Invited lecture at the
Mathematics Knowledge Management Symposium
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Heriot-Watt University, Edinburgh, Scotland

Memories of the AUTOMATH project

by N.G. de Bruijn

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THE AUTOMATH PROJECT

1972-1977

EINDHOVEN, THE NETHERLANDS

about 30 man-years

Support: ZWO + THE

ZWO: The Netherlands national science foundation
THE: Technological University Eindhoven

THE **LANGUAGE** AUTOMATH HAD BEEN DEFINED IN 1968
AND NEVER CHANGED ANY MORE.

ARISTOTLE (384-322 B.C.)

DISCOVERING MATHEMATICS,
FINDING PROOFS
require ingenuity

but

CHECKING PROOFS
can be mechanical (algorithmical)

CLAIM

COMPLETE FORMALIZATION

AND VERIFICATION

OF MATHEMATICS

IS NOT JUST

THEORETICALLY POSSIBLE,

IT IS FEASIBLE!

IT IS EVEN ATTRACTIVE,

IN PARTICULAR IN CASES WHERE

METICULOUS MATHEMATICAL PRECISION

GENERATES ELEGANCE.

CHECKING CORRECTNESS OF
A COMPLETE NETWORK OF

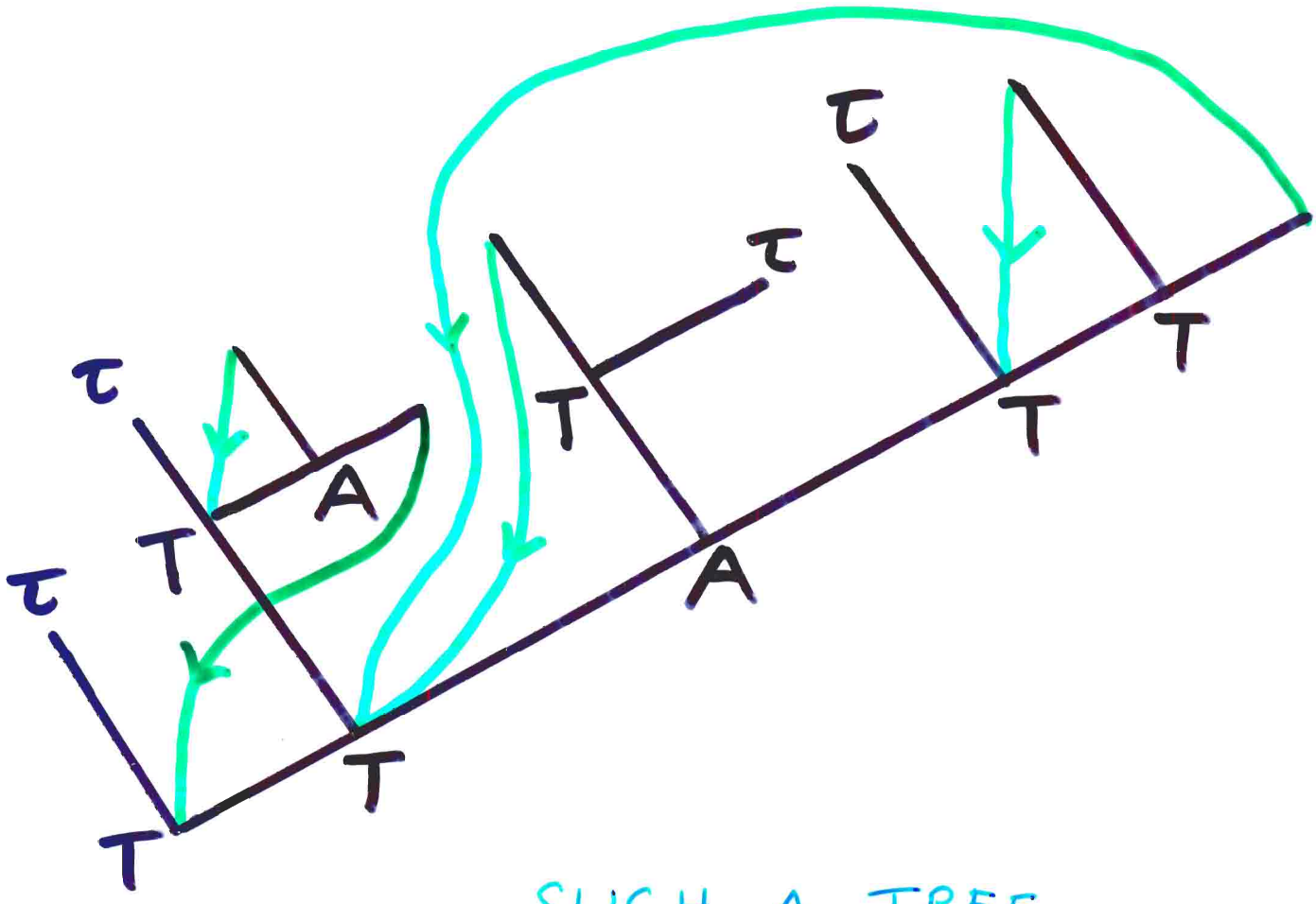
