeta \) in (2.14) (both being in \( W_0^{1,2}(\Omega) \)) gives

\[
\frac{u^n_w - u^{n-1}_w}{h}, \int_{C_\delta}^{(\zeta)} \frac{\tau \delta k_\delta}{k_{w,\delta} k_{n,\delta}} d\zeta \right) + (k_{n,\delta}(u^n_w) \nabla \theta_n(u^n_w), \nabla u^n_w) + (\nabla \theta_{n,\delta}(u^n_w), \frac{\tau \delta k_\delta}{k_{w,\delta} k_{n,\delta}}(u^n_w) \nabla u^n_w) = 0, \tag{2.37}
\]

and

\[
(k_{\delta,\delta}(u^n_w) \nabla p_n, \frac{\tau \delta k_\delta}{k_{w,\delta} k_{n,\delta}}(u^n_w) \nabla u^n_w) + (k_{n,\delta}(u^n_w) \nabla T_{\delta}(u^n_w) - T_{\delta}(u^{n-1}_w), \nabla \theta_n(u^n_w)) = 0. \tag{2.38}
\]

With \( G_{\alpha} \) defined in (2.3), in view of its monotonicity, we have

\[
(u^n_w - u^{n-1}_w) \cdot G_{\alpha}(u^n_w) \geq \Gamma_{\alpha}(u^n_w) - \Gamma_{\alpha}(u^{n-1}_w). \tag{2.39}
\]

Adding (2.37) and (2.38) and using (2.39) lead to

\[
\frac{1}{h} \int_{\Omega} (\Gamma_{\alpha}(u^n_w) - \Gamma_{\alpha}(u^{n-1}_w)) dx + (\nabla \theta_{n,\delta}(u^n_w), \frac{\tau \delta k_\delta}{k_{w,\delta} k_{n,\delta}}(u^n_w) \nabla u^n_w) + \frac{1}{h} (\nabla (T_{\delta}(u^n_w) - T_{\delta}(u^{n-1}_w)), \nabla \theta_n(u^n_w)) \leq 0. \tag{2.40}
\]

Summing up (2.40) for \( n = 1 \) to \( N^* \) gives

\[
\int_{\Omega} (\Gamma_{\alpha}(u^{N^*_w}) - \Gamma_{\alpha}(u^0_w)) dx + h \sum_{n=1}^{N^*} (\nabla \theta_{n,\delta}(u^n_w), \frac{\tau \delta k_\delta}{k_{w,\delta} k_{n,\delta}}(u^n_w) \nabla u^n_w) + \sum_{n=1}^{N^*} (\nabla (T_{\delta}(u^n_w) - T_{\delta}(u^{n-1}_w)), \nabla \theta_n(u^n_w)) \leq 0. \tag{2.41}
\]

Since \( \theta_{n,\delta}(u^n_w) = \frac{k_{n,\delta} k_{n,\delta}}{k_{\delta,\delta}}(u^n_w) \cdot \phi_{\alpha}(u^n_w), \) one has

\[
h \sum_{n=1}^{N^*} (\nabla \theta_{n,\delta}(u^n_w), \frac{\tau \delta k_\delta}{k_{w,\delta} k_{n,\delta}}(u^n_w) \nabla u^n_w) = \sum_{n=1}^{N^*} \left\| \nabla \phi \right\|^2. \tag{2.42}
\]

According to Lemma 2.2, we get

\[
\sum_{n=1}^{N^*} (\nabla (T_{\delta}(u^n_w) - T_{\delta}(u^{n-1}_w)), \nabla T_{\delta}(u^n_w)) = \frac{1}{2} \sum_{n=1}^{N^*} \left\| \nabla (T_{\delta}(u^n_w) - T_{\delta}(u^{n-1}_w)) \right\|^2 + \frac{1}{2} \left\| \nabla T_{\delta}(u^n_w) \right\|^2 - \frac{1}{2} \left\| \nabla T_{\delta}(u^0_w) \right\|^2. \tag{2.43}
\]

leading to

\[
\int_{\Omega} \Gamma_{\alpha}(u^{N^*_w}) + h \sum_{n=1}^{N^*} \left\| \nabla \phi \right\|^2 + \sum_{n=1}^{N^*} \left\| \nabla (T_{\delta}(u^n_w) - T_{\delta}(u^{n-1}_w)) \right\|^2 + \frac{1}{2} \left\| \nabla T_{\delta}(u^n_w) \right\|^2 \leq \Gamma_{\alpha}(u^0_w) + \frac{1}{2} \left\| \nabla T_{\delta}(u^0_w) \right\|^2. \tag{2.44}
\]
Recalling (1.21), (A1) and (A2), uniformly w.r. t. \( \delta \), it holds that

\[
\int_{\Omega} \Gamma_{\delta}(u^0) \, dx = \int_{\Omega} \int_{C_D} \int_{C_P} \frac{\tau_{\delta}k_{\delta}}{k_{\omega}} \, dz \, dv \, dx
\]

\[
= \int_{\Omega} \int_{C_D} \int_{C_P} \frac{\tau_{\delta}}{k_{\omega}} \, (z) \, dz \, dv \, dx + \int_{\Omega} \int_{C_D} \int_{C_P} \frac{\tau_{\omega}}{k_{\omega}} \, (z) \, dz \, dv \, dx
\]

\[
\leq \int_{\Omega} \int_{C_D} \int_{C_P} \frac{\tau_{\delta}}{k_{\omega}} \, (z) \, dz \, dv \, dx + \int_{\Omega} \int_{C_D} \int_{C_P} \frac{\tau_{\omega}}{k_{\omega}} \, (z) \, dz \, dv \, dx
\]

\[
= \int_{\Omega} \Gamma(u^0) \, dx
\]

\[
< \infty.
\]  

(2.45)

Furthermore, we also have

\[
\int_{\Omega} |\nabla T_{\delta}(u^0)|^2 \, dx = \int_{\Omega} \tau_{\delta}^2 |\nabla u^0|^2 \, dx \leq \int_{\Omega} \tau_{\delta}^2 |\nabla u^0|^2 \, dx = \int_{\Omega} |\nabla T(u^0)|^2 \, dx \leq C.
\]  

(2.46)

Therefore, we obtain

\[
\int_{\Omega} \Gamma_{\delta}(u^n) \, dx + \frac{1}{2} \sum_{n=1}^{N} \left| \nabla T_{\delta}(u^n) \right|^2 + \frac{1}{2} \sum_{n=1}^{N} \left| \nabla (T_{\delta}(u^n) - T_{\delta}(u^{n-1})) \right|^2 + \frac{1}{2} \sum_{n=1}^{N} \left| \nabla (u^n - u^{n-1}) \right|^2 \leq C.
\]  

(2.47)

By the definition of \( \tau_{\delta} \), this immediately gives

\[
\frac{1}{2} \left| \nabla u^n \right|^2 + \frac{1}{2} \sum_{n=1}^{N} \left| \nabla u^n \right|^2 \leq C.
\]  

(2.48)

To complete the proof, we use the Poincaré’s inequality

\[
\left| u^n \right|_{L^2(\Omega)} \leq \left| u^n \right|_{C_D(\Omega)} + \left| u^n \right|_{C_D(\Omega)} \leq C(\Omega) \left| \nabla u^n \right|_{L^2(\Omega)} + C \leq C.
\]  

(2.49)

Lemma 2.4. For any \( N^* \in 1, 2, ..., N \), we have

\[
\sum_{n=1}^{N^*} \left| \nabla (T_{\delta}(u^n) - T_{\delta}(u^{n-1})) \right|^2 + \sum_{n=1}^{N^*} \left| \nabla (u^n - u^{n-1}) \right|^2 \leq Ch,
\]  

(2.50)

\[
\sum_{n=1}^{N^*} \left| T_{\delta}(u^n) - T_{\delta}(u^{n-1}) \right|^2 + \sum_{n=1}^{N^*} \left| u^n - u^{n-1} \right|^2 \leq Ch,
\]  

(2.51)

\[
\left| \nabla p^n_\delta \right|^2 \leq C,
\]  

(2.52)

where \( C \) is independent on \( h \).

Proof. Testing in both (2.13) and (2.14) with \( h(T_{\delta}(u^n) - T_{\delta}(u^{n-1})) \), adding the resulting gives

\[
(u^n - u^{n-1}, T_{\delta}(u^n) - T_{\delta}(u^{n-1})) + h(k_{\omega}(u^n) \nabla p^n_\delta, \nabla (T_{\delta}(u^n) - T_{\delta}(u^{n-1})))
\]

\[
+ \left| \nabla k_{\omega}(u^n) \nabla (T_{\delta}(u^n) - T_{\delta}(u^{n-1})) \right|^2 + h(\nabla \theta_\delta(u^n), \nabla (T_{\delta}(u^n) - T_{\delta}(u^{n-1}))) = 0.
\]  