definitions
references
theories
books

is finally feasible in the computer age.

The system should be
SIMPLE and VERY GENERAL

For $\Delta\Lambda$ (the essence of AUTOMATH)
the core of the checker is less than
a page of Pascal code.

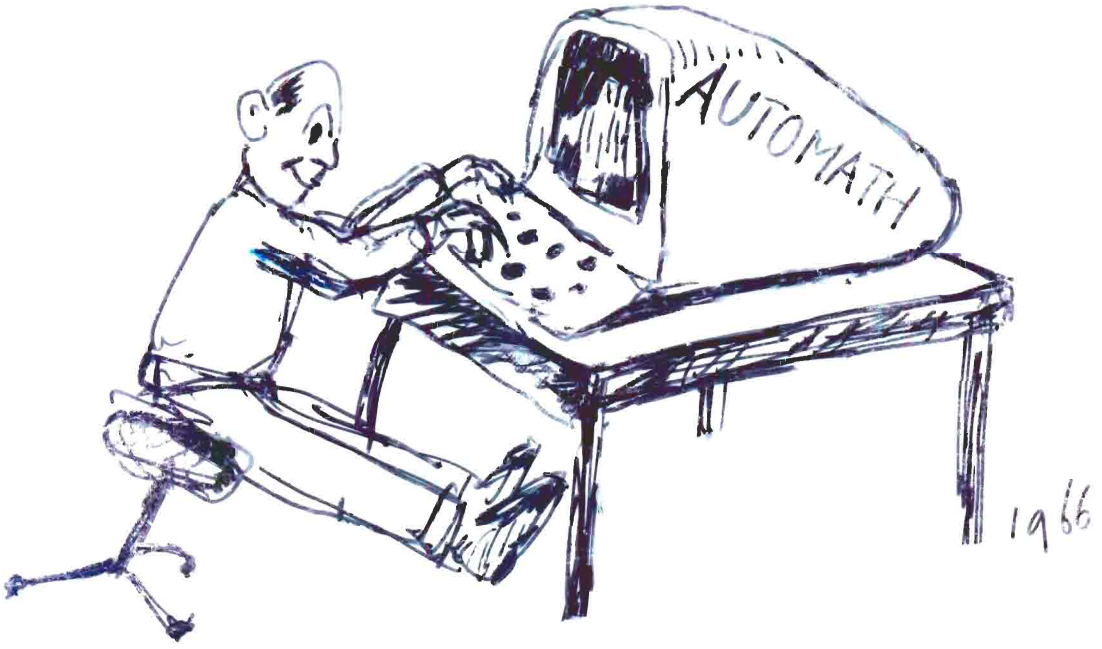
DELTA LAMBDA



SUCH A TREE
(WITH REFERENCE ARROWS)
MIGHT REPRESENT
A FULL MATHEMATICS
ENCYCLOPEDIA

DREAM OF THE SIXTIES

INTERACTIVE PRODUCTION AND VERIFICATION OF MATHEMATICS



NOW BEGINNING TO COME TRUE

ALL BY THEMSELVES, THE CREATIVE POWER OF COMPUTERS IS LIMITED (JUST LIKE OURS)

A2.2

THE AUTOMATH PROJECT
1972 - 1977

L.S. van Benthem Jutting
(translated Landau's Introduction to Analysis)

R.P. Nederpelt
(first proofs of strong normalisation)

D.T. van Daalen
(full theory of Automath)

I. Zandleven
(Chief programmer)

J. Zucker
(intro analysis in AUT-Π, using telescopes)

A. Kornaat
(general assistant)

R. de Vrijer
(some theory)

Assistant programmers, typists, students.

IMPORTANT DECISIONS

- Don't try Automatic Theorem Proving.
- Stay as close as possible to standard. mathematical presentation.
- Prefer the use of TYPED SETS, but don't forbid untyped sets.
- Try to keep the LOSS FACTOR constant, using the mechanism for definitions and abbreviations that made mathematics so successful. (AUTOMATH expands definitions and abbreviations only when strictly necessary).

- Don't put logic in the system;
  let the user start his books with it.
- Don't put induction and recursion in the system; consider it as book material, even when that might be slightly clumsier.
- Never prefer speed and efficiency
  over simplicity and generality.
- Test the feasibility and get experience
  by translating a sizable amount of
  mathematical material (the choice fell on Landau's Introduction to Analysis). Keep strictly to the text. Never replace it by
  a text that would suit Automath better.

WHAT PRECEDED AUTOMATH

The language of math used to be

acquired by OSMOSIS (sponge).

The rules of the game were

NEVER explained explicitly.

Yet there had been perfect authors.

Authors and readers had

NO support from

philosophy, logic, foundations.

Nevertheless:

mathematical literature offered

an almost perfect vehicle

for mathematical communication

in the twentieth century.

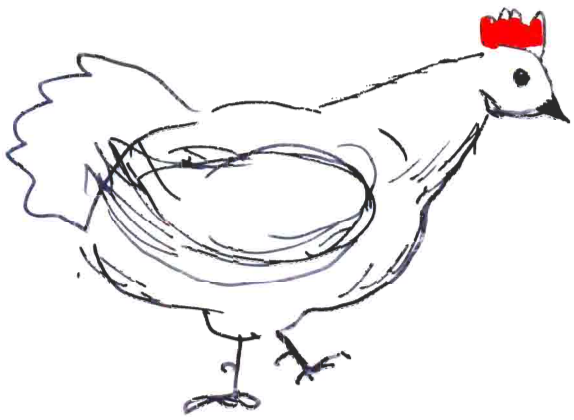
TEACHING AND LEARNING MATHEMATICS

IF YOU CAN'T EXPLAIN YOUR MATHEMATICS TO A MACHINE, IT IS AN ILLUSION TO THINK YOU CAN EXPLAIN IT TO A STUDENT

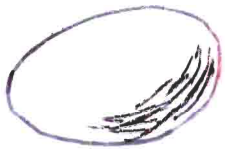
(OSMOSIS)

LOGIC TEACHING NEVER EXPLAINED THE RULES OF THE MATHEMATICS GAME

?