(2.53)
With $\psi = h p_{\delta}^n$ in (2.14), we have
\[
\begin{align*}
\quad h \left\| \sqrt{k_{\delta}(u_{\delta}^n)} \nabla p_{\delta}^n \right\|^2 &= -(k_{\omega \delta}(u_{\delta}^n) \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})), \nabla p_{\delta}^n) \\
&\leq \left\| \frac{k_{\omega \delta}(u_{\delta}^n)}{\sqrt{k_{\delta}(u_{\delta}^n)}} \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right\| \left\| \sqrt{k_{\delta}(u_{\delta}^n)} \nabla p_{\delta}^n \right\|, 
\end{align*}
\]
giving
\[
\begin{align*}
- h(k_{\omega \delta}(u_{\delta}^n) \nabla p_{\delta}^n, \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1}))) &\leq \left\| \frac{k_{\omega \delta}(u_{\delta}^n)}{\sqrt{k_{\delta}(u_{\delta}^n)}} \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right\|^2. 
\end{align*}
\]
Then, (2.53) becomes
\[
\begin{align*}
(u_{\delta}^n - u_{\delta}^{n-1}, T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) + \left\| \frac{k_{\omega \delta}(u_{\delta}^n)}{\sqrt{k_{\delta}(u_{\delta}^n)}} \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right\|^2 &+ h(\nabla \theta(u_{\delta}^n), \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1}))) \\
&\leq \left\| \frac{k_{\omega \delta}(u_{\delta}^n)}{\sqrt{k_{\delta}(u_{\delta}^n)}} \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right\|^2. 
\end{align*}
\]
In view of the definition $k_{\delta}$, one has
\[
\begin{align*}
\left\| \frac{k_{\omega \delta}(u_{\delta}^n)}{\sqrt{k_{\delta}(u_{\delta}^n)}} \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right\|^2 &- \left\| \frac{k_{\omega \delta}(u_{\delta}^n)}{\sqrt{k_{\delta}(u_{\delta}^n)}} \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right\|^2 \\
&= \left\| \frac{k_{\omega \delta}(u_{\delta}^n)k_{\delta}}{k_{\delta}} (u_{\delta}^n) \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right\|^2. 
\end{align*}
\]
Further, since
\[
\begin{align*}
h \left| \nabla \theta(u_{\delta}^n), \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right| &\leq \frac{1}{2} \left\| \frac{k_{\omega \delta}(u_{\delta}^n)k_{\delta}}{k_{\delta}} (u_{\delta}^n) \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right\|^2 \\
&+ \frac{h^2}{2} \left\| \frac{k_{\omega \delta}(u_{\delta}^n)k_{\delta}}{k_{\delta}} (u_{\delta}^n) p_{\delta}(u_{\delta}^n) \nabla u_{\delta}^n \right\|^2, 
\end{align*}
\]
(2.56) yields
\[
\begin{align*}
(u_{\delta}^n - u_{\delta}^{n-1}, T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1}))+ \frac{1}{2} \left\| \frac{k_{\omega \delta}(u_{\delta}^n)k_{\delta}}{k_{\delta}} (u_{\delta}^n) \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right\|^2 \\
&\leq \frac{h^2}{2} \left\| \frac{k_{\omega \delta}(u_{\delta}^n)k_{\delta}}{k_{\delta}} (u_{\delta}^n) p_{\delta}(u_{\delta}^n) \nabla u_{\delta}^n \right\|^2 \\
&\leq \frac{h^2}{2} \left\| \frac{k_{\omega \delta}(u_{\delta}^n)k_{\delta}}{k_{\delta}} (u_{\delta}^n) \nabla u_{\delta}^n \right\|^2. 
\end{align*}
\]
By (2.36) and (A1) and (A2), we have
\[
(u_{\delta}^n - u_{\delta}^{n-1}, T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1}))+ \left\| \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right\|^2 \leq C h^2. 
\]
This leads to
\[
\sum_{n=1}^{N} (u_{\delta}^n - u_{\delta}^{n-1}, T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1}))+ \sum_{n=1}^{N} \left\| \nabla (T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) \right\|^2 \leq C h. 
\]
By the definition of $T$, a $\xi$ exists, such that
\[
(u_{\delta}^n - u_{\delta}^{n-1}, T_{\delta}(u_{\delta}^n) - T_{\delta}(u_{\delta}^{n-1})) = \left\| \nabla \tau_{\delta}(\xi)(u_{\delta}^n - u_{\delta}^{n-1}) \right\|^2 \leq C h^2. 
\]
Similarly, a \( \tilde{\xi} \) exists, such that
\[
(u_\sigma^n - u_\sigma^{n-1}, T_\sigma(u_\sigma^n) - T_\sigma(u_\sigma^{n-1})) = \left\| \frac{1}{\sqrt{\tau_\sigma(\xi)}} (T_\sigma(u_\sigma^n) - T_\sigma(u_\sigma^{n-1})) \right\|^2 \leq Ch^2.
\] (2.63)

Then we obtain
\[
\sum_{n=1}^{N} \left\| T_\sigma(u_\sigma^n) - T_\sigma(u_\sigma^{n-1}) \right\|^2 + \sum_{n=1}^{N} \left\| u_\sigma^n - u_\sigma^{n-1} \right\|^2 \leq Ch.
\] (2.64)

Furthermore, according to (A1), (2.36) and (2.60), one also has
\[
\sum_{n=1}^{N} \left\| \nabla (u_\sigma^n - u_\sigma^{n-1}) \right\|^2 \leq Ch,
\] (2.65)

Finally, (2.52) follows from (2.54) and (2.60).

2.4. Existence of weak solutions to the regularized problem

To show the existence of a solution to Problem \( P_\sigma \), we consider linear interpolation in time:
\[
T_{\sigma N}(t) = T_\sigma(u_\sigma^{n-1}) + \frac{t - \tau_{n-1}}{h} (T_\sigma(u_\sigma^n) - T_\sigma(u_\sigma^{n-1})),
\] (2.66)
\[
U_{\sigma N}(t) = u_\sigma^{n-1} + \frac{t - \tau_{n-1}}{h} (u_\sigma^n - u_\sigma^{n-1}),
\] (2.67)
and piecewise constant functions in time
\[
T_{\sigma N}(t) = T_\sigma(u_\sigma^n), \quad U_{\sigma N}(t) = u_\sigma^n, \quad P_{\sigma N}(t) = p_\sigma^n,
\] (2.68)
for \( t \in (\tau_{n-1}, \tau_n), n = 1, 2, ..., N \). Clearly, \( T_{\sigma N} \in L^2(\Omega), \quad U_{\sigma N} \in L^2(0, T_M; V), \quad P_{\sigma N}(t) \in L^2(0, T_M; W_0^{1,2}(\Omega)) \).

We have the following result:

**Theorem 2.1.** Problem \( P_\sigma \) has a solution \((u_\sigma, p_\sigma)\).

**Proof.** According to the priori estimates in Lemma 2.3 and Lemma 2.4, we have
\[
\int_0^{T_M} \left\| T_{\sigma N}(t) \right\|_{L^2(\Omega)}^2 \, dt = \frac{N}{2} \int_{\tau_{n-1}}^{\tau_n} \left\| T_\sigma(u_\sigma^{n-1}) + \frac{t - \tau_{n-1}}{h} (T_\sigma(u_\sigma^n) - T_\sigma(u_\sigma^{n-1})) \right\|_{L^2(\Omega)}^2 \, dt
\]
\[
\leq 2 \sum_{n=1}^{N} \int_{\tau_{n-1}}^{\tau_n} \left( \left\| T_\sigma(u_\sigma^{n-1}) \right\|_{L^2(\Omega)}^2 + \left\| T_\sigma(u_\sigma^n) - T_\sigma(u_\sigma^{n-1}) \right\|_{L^2(\Omega)}^2 \right) \, dt \leq C,
\] (2.69)
\[
\int_0^{T_M} \left\| \nabla T_{\sigma N}(t) \right\|_{L^2(\Omega)}^2 \, dt \leq C,
\] (2.70)
\[
\int_0^{T_M} \left\| \partial_t U_{\sigma N}(t) \right\|_{L^2(\Omega)}^2 \, dt = \frac{1}{h} \sum_{n=1}^{N} \left\| u_\sigma^n - u_\sigma^{n-1} \right\|_{L^2(\Omega)}^2 \leq C,
\] (2.71)
\[
\int_0^{T_M} \left\| \partial_t T_{\sigma N}(t) \right\|_{L^2(\Omega)}^2 \, dt = \frac{1}{h} \sum_{n=1}^{N} \left\| T_\sigma(u_\sigma^n) - T_\sigma(u_\sigma^{n-1}) \right\|_{L^2(\Omega)}^2 \leq C,
\] (2.72)
and
\[
\int_0^{T_u} \| \partial_t \nabla T_{\delta}(t) \|^2_{L^2(\Omega)} \, dt = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left\| \frac{1}{h} \nabla \left( T_{\delta}(u_\delta^n) - T_{\delta}(u_\delta^{n-1}) \right) \right\|^2_{L^2(\Omega)} \, dt \\
= \frac{1}{h} \sum_{n=1}^N \left\| \nabla \left( T_{\delta}(u_\delta^n) - T_{\delta}(u_\delta^{n-1}) \right) \right\|^2_{L^2(\Omega)} \leq C. \quad (2.73)
\]

In the same way, one obtains similar estimates
\[
\int_0^{T_u} \| U_{\delta}(t) \|^2_{L^2(\Omega)} \, dt + \int_0^{T_u} \| U_{\delta}(t) \|^2_{L^2(\Omega)} \, dt + \int_0^{T_u} \| \partial_t U_{\delta}(t) \|^2_{L^2(\Omega)} \, dt + \int_0^{T_u} \| \partial_t U_{\delta}(t) \|^2_{L^2(\Omega)} \, dt \leq C. \quad (2.74)
\]

Therefore, the sequences \( \{ T_{\delta}(t) \}_{\delta \in \mathbb{N}} \) and \( \{ U_{\delta}(t) \}_{\delta \in \mathbb{N}} \) are uniformly bounded in \( W^{1,2}(0, T_u; W^{1,2}(\Omega)) \), so there exist two sub-sequences (still denoted by \( T_{\delta}(t) \) and \( U_{\delta}(t) \)) which converge weakly to some \( T_\ast(t) \in W^{1,2}(0, T_u; W^{1,2}(\Omega)) \) and \( u_\delta \in W^{1,2}(0, T_u; V) \).