LOGIC

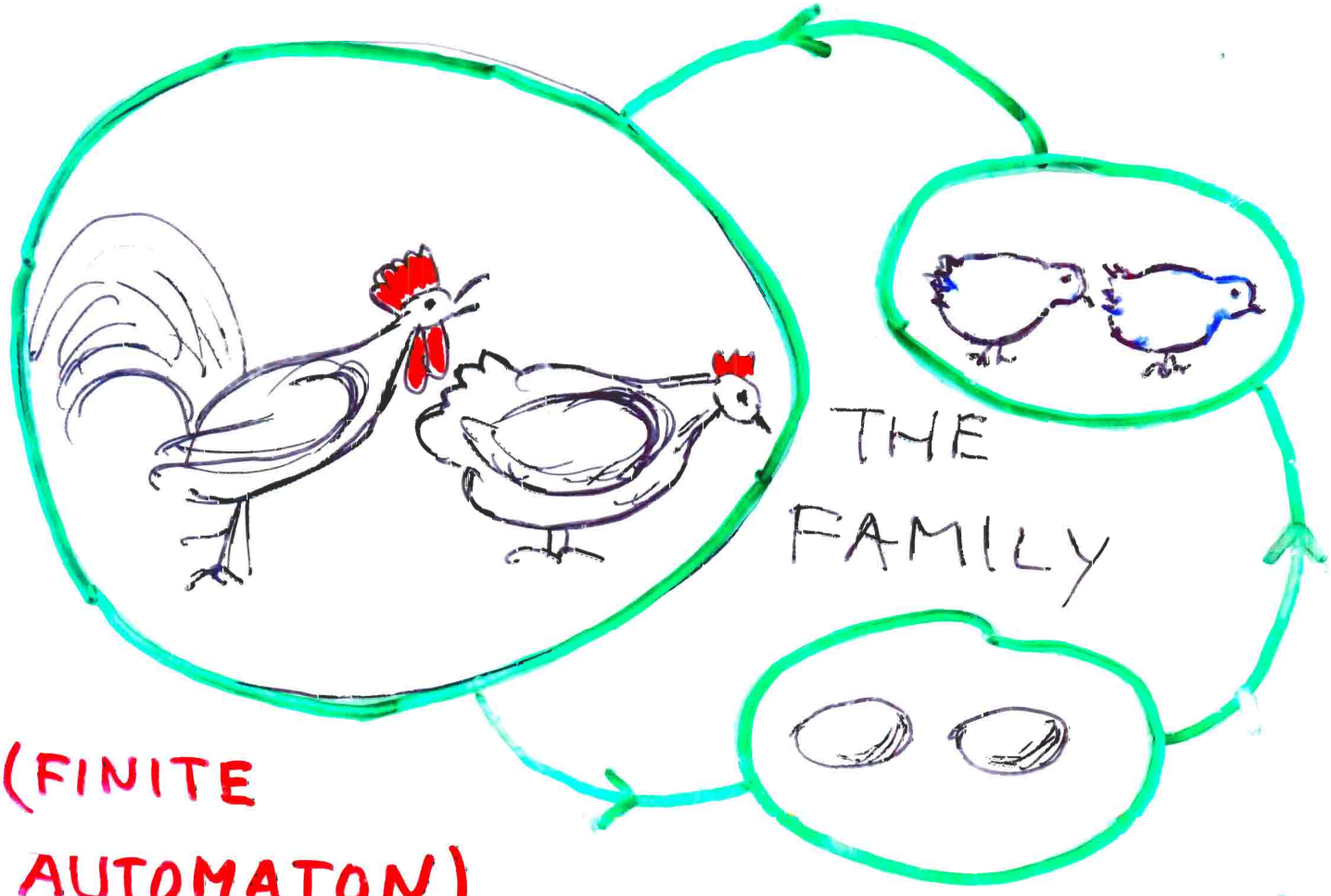


MATHEMATICS

OR

MATHEMATICS

LOGIC



(FINITE AUTOMATON)

A4.2

A mathematician's personal history and interests before starting Automath

Born 1918, The Hague.
Finished middle school 1934.
Depression, no future.
Studying mathematics (old-fashioned) in isolation 1934-1936.
Leiden University 1936-1939.
Assistant at Techn. Univ. Delft 1939-1944.

analytic number theory

Ph.D. 1943

algebraic number theory, modular functions

Philips Science Lab. (Eindhoven) 1944-1946.
Professor at Delft 1946-1952.

combinatorics, collaboration with Erdős

Professor at Amsterdam 1952-1960

asymptotics
Polya theory
functional analysis

Professor at Eindhoven 1960-

functional analysis
1962: programming combinatorics on an IBM 1820
Fourier theory for generalized functions
Algol

THINGS THAT PUSHED ME INTO THE DIRECTION OF AUTOMATH

1. Using λ-calculus notation in analysis courses.

2. A long tedious proof that I did not trust anymore.

3. A really hard problem in set theory and its 'mechanical' solution.

4. I was looking for original interesting programming tasks.

Previous experiences relevant for ideas on Automath

• 1937 Learning Naive Set Theory (Kamke)
"Do these large cardinals "exist"?
The power of language, talking coherently on things that make no sense at all.

• 1952-1960 Amsterdam, the world of Brouwer-Heyting-Beth.
Heyting: proof of $A \to B$.

• 1958 New Math!! I felt that it offered no understanding for the rules of the mathematics game.

• 1962- ... Playing with early computers.
Combinatorics programming.
Learning Algol in short courses by Van Wijngaarden and Dijkstra.
Inspirations from block structure and declarations in Algol '60
(Mathematicians had always **opened** blocks but had never bothered to **close** them explicitly).

• 1964 Teaching analysis with λ's (the notation only, no theory).

• Central problem for Van der Meiden's thesis
on Banach algebras, solved mechanically by natural deduction techniques (with flags).

Using λ-notation in courses on Functional Analysis

Notation suggested by Freudenthal:

$$\uparrow_{x \in S} \text{ instead of } \lambda x \in S.$$

Example:

Defining the Fourier operator:

$$\mathcal{F} := \uparrow_{f \in L_2} \uparrow_{x \in \mathcal{R}} \int_{-\infty}^{\infty} e^{-2\pi i x t} \ f(t) \ dt$$

Bourbaki **should** have done it.

GELFAND'S ELEGANT BANACH ALGEBRA SOLUTION OF WIENER'S TAUBERIAN THEOREM.

IT USED THE AXIOM OF CHOICE!!

The Maximal Ideals of the Banach algebrawere the points of a

were the points of a topological space.
My attempt to get rid of the axiom of choice:
Replace the topology by POINT-FREE topology.

There remained a hard set-theoretical problem that I could not do until I managed to solve it 'mechanically by flag-style natural deduction.

In 1966 I had to check a long proof in combinatorics, full of repetitions of a small number of simple ideas, and I felt the need to build a computer program to do the checking.

In Eindhoven I had a privileged position, being completely free to start any subject I liked.

Accordingly, I began, at the end of 1966, to work on computer verification of mathematics.

The first attempts already showed some of the AUTOMATH's characteristics of AUTOMATH.

For these first attempts see: Selected papers on Automath, p. 57-72.

Developing AUTOMATH was a very emotional affair.

It was a combination of natural deduction, type theory and lambda calculus. In particular, by the idea of 'proofs as objects', everything fell into its place. The whole stucture of mathematics became visible in a quite simple system.

One is always happy with a discovery, of course, but here it was so much more than discovering theorems and facts.

AUTOMATH was more than a system or a theory.

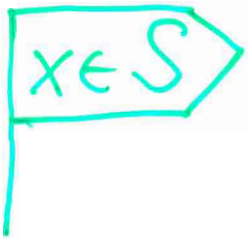
It was a LIFE STYLE.

Its philosophy was definitely anti-platonistic. There was no longer any question of the existence of mathematical objects. The only thing that could claim existence was mathematical language, being completely representable in the physical world.