For any \( \phi, \psi \in L^2(0, T_u; W^{1,2}(\Omega)) \), (2.13)-(2.14) give
\[
\int_0^{T_u} \int_\Omega \partial_t U_{\delta}(t) \phi \, dx dt - \int_0^{T_u} \int_\Omega k_{\ast}(\bar{U}_{\delta}(t)) \nabla \bar{P}_{\delta}(t) \phi \, dx dt + \int_0^{T_u} \int_\Omega k_{\ast}(\bar{U}_{\delta}(t)) \nabla \psi \phi \, dx dt = 0, \quad (2.75)
\]
\[
\int_0^{T_u} \int_\Omega k_{\ast}(\bar{U}_{\delta}(t)) \nabla \bar{P}_{\delta}(t) \phi \, dx dt + \int_0^{T_u} \int_\Omega k_{\ast}(\bar{U}_{\delta}(t)) \nabla \partial_t \bar{U}_{\delta}(t) \psi \, dx dt = 0. \quad (2.76)
\]

Clearly, \( T_{\delta}(t) \to T_\ast(t) \) strongly in \( L^2(Q) \). By Lemma 3.2 in [26], it follows that \( T_{\delta}(t) \to T_\ast(t) \) strongly in \( L^2(Q) \) as well and a similar conclusion can be drawn for \( \bar{U}_{\delta}(t) \) and \( u_\delta \). By the continuity of \( k_{\ast}, k_{\ast}, k_{\ast} \), we also have \( k_{\ast}(\bar{U}_{\delta}(t)) \to k_{\ast}(u_\delta) \), \( k_{\ast}(\bar{U}_{\delta}(t)) \to k_{\ast}(u_\delta) \), and \( k_{\ast}(\bar{U}_{\delta}(t)) \to k_{\ast}(u_\delta) \). Now we show that \( \partial_t \bar{U}_{\delta} = \partial_t \bar{U}_{\delta} \).

Since \( T_{\delta}(t) = T_{\delta}(u_\delta) = T_{\delta}(\bar{U}_{\delta}(t)) \) converges to \( T_\ast(t) \), by the definition of \( T_\ast(t) \), we also find that \( T_{\delta}(\bar{U}_{\delta}(t)) \to T_\ast(u_\delta) \), then we have \( T_\ast(t) = T_\ast(u_\delta) \).

Similarly, for \( \bar{P}_{\delta}(t) \), we have a \( p_\ast(t) \) such that
\[
\nabla \bar{P}_{\delta}(t) \to \nabla P_\ast \quad \text{weakly in } L^2(0, T_u; L^2(\Omega)) \).
\]

As in the proof of Lemma 2.1, we get
\[
k_{\ast}(\bar{U}_{\delta}(t)) \nabla \bar{P}_{\delta}(t) \to k_{\ast}(u_\delta) \nabla p_\ast \quad \text{weakly in } L^2(0, T_u; L^2(\Omega)),
k_{\ast}(\bar{U}_{\delta}(t)) \nabla \bar{P}_{\delta}(t) \to k_{\ast}(u_\delta) \nabla p_\ast \quad \text{weakly in } L^2(0, T_u; L^2(\Omega)),
k_{\ast}(\bar{U}_{\delta}(t)) \nabla \partial_t \bar{U}_{\delta}(t) \to k_{\ast}(u_\delta) \nabla \partial_t \bar{U}_{\delta}(t) \quad \text{weakly in } L^2(0, T_u; L^2(\Omega)),
k_{\ast}(\bar{U}_{\delta}(t)) \nabla \partial_t \bar{U}_{\delta}(t) \to k_{\ast}(u_\delta) \nabla \partial_t \bar{U}_{\delta}(t) \quad \text{weakly in } L^2(0, T_u; L^2(\Omega)).
\]

Combining the results, we obtain that \( (u_\delta, p_\ast) \) is the solution pair of Problem \( P_\delta \).

\[
2.5. \quad \textbf{Uniqueness of Problem } P_\delta
\]

After having obtained the existence of a weak solution, we show its uniqueness. To do so, we consider the following system:
\[
\partial_t s - \nabla \cdot (k_{\ast}(s) \nabla p_\ast) = 0, \quad (2.77)
\]
\[
-\partial_t s - \nabla \cdot (k_{\ast}(s) \nabla p_\ast) = 0, \quad (2.78)
\]
\[
p_\ast - p_\ast = p_{\ast}(s) - p_{\ast}(C_D) - \partial_t \bar{s}(s). \quad (2.79)
\]

Formally, this is equivalent to (2.6) - (2.7), in the sense that \( u_\delta = s \) and the global pressure \( p \) is given by (1.9) or (1.10), often finding \( p_{\ast}, p_\ast \) in (2.77) - (2.79).

Clearly, the initial and boundary conditions should be compatible with the original ones:
\[
s(0, \cdot) = u^0_\ast \quad \text{in } \Omega, \quad (2.80)
\]
Proof. The proof follows the ideas in [19], see [29] for the underlying ideas.

Problem P: Given $s(0, \cdot) = u^0$, find $p_n \in L^2(0, T_M; W^{1,2}_0(\Omega))$, $p_n \in L^2(0, T_M; W^{1,2}_0(\Omega))$ and $s \in L^2(0, T_M; L^2(\Omega))$, such that

\begin{align*}
  (\partial_t s, \phi) + (k_{\alpha\phi}(s) \nabla p_n, \nabla \phi) &= 0, \quad (2.82) \\
  -(\partial_t s, \psi) - (k_{\alpha\psi}(s) \nabla p_n, \nabla \psi) &= 0, \quad (2.83) \\
  (p_n - p_n(x, \cdot), \rho) &= (p_{\alpha\rho}(s) - p_{\alpha\rho}(C_D), \rho) - (\partial_t T_{\alpha}(s, \cdot), \rho), \quad (2.84)
\end{align*}

for all $\phi, \psi \in L^2(0, T; W^{1,2}_0(\Omega))$ and $\rho \in L^2(0, T; L^2(\Omega))$.

The equivalence above can be made more precisely.

Lemma 2.5. Problem $P_e$ and Problem $P_{\delta}$ are equivalent. Specifically, $(u_{\delta}, p_{\delta})$ is a solution to Problem $P_{\delta}$ if and only if $(s, p_{n}, \bar{p}_n)$ solves Problem $P_e$, with $s = u_{\delta}$, $p_n = p_{\delta} + \int_{C_D} f_{\alpha\delta}(z) p_{\alpha\delta}(z) dz - p_{\delta}(u_{\delta}) + p_{\alpha\delta}(C_D) + \tau(u_{\delta}) \partial_t u_{\delta}$, $p_n = p_{\delta} + \int_{C_D} f_{\alpha\delta}(z) p_{\alpha\delta}(z) dz$.

Proof. The proof follows the ideas in [19], see [29] for the underlying ideas.

Theorem 2.2. There exists at most one solution $(u_{\delta}, p_{\delta})$ for Problem $P_{\delta}$.

Proof. Uniqueness for Problem $P_e$ is proved in [12]. Then, by equivalence, Problem $P_{\delta}$ has at most one solution.

3. Existence of Problem P

Below we extend the existence result to a class of degenerate equations. Specifically, we assume the following wing asymptotic behaviors for the nonlinear functions $k_{\alpha\phi}, k_{\alpha\psi}, p_{\alpha\rho}$ and $\tau$.

A5: There exist $\alpha > 0$, $\beta > 0$, $\lambda > 0$, $\omega > 0$, and for different dimensions, we assume the followings

\begin{align*}
  d = 3: & \quad \alpha \geq \lambda > \alpha/3 + 10/3, \quad \omega > 5/2, e := \omega + \beta > 5, \\
  d = 2 \text{ or } d = 1: & \quad \alpha \geq \lambda > 4, \quad \omega > 2, e > 4,
\end{align*}

and the followings hold

\begin{align*}
  \lim_{u \to 0} k_{\alpha\phi}(u) \cdot u^{-\alpha} &= C_1, \quad (3.1) \\
  \lim_{u \to 0} p_{\alpha\rho}'(u) \cdot u^2 &= C_2, \quad (3.2) \\
  \lim_{u \to 1} k_{\alpha\phi}(u) \cdot (1 - u)^{-\beta} &= C_3, \quad (3.3) \\
  \lim_{u \to 1} \tau(u) \cdot (1 - u)^{-\omega} &= C_4, \quad (3.4)
\end{align*}

for some constants $C_1, C_2, C_3, C_4 \in (0, +\infty)$.

To avoid unnecessary technical complications, we restrict the proof to the cases $k_{\alpha\phi} = u^\alpha$, $k_{\alpha\psi} = (1 - u)^\beta$, $p_{\alpha\rho}'(u) = -u^4$, and $\tau(u) = (1 - u)^{-\omega}$. We note that these nonlinearities are commonly encountered in the porous media literature [4, 7]. Less standard is the function $\tau$. We refer to [6, 9], where this type of behavior is proposed.

Since in (A1) we assume that the product $k_{\alpha\phi} \cdot p_{\alpha\rho}'$ is uniformly bounded in $\mathbb{R}$, this implies $\alpha \geq \lambda$. According to the definition of $\tau$ and its regularization, one has $T(u)$ and $T_{\delta}(u)$ as followings:

\begin{align*}
  T(u) = \begin{cases} 
  u + \frac{1}{\omega - 1}, & \text{if } u < 0, \\
  \frac{1}{\omega - 1} (1 - u)^{1 - \omega}, & \text{if } 0 \leq u < 1, \\
  +\infty, & \text{if } u > 1,
  \end{cases}
\end{align*}

(3.5)
Let suppose the hypotheses (Lemma 3.1. Then we have the following results

Here we define the characteristic function

\[ \chi_{(a,b)} = \begin{cases} 1, & \text{if } v \in (a, b), \\ 0, & \text{if } v \notin (a, b). \end{cases} \]  

(3.7)

Then we have the following results

**Lemma 3.1.** Let suppose the hypotheses (A1), (A2) and (A5). Then there exists a constant \( C > 0 \), independent on \( \delta \), such that for the first component of the weak solution pair \((u_0, p_0)\) to Problem \( P_0 \), one has

\[ \|((u_0) + \delta)^{2-\omega}\|_{L^2(0,T,\mathbb{C}^1(\Omega))} \leq C, \]  

(3.8)

\[ \|((1 - u_0) + \delta)^{2-\omega}\|_{L^2(0,T,\mathbb{C}^1(\Omega))} \leq C, \]  

(3.9)

\[ \|\nabla T_\delta(u_0)\|_{L^2(0,T;\mathbb{C}^1(\Omega))} + \|\nabla u_0\|_{L^2(0,T;\mathbb{C}^1(\Omega))} \leq C, \]  

(3.10)

\[ \left\| \sqrt{-p'_{\alpha}T_\delta \nabla u_0} \right\|_{L^2(0,T;\mathbb{C}^1(\Omega))} \leq C. \]  

(3.11)

**Proof.** With \( t \in (0, T_M) \), taking the test function \( \phi = \chi_{(0,t)} \int_{C_0}^{u_0} \frac{\tau_0 k_0}{k_0 n_0} (z) dz \) in (2.10) and \( \psi = \chi_{(0,t)} \int_{C_0}^{u_0} \frac{\tau_0}{k_0} (z) dz \) in (2.11) gives

\[ \int_0^t (\partial_t u_0, \int_{C_0}^{u_0} \frac{\tau_0 k_0}{k_0 n_0} (z) dz) dt - \int_0^t (\nabla p_0, \frac{\tau_0 k_0}{k_0 n_0} \nabla u_0) dt + \int_0^t \left\| \sqrt{-p'_{\alpha}T_\delta \nabla u_0} \right\|^2 dt = 0, \]  

(3.12)

\[ \int_0^t (\nabla p_0, \frac{\tau_0 k_0}{k_0 n_0} \nabla u_0) dt + \int_0^t (\nabla T_\delta(u_0), \nabla T_\delta(u_0)) dt = 0. \]  

(3.13)

Adding (3.12) and (3.13) gives

\[ \int_0^t \int_\Omega (\partial_t u_0, \int_{C_0}^{u_0} \frac{\tau_0 k_0}{k_0 n_0} (z) dz) dxdz + \int_0^t \left\| \sqrt{-p'_{\alpha}T_\delta \nabla u_0} \right\|^2 dt + \frac{1}{2} \int_0^t \int_\Omega \frac{d}{dt} \|\nabla T_\delta(u_0)\|^2 dt = 0. \]  

(3.14)

By the definition of \( k_\delta \), one has

\[ \int_\Omega (\partial_t u_0, \int_{C_0}^{u_0} \frac{\tau_0 k_0}{k_0 n_0} (z) dz) dxdz = \int_\Omega (\partial_t u_0, \int_{C_0}^{u_0} \tau_\delta (z) dz) dxdz + \int_\Omega (\partial_t u_0, \int_{C_0}^{u_0} \tau_\delta (z) dz) dxdz. \]  

(3.15)

The following identities hold a.e.

\[ \partial_t u_0 \int_{C_0}^{u_0} \tau_\delta (z) dz = \partial_t \left( u_0 \int_{C_0}^{u_0} \tau_\delta (z) dz - \int_{C_0}^{u_0} \tau_\delta (z) dz \right) \]  

(3.16)

\[ \partial_t u_0 \int_{C_0}^{u_0} \tau_\delta (z) dz + (u_0 - 1) \int_{C_0}^{u_0} \tau_\delta (z) dz - \int_{C_0}^{u_0} \tau_\delta (z) dz \]  

(3.17)
As in [27], we define the functions \( E_{\tilde{w}d}, E_{\tilde{w}o} : \mathbb{R} \to \mathbb{R} \)

\[
\tilde{E}_{w}(y) = \int_{C_{\xi}} \int_{k_{w}} \frac{\tau_{d}}{1-(1-z)^{2}} \text{d}z \text{d}v + \frac{(C_{D} + \delta)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \frac{(C_{D} + \delta)^{1-\alpha}}{\alpha-1} C_{D}
\]

\[
\tilde{E}_{w}(y) = \int_{y}^{\gamma} \int_{C_{\xi}} \int_{k_{w}} \frac{\tau_{d}}{1-(1-z)^{2}} \text{d}z \text{d}v + \frac{(C_{D} + \delta)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \frac{(C_{D} + \delta)^{1-\alpha}}{\alpha-1} C_{D},
\]  

(3.18)

\[
\tilde{E}_{\tilde{w}}(y) = \int_{C_{\xi}} \int_{k_{w}} \frac{\tau_{d}}{1-(1-z)^{2}} \text{d}z \text{d}v + \frac{(C_{D} + \delta)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \frac{(C_{D} + \delta)^{1-\alpha}}{\alpha-1} C_{D},
\]

(3.19)

and

\[
\tilde{E}_{\tilde{w}}(y) = \int_{y}^{\gamma} \int_{C_{\xi}} \int_{k_{w}} \frac{\tau_{d}}{1-(1-z)^{2}} \text{d}z \text{d}v + \frac{(C_{D} + \delta)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \frac{(C_{D} + \delta)^{1-\alpha}}{\alpha-1} C_{D},
\]  

(3.20)

The choice of these terms is justified by the following calculations.

Recalling (A5), we have

\[
\int_{C_{\xi}} \int_{C_{\xi}} \frac{1}{k_{w}} (z) \text{d}z \text{d}v = \frac{(C_{D} + \delta)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \frac{(C_{D} + \delta)^{1-\alpha}}{\alpha-1} C_{D}.
\]

Similarly, for \( u_{\delta} < 0 \), one has

\[
\int_{C_{\xi}} \int_{C_{\xi}} \frac{1}{k_{w}} (z) \text{d}z \text{d}v = \frac{(C_{D} + \delta)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \frac{(C_{D} + \delta)^{1-\alpha}}{\alpha-1} C_{D},
\]

(3.22)

and for \( u_{\delta} > 1 - \delta \), we get

\[
\int_{C_{\xi}} \int_{C_{\xi}} \frac{1}{k_{w}} (z) \text{d}z \text{d}v = \frac{(C_{D} + \delta)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \frac{(C_{D} + \delta)^{1-\alpha}}{\alpha-1} C_{D},
\]

(3.23)

Note that the calculations above hold for the choice \( k_{u}(\alpha) = u^{-\alpha} \). If, instead, \( k_{w} \) behaves like in (3.1), there the expressions on the right in the above are dominating terms, the reminders being regular w.r.t \( \delta \).

In this way \( \tilde{E}_{w}(u_{\delta}) \) becomes

\[
\tilde{E}_{\tilde{w}}(u_{\delta}) = \begin{cases} 
\frac{\delta^{-\alpha} u_{\delta}^{2} + (C_{D} + \delta)^{1-\alpha} u_{\delta}}{\alpha-1} + \frac{\delta^{-\alpha}}{(1-\alpha)(2-\alpha)}, & \text{for } u_{\delta} < 0, \\
\frac{\delta^{-\alpha} u_{\delta}^{2} + (C_{D} + \delta)^{1-\alpha} u_{\delta}}{\alpha-1} + \frac{1}{(1-\alpha)(2-\alpha)}, & \text{for } 0 \leq u_{\delta} \leq 1 - \delta, \\
\frac{\delta^{-\alpha}}{(1-\alpha)(2-\alpha)}, & \text{for } u_{\delta} > 1 - \delta.
\end{cases}
\]

(3.24)

We note that

\[
\tilde{E}_{w}(u_{\delta}) \geq \tilde{E}_{w}^{0}(u_{\delta}) := \begin{cases} 
\frac{\delta^{-\alpha} u_{\delta}^{2} + (C_{D} + \delta)^{1-\alpha} u_{\delta}}{(1-\alpha)(2-\alpha)}, & \text{for } u_{\delta} < 0, \\
\frac{1}{2} \left( \frac{u_{\delta}^{2}}{(1-\alpha)(2-\alpha)} \right) + \frac{1}{(1-\alpha)(2-\alpha)}, & \text{for } 0 \leq u_{\delta} \leq 1 - \delta, \\
\frac{1}{2} + \frac{1}{(1-\alpha)(2-\alpha)}, & \text{for } u_{\delta} > 1 - \delta.
\end{cases}
\]

(3.25)

whereas \( \tilde{E}_{w}^{0}(u_{\delta}) \geq C(u_{\delta} + \delta)^{2-\alpha} (\alpha > 0 \text{ independent on } \delta) \).

In the same way, for \( \tilde{E}_{w}^{0} \), since \( \frac{\tau_{d}}{k_{w}(\alpha)} = (1 - z + \delta)^{-\gamma} \) (see (A5)), for \( \delta \leq u_{\delta} \leq 1 \), one has

\[
\int_{1}^{u_{\delta}} \int_{C_{\xi}} \frac{\tau_{d}}{k_{w}(\alpha)} \text{d}z \text{d}v = \frac{(1 - u_{\delta} + \delta)^{-\gamma}}{(1-\alpha)(2-\alpha)} + \frac{(1 - C_{D} + \delta)^{-\gamma}}{\alpha-1}(1 - u_{\delta}) - \frac{\delta^{-\alpha}}{(1-\alpha)(2-\alpha)}.
\]

(3.26)
Further, for \( u_\delta < \delta \), one gets
\[
\int_1^u \int_{C_0} \frac{\tau_\delta}{k_{\delta}} dz dv = \frac{(u_\delta - \delta)^2}{2} + \frac{1}{(1 - e)(2 - e)} + \frac{1 - \delta}{e - 1} + (1 + \delta - C_D)^{1-a} - 1\frac{1 - u_\delta}{(1 - e)(2 - e)}. \tag{3.27}
\]
and for \( u_\delta > 1 \), we have
\[
\int_1^u \int_{C_0} \frac{\tau_\delta}{k_{\delta}} dz dv = \frac{\delta^2 - e}{2} (u_\delta - 1)^2 + (\frac{\delta^2 - e}{e - 1} - (1 + \delta - C_D)^{1-a}\frac{1}{e - 1})(u_\delta - 1). \tag{3.28}
\]
Then \( E_{\delta}(v) \) rewrites
\[
E_{\delta}(u_\delta) = \begin{cases} (\frac{u_\delta - \delta^2}{2} + \frac{1}{(1 - e)(2 - e)} + (\frac{\delta^2 - e}{e - 1} + (1 + \delta - C_D)^{1-a} - 1)(1 - u_\delta), & \text{for } u_\delta < \delta, \\ (\frac{1}{(1 - e)(2 - e)} + (\frac{\delta^2 - e}{e - 1} + (1 + \delta - C_D)^{1-a}) (u_\delta - 1) + (\frac{\delta^2 - e}{(1 - e)(2 - e)}), & \text{for } u_\delta > 1, \\ \end{cases} \tag{3.29}
\]
so
\[
E_{\delta}(u_\delta) \geq E_{\delta}^0(u_\delta) := \begin{cases} (\frac{u_\delta - \delta^2}{2} + \frac{1}{(1 - e)(2 - e)} + (\frac{\delta^2 - e}{e - 1} + (1 + \delta - C_D)^{1-a}) (u_\delta - 1), & \text{for } u_\delta < \delta, \\ (\frac{1}{(1 - e)(2 - e)} + (\frac{\delta^2 - e}{e - 1} + (1 + \delta - C_D)^{1-a}) (u_\delta - 1), & \text{for } u_\delta > 1, \\ \end{cases} \tag{3.30}
\]
and
\[
E_{\delta}^0(u_\delta) \geq C[|1 - u_\delta| + \delta^2 - e], \tag{3.31}
\]
with \( C > 0 \) independent on \( \delta \).