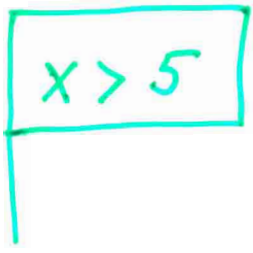
MATHEMATICAL DEDUCTION

CONTEXT INDICATED BY
NESTED FLAGS AND FLAG POLES

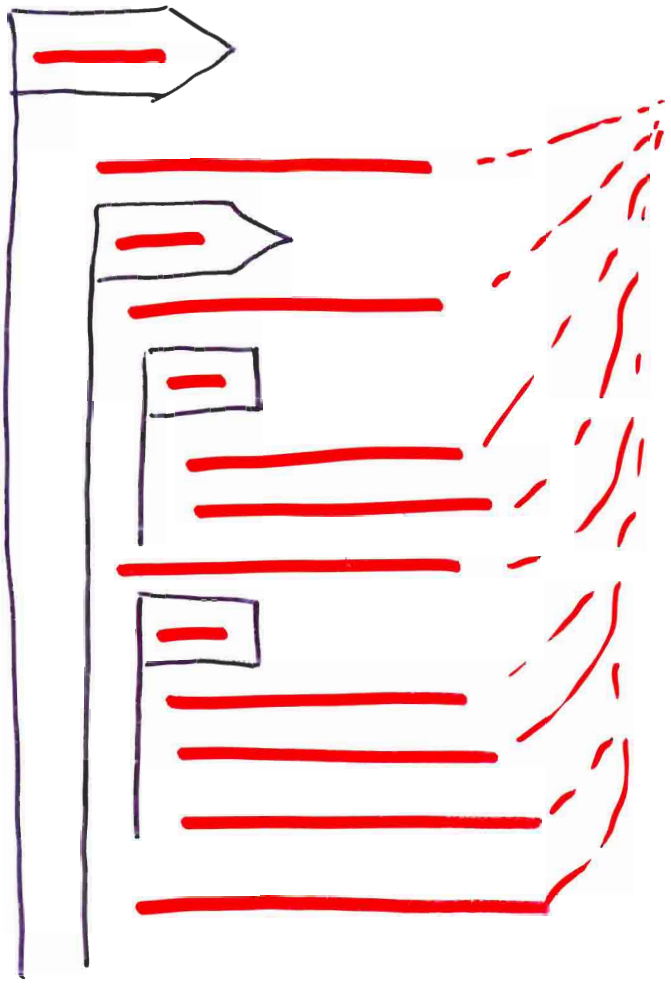
INTRO VARIABLE



ASSUMPTION



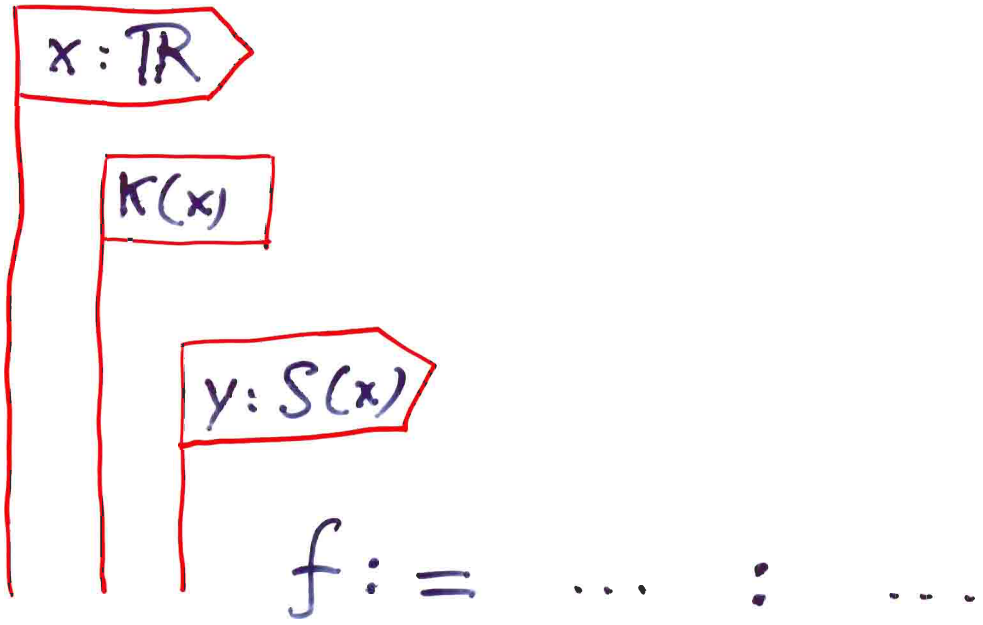
BOOK STRUCTURE



LINES :
DEFINITIONS
OR
FACTS
(PROVED
PROPOSITIONS
AND
AXIOMS)

LC
VI '99

A THEOREM



AN APPLICATION

MACHINE WANTS

- A VALUE FOR x a...
- A PROOF FOR K(a...)
- A VALUE FOR y

AND DOES :

- A TYPE CHECK
- A PROOF CHECK
- A TYPE CHECK

ANOTHER PARALLEL

$x : \mathbb{R}$

$f := \ldots : \mathbb{Z}$

18TH CENTURY FUNCTION

$\varphi := \lambda_{x:\mathbb{R}} f(x)$

$x :$ PROOF OF (a)

$\ldots := \ldots :$ PROOF OF (b)

$\ldots :$ PROOF OF $(a \to b)$

HEYTING'S IDEA

THIS LED TO THE CONCEPTS

- dependent types
- proofs as objects ('propositions as types)'

$$p(c(x,\ldots),\ldots) \ : \ \text{PROOFOF}(y > 5)$$

(The $p(c(x,\ldots),\ldots)$ is an instantiated reference
to a previous proof.)