Substitute \( E_{\delta} + E_{\delta} \) into (3.14) instead of \( \int \int_{k_{\delta}} \frac{\tau_\delta}{k_{\delta}} (z) dz dx \), we have
\[
\int \Omega E_{\delta}(u_\delta(t)) dx + \int \Omega E_{\delta}(u_\delta(t)) dx + \int_0^t \| \nu_\delta \tau_\delta \nabla u_\delta \|^2 dt + \frac{1}{2} \| \nabla T_\delta(u_\delta(t)) \|^2 = \int \Omega E_{\delta}(u_\delta) dx + \int \Omega E_{\delta}(u_\delta) dx + \frac{1}{2} \| \nabla T_\delta(u_\delta) \|^2. \tag{3.32}
\]
As the proof in Lemma 2.3, one has
\[
T_\delta(u_\delta) = \int_{C_\delta} \int_{k_{\delta}} \tau_\delta (z) dz dv + \int_{C_\delta} \int_{k_{\delta}} \tau_\delta (z) dz dv \leq C, \tag{3.33}
\]
and
\[
\int \Omega \| \nabla T_\delta(u_\delta) \|^2 dx = \int \Omega \tau_\delta \nabla u_\delta \|^2 dx \leq \int \Omega \tau_\delta \nabla u_\delta \|^2 dx = \int \Omega |\nabla T(u_\delta)|^2 dx \leq C. \tag{3.34}
\]
These lead to
\[
\int \Omega E_{\delta}(u_\delta) dx = \int \Omega \int_{C_\delta} \int_{k_{\delta}} \tau_\delta (z) dz dv + \int \Omega \frac{(C_D + \delta)^{2-a}}{(\alpha - 1)} + \frac{(C_D + \delta)^{1-a} C_D}{(\alpha - 1)} dx \leq C. \tag{3.35}
\]
Similarly, we also have
\[
\int \Omega E_{\delta}(u_\delta) dx = \int \Omega \int_{C_\delta} \int_{k_{\delta}} \tau_\delta (z) dz dv + \int_\Omega \frac{\delta^2 - e}{(1 - e)(2 - e)} dx \leq \int \Omega \int_{C_\delta} \int_{k_{\delta}} \tau_\delta (z) dz dv + \int_\Omega (\frac{1}{(1 - e)(2 - e)} dx + \int \Omega \frac{(C_D + \delta)^{2-e} - C_D}{(e - 1)} dx + \int_\Omega \frac{(C_D + \delta)^{1-a} (1 - C_D)}{(1 - e)(2 - e)} dx \leq C. \tag{3.36}
\]
Further, observe that $E_{u_0}$ is convex, allowing a minimum at $u_0 = C_D$, then we have
$E(u) \geq \frac{(C_A \phi^2 C_D^2)}{(1 - \alpha q^2 a)} + \frac{(C_A \phi^2 C_D^2)}{q^2 a - 1} C_D > 0$.
And $E_{u_0}$ is also positive obtained from (3.30) and (3.31). These give the estimates from (3.32):
\[ \int_\Omega E_{u_0}(u_0) + \int_\Omega E_{\bar{u}_0}(u_0) \leq C. \] (3.37)

Since $\tau_\delta$ is far away from 0, then one has
\[ \int_\Omega \bar{E}_{u_0}(u_0) \leq C. \] (3.38)

These lead to
\[ \int_0^T \| \sqrt{-p_\delta T_\delta \nabla u_0} \|^2 dt \leq C, \] (3.39)
\[ \int_\Omega (|u_0| + \tilde{\delta})^{2-\alpha} dx \leq C, \] (3.40)
\[ \int_\Omega (\|1 - u_0\| + \tilde{\delta})^{2-\epsilon} dx \leq C, \] (3.41)
\[ \|\nabla T_\delta(u_0)\|^2 + \|\nabla u_0\|^2 \leq C. \] (3.42)

**Lemma 3.2.** Under the assumptions (A1), (A2) and (A5), there exists a constant $C > 0$, independent on $\delta$, such that the weak solution pair $(u_0, p_0)$ of Problem $P_\delta$ satisfies:
\[ \int_0^T \| \partial_\delta u_0 \|^2 dt + \int_0^T \| \frac{1}{\sqrt{T_\delta(u_0)}} \partial_\delta T_\delta(u_0) \|^2 dt + \int_0^T \| \sqrt{k_{w_0}} \partial_\delta \nabla T_\delta(u_0) \|^2 dt \leq C, \] (3.43)
\[ \int_0^T \| \sqrt{k_{w_0}} \nabla p_0 \|^2 + \int_0^T \| \sqrt{k_{w_0}} \nabla (p_\delta + \partial_\delta T_\delta(u_0)) \|^2 \leq C. \] (3.44)

**Proof.** Testing by $\partial_\delta T_\delta(u_0)$ both in (2.10) and (2.11), adding the resulting gives
\[ \int_0^T (\partial_\delta u_0, \partial_\delta T_\delta(u_0)) dt + \int_0^T (k_{w_0} \nabla p_0, \nabla \partial_\delta T_\delta(u_0)) dt + \int_0^T (\nabla \theta_0(u_0), \nabla \partial_\delta T_\delta(u_0)) dt + \int_0^T \| \sqrt{k_{w_0}} \nabla \partial_\delta T_\delta(u_0) \|^2 dt = 0. \] (3.45)

Further, taking $\psi = p_0$ in (2.11) gives
\[ \| \sqrt{k_0} \nabla p_0 \|^2 = -(k_{w_0} \nabla \partial_\delta T_\delta(u_0), \nabla p_0) \leq \left\| \frac{k_{w_0}}{\sqrt{k_0}} \nabla \partial_\delta T_\delta(u_0) \right\| \left\| \sqrt{k_0} \nabla p_0 \right\|, \] (3.46)
implying
\[ -(k_{w_0} \nabla p_0, \nabla \partial_\delta T_\delta(u_0)) \leq \left\| \frac{k_{w_0}}{\sqrt{k_0}} \nabla \partial_\delta T_\delta(u_0) \right\|^2. \] (3.47)

Then (3.45) becomes
\[ \int_0^T (\partial_\delta u_0, \partial_\delta T_\delta(u_0)) dt + \int_0^T \left\| \sqrt{k_{w_0}} \nabla \partial_\delta T_\delta(u_0) \right\|^2 dt + \int_0^T (\nabla \theta_0(u_0), \nabla \partial_\delta T_\delta(u_0)) dt \leq \int_0^T \left\| \frac{k_{w_0}}{\sqrt{k_0}} \nabla \partial_\delta T_\delta(u_0) \right\|^2 dt. \] (3.48)
Further, one has
\[
\left\| \sqrt{k_o} \nabla \partial_t T_\delta(u_0) \right\|^2 - \left\| \frac{k_o}{\sqrt{k_\delta}} \nabla \partial_t T_\delta(u_0) \right\|^2 = \left\| \sqrt{k_o} \nabla \partial_t T_\delta(u_0) \right\|^2, \tag{3.49}
\]
and
\[
\left\| (\nabla \theta(u_0), \nabla \partial_t T_\delta(u_0)) \right\| \leq \frac{1}{2} \left\| \sqrt{k_o} \nabla \partial_t T_\delta(u_0) \right\|^2 + \frac{1}{2} \left\| \frac{k_o}{\sqrt{k_\delta}} (-p_\delta') \nabla u_0 \right\|^2. \tag{3.50}
\]
Then (3.48) leads to
\[
\int_0^T \left( \partial_t u_\delta, \partial_t T_\delta(u_0) \right) dt + \frac{1}{2} \int_0^T \left\| \sqrt{k_o} \nabla \partial_t T_\delta(u_0) \right\|^2 dt \leq \frac{1}{2} \int_0^T \left\| \frac{k_o}{\sqrt{k_\delta}} p_\delta' \nabla u_0 \right\|^2 dt \\
\leq \frac{1}{2} \int_0^T \left\| k_o \nabla \partial_t T_\delta(u_0) \right\|^2 dt. \tag{3.51}
\]
By using \( \left\| p_\delta' \nabla u_0 \right\|^2 \leq C, \left\| k_o \nabla p_\delta' \right\| \leq C, \) and since by (A1), \( \frac{1}{\tau_\delta} \) is bounded, we have
\[
\int_0^T \left( \partial_t u_\delta, \partial_t T_\delta(u_0) \right) dt + \frac{1}{2} \int_0^T \left\| \sqrt{k_o} \nabla \partial_t T_\delta(u_0) \right\|^2 dt \leq C. \tag{3.52}
\]
Then, by (3.50), this particularly implies
\[
\int_0^T \left\| (\nabla \theta(u_0), \nabla \partial_t T_\delta(u_0)) \right\| \leq C. \tag{3.53}
\]
Clearly,
\[
\int_0^T \left\| \sqrt{k_o} \nabla \partial_t u_\delta \right\|^2 dt = \int_0^T \left\| \frac{1}{\sqrt{k_\delta}} \nabla \theta T_\delta(u_0) \right\|^2 dt = \int_0^T \left( \partial_t u_\delta, \partial_t T_\delta(u_0) \right) dt \leq C. \tag{3.54}
\]
Testing again (2.10) with \( \phi = \partial_t T_\delta(u_0) \), we have
\[
\int_0^T \left( \partial_t u_\delta, \partial_t T_\delta(u_0) \right) dt - \int_0^T \left( k_o \nabla p_\delta, \nabla \partial_t T_\delta(u_0) \right) dt + \int_0^T \left( \nabla \theta(u_0), \nabla \partial_t T_\delta(u_0) \right) dt = 0. \tag{3.55}
\]
Choosing now \( \psi = p_\delta + \partial_t T_\delta(u_0) \) in (2.11) gives
\[
\int_0^T \left( k_o \nabla p_\delta, \nabla \partial_t u_\delta \right) dt + \int_0^T \left( k_o \nabla \partial_t T_\delta(u_0) \right) dt + \int_0^T \left\| \sqrt{k_o} \nabla \left( p_\delta + \partial_t T_\delta(u_0) \right) \right\|^2 dt = 0. \tag{3.56}
\]
Adding the equations (3.55) and (3.56), and using (3.53), we find
\[
\int_0^T \left\| \sqrt{k_o} \nabla \partial_t u_\delta \right\|^2 dt + \int_0^T \left\| k_o \nabla p_\delta \right\|^2 dt + \int_0^T \left\| \sqrt{k_o} \nabla \left( p_\delta + \partial_t T_\delta(u_0) \right) \right\|^2 dt \leq \int_0^T \left\| (\nabla \theta(u_0), \nabla \partial_t T_\delta(u_0)) \right\| dt \leq C, \tag{3.57}
\]
which concludes the proof. \( \square \)

Furthermore, by (A1), from (3.11), one gets
\[
\int_0^T \left\| -p_\delta'(u_0) \nabla u_0 \right\|^2 dx dt + \int_0^T \left\| \sqrt{\tau_\delta} u_0 \nabla u_0 \right\|^2 dx dt \leq C. \tag{3.58}
\]
With the notation:
\[
[u_0]_{1-\delta} = \max(0, \min[1 - \delta, u_0]).
\]
\[
\n\int_0^{[u_n]_0^{-\delta}} \frac{1}{(z + \delta)^{-1/2}} dz = \frac{1}{1 - \delta/2} \left( ([u_n]_0^{-\delta} + \delta)^{1-\delta/2} - \delta^{1-\delta/2} \right),
\]

we have
\[
\nabla (([u_0]_0^{-\delta} + \delta)^{1-\delta/2} \in L^2(Q),
\]

and by (3.58) it is bounded uniformly w.r.t. \(\delta\). Further, since the trace of \([u_3]^{-\delta} + \delta\) on \(\partial\Omega\) is \(C_D + \delta\), applying the Poincaré's inequality for \(\left( [u_3]_0^{-\delta} + \delta \right)^{1-\delta/2} - \left( C_D + \delta \right)^{1-\delta/2}\), one immediately obtains that
\[
\|([u_3]_0^{-\delta} + \delta)^{1-\delta/2} \|_{L^2(0,T;W^{1,2}(\Omega))} \leq C,
\]

for some \(\delta\) - independent \(C\).

By Sobolev Embedding Theorem, one obtains
\[
([u_3]_0^{-\delta} + \delta)^{1-\delta/2} \in L^2(0,T;L^C(\Omega)), \quad \text{if } d = 1,
\]
\[
([u_3]_0^{-\delta} + \delta)^{1-\delta/2} \in L^2(0,T;L^{r}(\Omega)), \quad \text{for any } r \in (1, +\infty), \quad \text{if } d = 2,
\]
\[
([u_3]_0^{-\delta} + \delta)^{1-\delta/2} \in L^2(0,T;L^{\frac{2r}{r-2}}(\Omega)), \quad \text{if } d > 2,
\]

and the respective norms are bounded uniformly w.r.t. \(\delta\).

Similarly, for
\[
[u_n]^3 := \max(\delta, \min(1, u_n)),
\]

one has
\[
(1 - [u_3]^3 + \delta)^{1-\delta} \in L^2(0,T;C(\bar{\Omega})), \quad \text{if } d = 1,
\]
\[
(1 - [u_3]^3 + \delta)^{1-\delta} \in L^2(0,T;L^r(\Omega)), \quad \text{for any } r \in (1, +\infty), \quad \text{if } d = 2,
\]
\[
(1 - [u_3]^3 + \delta)^{1-\delta} \in L^2(0,T;L^{\frac{2r}{r-2}}(\Omega)), \quad \text{if } d > 2,
\]

**Lemma 3.3.** For \(\gamma, \gamma_n, \gamma_t\), chosen appropriately, the functions \(k_\alpha([u_3]_0^{-\delta})^{-\gamma}, \left( k_\alpha([u_3]^3) \right)^{-\gamma}, \) and \((\tau_\delta([u_3]^3))^{-\gamma}\) are in \(L^1(Q)\) and have uniformly bounded norms w.r.t. \(\delta\).

**Proof.** We detail the proof for \(k_\alpha\), the arguments for \(\tau_\delta\) being identical to those in [27]. To do so, we consider the cases \(d = 1, 2,\) and \(3\) separately \((d > 3\) being similar to \(d = 3\). We start with the case \(d = 3\), and choose \(\gamma_n = \frac{1}{\delta} (\frac{2}{2} + \frac{2}{3} - \frac{2}{1})\). (A5), one gets \(\gamma_n > 1\). Applying Hölder inequality, for \(p = 3, q = \frac{2}{3}, \) one gets for a.e. \(t:\)

\[
\int_\Omega (1 - [u_3]^3 + \delta)^{-\gamma} \partial dxdt = \left( \int_\Omega (1 - [u_3]^3 + \delta)^{1-\delta/2} dx \right) dt \cdot \left( \int_\Omega (1 - [u_3]^3 + \delta)^{1-\delta/2} dx \right)^{1/3} \cdot \left( \int_\Omega (1 - [u_3]^3 + \delta)^{1-\delta/2} dx \right)^{2/3}.\]

Due to (3.9) and (3.67), we have \((1 - [u_3]^3 + \delta)^{1-\delta/2} \in L^\infty(0,T;L^1(\Omega))\) and \((1 - [u_3]^3 + \delta)^{1-\delta} \in L^2(0,T;L^C(\Omega))\), and the norms are bounded uniformly w.r.t. \(\delta\).

This implies:
\[
\int_0^{T_n} \int_\Omega (1 - [u_3]^3 + \delta)^{-\gamma} \partial dxdt \leq \int_0^{T_n} \left( \int_\Omega (1 - [u_3]^3 + \delta)^{1-\delta/2} dx \right)^{1/3} \cdot \left( \max_{0 \leq t \leq T_n} \int_\Omega (1 - [u_3]^3 + \delta)^{1-\delta/2} dx \right)^{2/3} \leq C.
\]
With \( \gamma_r = \frac{1}{10} (\frac{\alpha}{2} + \frac{4e}{\alpha} - \frac{10}{2} \beta) \), the estimate for \( (\tau_d((u_0)_0))^{\gamma_r} \) follows similarly.

For \( d = 2 \), we choose any \( r > \max\{2\alpha - 4, \beta, (\alpha - 2)/(\omega - 2), (2\alpha - 4)/(\alpha - 4)\} \) and define

\[
\gamma_n = -\frac{1}{r} \left( 4 - \frac{4}{r} - \omega - \epsilon(1 - \frac{2}{r}) \right),
\]

respectively.

\[
\gamma_r = -\frac{1}{\omega} \left( 4 - \frac{4}{r} - \omega - \epsilon(1 - \frac{2}{r}) \right),
\]

and we apply Hölder inequality for \( p = \frac{q}{r} \) and \( q = \frac{r}{r-\epsilon} \) to obtain for a.e. \( t \)

\[
\int_{\Omega} (1 - [u_0]^1_0 + \delta)^{-\gamma_d} dx dt \leq \left( \int_{\Omega} (1 - [u_0]^1_0 + \delta)^{1-\omega/2r} dx \right)^{2/\epsilon} \cdot \left( \int_{\Omega} (1 - [u_0]^1_0 + \delta)^{2-\epsilon} dx \right)^{2/\epsilon} ,
\]

and

\[
\int_{\Omega} (1 - [u_0]^1_0 + \delta)^{-\gamma_d} dx dt \leq \left( \int_{\Omega} (1 - [u_0]^1_0 + \delta)^{1-\omega/2r} dx \right)^{2/\epsilon} \cdot \left( \int_{\Omega} (1 - [u_0]^1_0 + \delta)^{2-\epsilon} dx \right)^{2/\epsilon} .
\]

Then, the proof continues as before.

Finally, for \( d = 1 \), we take \( \gamma_n = (\omega + \epsilon - 4)/\beta \) and \( \gamma_r = (\omega + \epsilon - 4)/\omega \). Similarly, by assumption (A5), we have \( \gamma_n, \gamma_r > 1 \). Then, using (3.9), (3.65) to estimate for a.e. \( t \)

\[
\int_{\Omega} (1 - [u_0]^1_0 + \delta)^{-\gamma_d} dx = \int_{\Omega} (1 - [u_0]^1_0 + \delta)^{1-\omega/2r} (1 - [u_0]^1_0 + \delta)^{2-\epsilon} dx \\
\leq \left( \max(1 - [u_0]^1_0 + \delta)^{1-\omega/2} \right)^2 \int_{\Omega} (1 - [u_0]^1_0 + \delta)^{2-\epsilon} dx \in L^1(0, T_M),
\]

and

\[
\int_{\Omega} (1 - [u_0]^1_0 + \delta)^{-\gamma_d} dx = \int_{\Omega} (1 - [u_0]^1_0 + \delta)^{1-\omega/2r} (1 - [u_0]^1_0 + \delta)^{2-\epsilon} dx \\
\leq \left( \max(1 - [u_0]^1_0 + \delta)^{1-\omega/2} \right)^2 \int_{\Omega} (1 - [u_0]^1_0 + \delta)^{2-\epsilon} dx \in L^1(0, T_M),
\]

and the proof follows again as before.

Now we obtain further estimates for \( (u_0, p_3) \).