All this happens in PAL (lambda-free Automath) already.
PAL's mathematics is roughly the mathematics from
before about 1800. It took the whole 19-th century
to let the idea of a FUNCTION develop from a
metamathematical notion to a mathematical object.

For this, lambda calculus was en essential tool,
but it it is taking a very long time to let
lambda calculus conquer the whole field of mathematics.

Mathematical Deduction

Books with the following kinds of lines:

Information-carrying lines (indicated below as $\cdots$):

C: constructing objects, propositions or types

A: claiming a statement without proof (axiom)

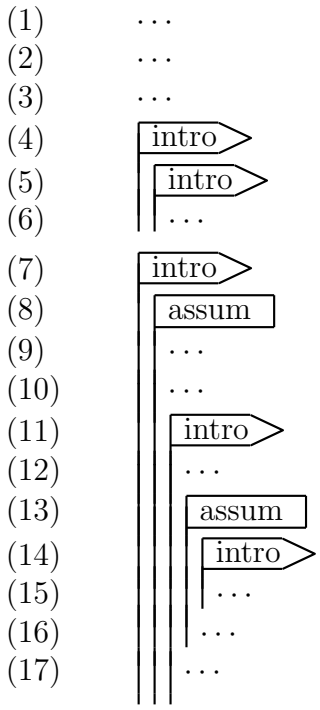
P: declaring objects, propositions or types as primitives

T: giving a statement with proof

Context-changing lines:

assum: introducing an assumption

intro: Introducing a (typed) variable or a variable type



Long Semipal

(3)	impl(x,y)	:=	PN
(4)	con	:=	PN
(5)	non(x)	:=	impl(x,con)
(6)	or(x,y)	:=	impl(non(x),y)
(7)	and(x,y)	:=	non(or(non(x),non(y)))
(8)	equiv(x,y)	:=	or(and(x,y),and(non(x),non(y)))

Semipal, presented with flags

(1) x
(2) y
(3) impl := PN
(4) con := PN
(5) non := impl(con)
(6) or := impl(non, y)
(7) and := non(or(non, non(y)))
(8) equiv := or(and, and((non, non(y)))

Semipal, presented with context indicators

(1)	∘	x	:=	—
(2)	x ∘	y	:=	—
(3)	y ∘	impl	:=	PN
(4)	∘	con	:=	PN
(5)	x ∘	non	:=	impl(con)
(6)	y ∘	or	:=	impl(non,y)
(7)	y ∘	and	:=	non(or(non,non(y)))
(8)	y ∘	equiv	:=	or(and,and(non,non(y)))

Survey of kinds of lines

Kinds of lines in AUTOMATH (as well as in PAL), with interpretation. They can occur in any context.

On the left is a code for the kind of line; it is not a part of the line. The letters x, c, f stand for identifiers; p, q stand for expressions.

(2tEB)	x :=	——	:	**type**	introducing a type variable
(2tPN)	c :=	PN	:	**type**	introducing primitive type
(2tDF)	f :=	p	:	**type**	defining a new type
(2pEB)	x :=	——	:	**prop**	introducing a prop variable
(2pPN)	c :=	PN	:	**prop**	introducing primitive prop
(2pDF)	f :=	p	:	**prop**	defining a new prop
(3tEB)	x :=	——	:	q	introducing an object variable
(3tPN)	c :=	PN	:	q	introducing a primitive object
(3tDF)	f :=	p	:	q	defining a new object
(3pEB)	x :=	——	:	q	making an assumption
(3pPN)	c :=	PN	:	q	proclaiming an axiom
(3pDF)	f :=	p	:	q	proving a theorem

Note: A prop is a *proof class*. It should not be seen as a proposition, but as the class of all proofs of a particular proposition.

In group 3p the letters x, c, f do not correspond to things that occur in formulas in ordinary mathematical texts. There they are replaced by *references* used in explanations.