**Lemma 3.4.** Let \( d = 1, 2, \) or \( 3 \) and assume (A1), (A2) and (A5). There exist \( r_1, r_2, r_3 \in (1, 2) \) and \( C > 0 \) independent on \( \delta \), such that the weak solution pair \( (u_0, p_0) \) satisfies for all \( \delta > 0 \)

\[
||\partial_t T_\delta(u_0)||_{L^1(O)} + ||\nabla p_0||_{L^1(O)} + ||\nabla(p_3 + \partial_t T_\delta(u_0))||_{L^1(O)} \leq C.
\]

**Proof.** The proof uses the estimates in Lemma 3.3 and distinguishes as before three cases, \( d = 1, 2, \) and 3. We start with the latter.

By (3.43), one has \( \frac{1}{\sqrt{\tau}} \partial_t T_\delta(u_0) \in L^2(Q) \). So here we show for any \( r_1 \in (1, 2) \), one has \( \partial_t T_\delta(u_0) \in L^r(Q) \). Moreover, for appropriately chosen \( r_1 \), the corresponding norm is bounded uniformly w.r.t. \( \delta \). To see this, we apply Hölder inequality to get

\[
\int_0^{T_M} \int_{\Omega} |\partial_t T_\delta(u_0)|^r dx dt = \int_0^{T_M} \int_{\Omega} \left( \frac{1}{\sqrt{\tau}} \partial_t T_\delta(u_0) \right)^r \left( \frac{1}{\sqrt{\tau}} \partial_t T_\delta(u_0) \right)^{r_1/2} dx dt \\
\leq \left( \int_0^{T_M} \int_{\Omega} \left( \frac{1}{\sqrt{\tau}} \partial_t T_\delta(u_0) \right)^{r_1/2} dx dt \right)^{2/\epsilon} \cdot \left( \int_0^{T_M} \int_{\Omega} \left( \frac{1}{\sqrt{\tau}} \partial_t T_\delta(u_0) \right)^{1-r_1/2} dx dt \right)^{2/\epsilon} .
\]
The first integral on the right hand side is bounded by (3.43), the second we recall Lemma 3.3 and choose \( r_1 \) such that \( r_{1/5} = \gamma = \frac{1}{5} (\frac{22}{7} + \frac{3}{4} - \frac{1}{2}) \), which, by (A5), \( r_1 = \frac{22}{7} \left( \frac{5}{2} - \frac{1}{2} \right) \), satisfies \( r_1 \in (1, 2) \).

Similarly, one also has the estimate
\[
\int_0^{T_u} \int_\Omega |\nabla p_\delta|^2 dx dt = \int_0^{T_u} \int_\Omega \left| \sqrt{k_\alpha} \nabla p_\delta \right|^2 dx dt \\
\leq \left( \int_0^{T_u} \int_\Omega k_\alpha |\nabla p_\delta|^2 dx dt \right)^{r_1/2} \cdot \left( \int_0^{T_u} \int_\Omega k_\alpha^{-r_1/2} dx dt \right)^{1-r_1/2} 
\]
(3.78)

We obtain
\[
r_2 = \frac{2(3\omega + 2e - 10)}{(5e - 10)},
\]
and following (A5), one has \( r_2 \in (1, 2) \).

Similarly, one has
\[
\int_0^{T_u} \int_\Omega |\nabla (p_\delta + \partial_t T_\delta(u_\delta))|^2 dx dt \\
\leq \left( \int_0^{T_u} \int_\Omega k_\alpha |\nabla (p_\delta + \partial_t T_\delta(u_\delta))|^2 dx dt \right)^{r_1/2} \cdot \left( \int_0^{T_u} \int_\Omega k_\alpha^{-r_1/2} dx dt \right)^{1-r_1/2} 
\]
(3.79)

then for this case, we have
\[
r_3 = \frac{2(3\lambda + 2\alpha - 10)}{3\alpha + 5\alpha - 10} \in (1, 2),
\]
when \( \alpha > 5 \) and \( \lambda > 10/3 + \alpha/3 \), this implies \( \nabla (p_\delta + \partial_t T_\delta(u_\delta)) \in L^\infty(Q) \).

**Case 2: \( d = 2 \)**

Similarly, for two dimension case, we show for any \( r_1 \in (1, 2) \), one has \( \partial_t T_\delta(u_\delta) \in L^\infty(Q) \) and the corresponding norm is bounded uniformly w.r.t. \( \delta \). Since one has
\[
\int_0^{T_u} \int_\Omega |\partial_t T_\delta(u_\delta)|^2 dx dt \leq \left( \int_0^{T_u} \int_\Omega \left| \frac{1}{T_\delta} \partial_t T_\delta(u_\delta) \right|^2 dx dt \right)^{r_1/2} \cdot \left( \int_0^{T_u} \int_\Omega k_\alpha^{-r_1/2} dx dt \right)^{1-r_1/2} .
\]
(3.80)

Then, for any \( r > 2(e - 2)/(2\omega + e - 4) \), we solve
\[
r_1 = \frac{2(\omega + e - 2(e - 2)/r - 4)}{(2\omega + e - 2(e - 2)/r - 4)} \in (1, 2),
\]
which implies \( \partial_t T_\delta(u_\delta) \in L^r(Q) \).

Using the same way, we also get \( \nabla p_\delta \in L^{\delta r}(Q) \) for
\[
r_2 = \frac{2(\omega + e - 2(e - 2)/r - 4)}{(\omega + e + \beta - 2(e - 2)/r - 4)},
\]
and \( \nabla (p_\delta + \partial_t T_\delta(u_\delta)) \in L^\infty(Q) \) for
\[
r_3 = \frac{2(4 - \lambda - \alpha + 2(\alpha - 2)/r)}{(4 - \lambda - 2\alpha + 2(\alpha - 2)/r)}.
\]

**Case 3: \( d = 1 \)**

The proof follows as before, we have \( \partial_t T_\delta(u_\delta) \in L^\infty(Q) \) for
\[
r_1 = \frac{2(\omega + e - 4)}{(2\omega + e - 4)} \in (1, 2),
\]
\( \nabla p_\delta \in L^{\delta r}(Q) \), for
\[
r_2 = \frac{2(\omega + e - 4)}{(\omega + e + \beta - 4)} \in (1, 2),
\]
22
and furthermore, \( \nabla (p_\delta + \partial_t T_\delta (u_\delta)) \in L^r(Q) \) for
\[
\delta = \frac{2(4 - \lambda - \alpha)}{(4 - \lambda - 2\alpha)} \in (1, 2).
\]

Then we have the estimates
\[
\| \partial_t T_\delta (u_\delta) \|_{L^2(Q)} \leq C, \quad \text{with } r_1 \in (1, 2),
\]
\[
\| \nabla p_\delta \|_{L^2(Q)} \leq C, \quad \text{with } r_2 \in (1, 2),
\]
\[
\| \nabla (p_\delta + \partial_t T_\delta (u_\delta)) \|_{L^2(Q)} \leq C, \quad \text{with } r_3 \in (1, 2).
\]

\[
\text{Proof.}
\]

With \( r^* = \min\{r_1, r_2, r_3\} \), by Lemma 3.1, 3.2 and 3.4, one obtains the existence of a subsequence \( \delta \searrow 0 \) (still denoted by \( \delta \)) and of \( u \in W^{1,2}(Q), T^* \in W^{1,r'}(0, T_M; W^{1,r'}(\Omega)) \) and \( p \in L^2(0, T_M; W^{1,r'}(\Omega)) \), such that
\[
u \delta \longrightarrow u \quad \text{strongly in } L^2(Q),
\]
\[
\partial_t \nu \delta \longrightarrow \partial_t u \quad \text{weakly in } L^2(Q),
\]
\[
\nabla \nu \delta \longrightarrow \nabla u \quad \text{weakly in } L^2(Q),
\]
\[
T_\delta \longrightarrow T^* \quad \text{weakly in } W^{1,r'}(Q),
\]
\[
T_\delta \longrightarrow T^* \quad \text{strongly in } L^p(Q),
\]
\[
\nabla \partial_t T_\delta \longrightarrow \nabla \partial_t T^* \quad \text{weakly in } L^r(Q),
\]
\[
p_\delta \longrightarrow p \quad \text{weakly in } W^{1,r'}(Q).
\]

where \( q = +\infty \), if \( d = 1, q = \frac{d+r'}{d-r} \), if \( d = 2 \) or \( d = 3 \) (see [17]).

In the remaining, we prove that \( T^* = T(u) \) a.e., and that \( (u, p) \) is a solution pair to Problem P. But before doing so, we also prove that the limit \( u \) above is essentially bounded by \( 0 \) and \( 1 \).

**Theorem 3.1.** The limit \( u \in W^{1,2}(Q) \) satisfies \( 0 \leq u \leq 1 \) a.e. in \( Q \).

**Proof.** Given \( t \in (0, T_M] \), let \( \Omega_{\nu(t)} \) be the support of \( [u_\delta(t, \cdot) + \epsilon]_+ \) (the negative cut of \( u_\delta(t, \cdot) + \epsilon \)). As follows from Lemma 3.1, a \( C > 0 \) exists such that, for all \( \delta > 0 \), one has
\[
\int_{\Omega} \tilde{E}_{\nu\delta}(u_\delta) dx = \int_{\Omega} \int_{C_D} \int_{C_D} ^{u_\delta} \frac{1}{k_{\nu\delta}} (z) dz dv dx + \int_{\Omega} \int_{C_D} \int_{C_D} ^{u_\delta} \frac{(C_D + \delta)z^{2-a}}{(1-a)(2-a)} + \frac{(C_D + \delta)^{1-a}}{(a-1)}C_D dx \leq C,
\]
\[
\int_{\Omega} \tilde{E}_{\nu}(u) dx \leq \int_{\Omega} \int_{C_D} \int_{C_D} ^{u_\delta} \frac{1}{k_{\nu\delta}} (z) dz dv dx + \int_{\Omega} \int_{C_D} \int_{C_D} ^{u_\delta} \frac{\delta^{2-a}}{(1-a)(2-a)} dx \leq C.
\]

Since the constant arguments in the last two integrals are positive, this gives
\[
\int_{\Omega} \int_{C_D} \int_{C_D} ^{u_\delta} \frac{1}{k_{\nu\delta}} (z) dz dv dx + \int_{\Omega} \int_{C_D} \int_{C_D} ^{u_\delta} \frac{\delta^{2-a}}{(1-a)(2-a)} dx \leq C.
\]

Then we get
\[
C \geq \int_{\Omega} \int_{0} ^{u_\delta} \int_{\Omega} \int_{k_{\nu\delta}} \frac{1}{k_{\nu\delta}} (z) dz dv dx
\]
\[
= \int_{\Omega} \int_{0} ^{u_\delta} \frac{(C_D + \delta)^{1-a} - \delta^{1-a}}{(a-1)} u_\delta + \frac{\delta^{2-a}}{(1-a)(2-a)} dx
\]
\[
\geq \int_{\Omega \subset (0)} \int_{0} ^{u_\delta} \int_{\Omega} \int_{k_{\nu\delta}} \frac{1}{k_{\nu\delta}} (z) dz dv dx
\]
\[
= \frac{\delta^{2-a}}{2} \epsilon + \frac{\delta^{1-a} - (C_D + \delta)^{1-a}}{(a-1)} \epsilon + \frac{\delta^{2-a}}{(1-a)(2-a)} dx.
\]
Let now $\delta \searrow 0$, this immediately implies that

$$\text{meas}(\Omega^{t}_{\delta}(t)) = 0,$$  \hfill (3.95)

with $\Omega^{t}_{\delta}(t)$ having the same definition as $\Omega^{t}_{\delta}(t)$, but now for the function $u$.

Since $u_{\delta} \rightharpoonup u$ in $C((0, T_{M}); L^{2}(\Omega))$ (by (3.85), (3.86) and the compact embedding see [1] and Theorem 4.4). Thus $u_{\delta} \rightarrow u$ a.e., in $\Omega$, for all $t$. This holds for every $\epsilon > 0$, hence $u \geq 0$. Similarly, if $u_{\delta} > 1 + \epsilon$, use the bounds on $\int_{\Omega} \int_{0}^{1} \frac{T_{\delta}}{C_{n} \delta} d\omega d\tau$, we obtain $u \leq 1$. \hfill \Box

**Remark 4.** For two phase flow model, $0 \leq u \leq 1$ means that the saturation remains in the physical range. Note that this only holds due to the degeneracy encountered for $u = 0$ or $u = 1$.

Finally, we obtain the existence of a solution for Problem P.

**Theorem 3.2.** Under the assumptions (A1), (A2) and (A5), there exists a solution pair $(u, p)$ for Problem P.

**Proof.** We start by identifying $T^{*}$ as $T(u)$. To do so, define $[v]^{1-} = \min[1, v]$ and let $T(f) = T^{*} f$ a.e. We proceed as in [28] and consider first the inverse function of $T^{*}$. According to the definition of $T$ and $T_{\delta}$, one has

$$f = T^{-1}(T^{*}) = \begin{cases} T^{*} - \frac{1}{\omega - 1}, & \text{if } T^{*} < \frac{1}{\omega - 1}, \\ 1 - \frac{1}{(\omega - 1)^{T^{*}}}, & \text{if } T^{*} \geq \frac{1}{\omega - 1}. \end{cases}$$  \hfill (3.96)

Clearly,

$$[u_{\delta}]^{1-} = T_{\delta}^{-1}(T_{\delta}) = \begin{cases} T_{\delta} - \frac{1}{\omega - 1} + \delta, & \text{if } T_{\delta} < \frac{1}{\omega - 1}, \\ 1 - \frac{1}{(\omega - 1)^{T_{\delta}}}, & \text{if } T_{\delta} \geq \frac{1}{\omega - 1}. \end{cases}$$  \hfill (3.97)

Now we prove that $[u_{\delta}]^{1-} \rightharpoonup f$ strongly in $L^{2}(Q)$, and hence a.e. in $Q$. Since $T^{-1}(\cdot)$ is Lipschitz continuous, by (3.97) one has

$$\int_{Q} |f - [u_{\delta}]^{1-}| d\omega d\tau = \int_{Q} |T^{-1}(T^{*}) - T^{-1}(T_{\delta})| d\omega d\tau \leq C \int_{Q} |T^{*} - T_{\delta}| d\omega d\tau + \int_{Q} |T^{-1}(T_{\delta}) - [T_{\delta}]^{1-}| d\omega d\tau.$$  \hfill (3.98)

Clearly, the first integral above approaches 0 as $\delta \searrow 0$. For the second we note that, by (3.96) and (3.97), one has $T_{\delta}^{-1}(T_{\delta}) - T^{-1}(T_{\delta}) = \delta$, for any argument $T_{\delta}$. With this, the second integral also approaches 0 as $\delta \searrow 0$. This means that

$$[u_{\delta}]^{1-} \rightharpoonup f \text{ a.e. in } Q.$$  \hfill (3.99)

However $u_{\delta} \rightarrow u$ a.e. in $Q$ (by the strong convergence in $L^{2}(Q)$). This immediately gives $[u_{\delta}]^{1-} \rightarrow [u]^{1-}$ a.e.. In the view of Theorem 3.1, we also have $[u]^{1-} = u$ a.e.. Therefore, $u = f$ a.e., and consequently, we have

$$T(u) = T^{*}.$$  \hfill (3.100)

Having identified $T^{*}$ by $T(u)$, (3.89) gives $\nabla \delta T_{\delta}(u_{\delta}) \rightarrow \nabla \delta T(u)$.

Then, according to $k_{n} (\cdot)$ Lipschitz continuous, one has

$$k_{n}(u_{\delta}) \rightarrow k_{n}(u) \text{ a.e. in } Q.$$  

Further, since $k_{n}(u)$ converges pointwise to $k_{n}(u)$, and

$$|k_{n}(u)| \leq C,$$
uniformly with w.r.t. $\delta$. Then by Dominated convergence theorem, we obtain

$$k_{\delta}(u) \to k(u) \quad \text{a.e. in $Q$.}$$

Therefore, we have

$$k_{\delta}(u_0) \to k(u) \quad \text{strongly in $L^2(Q)$.}$$

Similarly, we also have $\sqrt{k_{\delta}(u_0)}$ converges to $\sqrt{k(u)}$ strongly in $L^2(Q)$. In the same fashion, since $k_w, k$ and $\theta$ are Lipschitz continuous, on also obtains

$$k_{\delta}(u_0) \to k(u),$$

$$k_{\delta}(u_0) \to k(u),$$

$$\theta_{\delta}(u_0) \to \theta(u),$$

strongly in $L^2(Q)$.

By the (3.44), a $g \in L^2(Q)$ exists, such that

$$\sqrt{k_{\delta}(u_0)} \nabla (p_\delta + \partial_\delta T_\delta(u_0)) \to g \quad \text{weakly in $L^2(Q)$, for $\delta \searrow 0$.} \quad (3.101)$$

To identify $g$, we consider $\phi \in C_0^\infty(\Omega)$ arbitrarily, and note that $\phi \sqrt{k_{\delta}(u_0)} \to \phi \sqrt{k(u)}$ strongly in $L^q(Q)$, for any $q \in [1, \infty)$. This follows as above, using the uniform boundness of $k_{\delta}, k_w$ and the pointwise convergence of $k_{\delta}(u_0)$ to $k(u)$. Further, by (3.89) and (3.90), one gets $\nabla (p_\delta + \partial_\delta T_\delta(u_0)) \to \nabla (p + \partial T(u))$ weakly in $L^r(Q)$. Taking $q$ such that $1/q + 1/r^* = 1$ gives by weak-strong convergence argument

$$\int_0^{T_u} \int_Q (k_{\delta} \nabla (p_\delta + \partial_\delta T_\delta(u_0)), \nabla \phi) dt \to \int_0^{T_u} \int_Q (k_w \nabla (p + \partial T(u)), \nabla \phi) dt. \quad (3.102)$$

This is sufficient to identify $g = \sqrt{k_{\delta}(u)} \nabla (p + \partial T(u))$.

We have now all the ingredients to pass to the limit ($\delta \searrow 0$) in the integrals appearing in Problem $P_\delta$. Using the convergence results above, it is straightforward to show that $(u, p)$ solves Problem P. \hfill \square

4. Conclusions

In this paper, we prove the existence of a weak solution to degenerate elliptic parabolic system modeling two-phase flow in porous media, and including dynamic effects in the capillary pressure. The major difficulty is the degeneracy of the non-linear third order derivative term. We get the estimate for the third order derivative term by applying the structures of relative permeabilities, capillary pressure and the dynamic damping factor. By compactness arguments, we show the existence of a solution for the original problem.

Acknowledgments

The work of X. Cao is supported by CSC (China Scholarship Council). This support is gratefully acknowledged. The authors are members of the International Research Training Group NUPUS funded by the German Research Foundation DFG (GRK 1398) and The Netherlands Organization for Scientific Research NWO (DN 81-754).

References

<table>
<thead>
<tr>
<th>Number</th>
<th>Author(s)</th>
<th>Title</th>
<th>Month</th>
</tr>
</thead>
<tbody>
<tr>
<td>14-33</td>
<td>E.N.M. Cirillo, A. Muntean, R.A. van Santen, A. Sengar</td>
<td>Residence time estimates for asymmetric simple exclusion dynamics on stripes</td>
<td>Nov. ’14</td>
</tr>
<tr>
<td>14-34</td>
<td>B.S. van Lith, J.H.M. ten Thije Boonkamp, W.L. IJzerman, T.W. Tukker</td>
<td>Existence and uniqueness of solutions to Liouville’s equation and the associated flow for Hamiltonians of bounded variation</td>
<td>Dec. ’14</td>
</tr>
<tr>
<td>15-02</td>
<td>J.H.M. Evers, I.A. Zisis, B.J. van der Linden, M.H. Duong</td>
<td>From continuum mechanics to SPH particle systems and back: Systematic derivation and convergence</td>
<td>Febr. ’15</td>
</tr>
<tr>
<td>15-03</td>
<td>X. Cao, I.S. Pop</td>
<td>Degenerate two-phase flow in porous media including dynamic effects in the capillary pressure: existence of a weak solution</td>
<td>Febr. ’15</td>
</tr>
</tbody>
</table>