Example of a book written in PAL

(1) $\boxed{\alpha : \mathbf{type}}$

(2) $\quad\boxed{x : \alpha}$

(3) $\quad\quad\boxed{y : \alpha}$

(4) $\quad\quad\mathrm{EQ} :=\ \mathrm{PN}\ : \mathbf{prop}$

(5) $\quad\quad a :=\ \mathrm{PN}\ : \mathrm{EQ}(x, x)$

(6) $\quad\quad\boxed{u : \mathrm{EQ}(x, y)}$

(7) $\quad\quad\quad\boxed{z : \alpha}$

(8) $\quad\quad\quad\quad\boxed{v : \mathrm{EQ}(z, y)}$

(9) $\quad\quad\quad\quad b :=\ \mathrm{PN}\ : \mathrm{EQ}(z, x)$

(10) $\quad\quad m := a(y) : \mathrm{EQ}(y, y)$

(11) $\quad\quad\quad c := b(y, m) : \mathrm{EQ}(y, x)$

(12) $\quad\quad\quad\quad\boxed{w : \mathrm{EQ}(y, z)}$

(13) $\quad\quad\quad\quad d := c(y, z, w) : \mathrm{EQ}(y, z)$

(14) $\quad\quad\quad\quad e := b(d) : \mathrm{EQ}(z, x)$

(15) $\quad\quad\quad\quad f := c(z, x, e) : \mathrm{EQ}(x, z)$

(16) $\quad\quad\quad\quad g := b(z, y, d, x, u) : \mathrm{EQ}(x, z)$

(17) $\quad\quad\quad\quad h := c(z, x, b(c(y, z, w))) : \mathrm{EQ}(x, z)$

(18) $\quad\quad\quad\quad k := b(z, y, c(y, z, w), x, u) : \mathrm{EQ}(x, z)$

(19) $\quad\quad\quad\quad l := b(z, x, b(b(y, z, w, z, a(z))), x, a) : \mathrm{EQ}(x, z)$

(20) $\quad\quad\quad\quad m := b(z, y, b(y, z, w, z, a(z), x, u) : \mathrm{EQ}(x, z)$

Interpretation. If $\equiv$ satisfies (i) $x \equiv x$ and (ii) $\frac{x \equiv y \quad z \equiv y}{z \equiv x}$ then it satisfies (iii) $\frac{x \equiv y}{y \equiv x}$ and (iv) $\frac{x \equiv y \quad y \equiv z}{x \equiv z}$.

Proof. Replacing x by y in (i) we get $x \equiv y$ (This is expressed by m). Replacing z by y in (ii) we get $\frac{x \equiv y \quad y \equiv y}{y \equiv x}$ (this is expressed by c, which uses b and m).

In order to prove (iv) we assume $x \equiv y$ and $y \equiv z$, and our goal is $x \equiv z$. We have two different proofs. In the first one we start by turning $y \equiv z$ into $z \equiv y$ by application of (iii) (updating c), then we apply (iv) (updating now just means fulfilling the assumption v by the proof d). So we have reached $z \equiv x$, and we have to apply an updated version of (ii) in order to get to the goal $x \equiv z$.

The second proof starts by writing (ii) in the form $\frac{z \equiv y \quad y \equiv z}{x \equiv z}$ (just interchanging x and z) and this time we have to appeal to d only once.

The proofs f and g are essentially different. We evaluate their normal forms. First, by elimination of d and e, h and k are obtained from f and g, respectively. Eliminating c we get l and m as normal forms of f and g, respectively.

NATURAL DEDUCTION

WAS NOT A NEW IDEA.

ON THE CONTRARY:

IT WAS THE WAY PEOPLE REASONED

BEFORE IT WAS TRIED TO EXPLAIN REASONING

BY MEANS OF THE

ALGEBRA OF TRUTH VALUES.

BOOLEAN LOGIC IS

METATHEORY OF CLASSICAL REASONING.

NOT REASONING ITSELF.

E. LANDAU GRUNDLAGEN DER ANALYSIS 1927

L. S. v. BENTHEM JUTTING 1975
TRANSLATION INTO AUTOMATH

	LANDAU	JUTTING
NUMBER CHAR.	10^5	10^6
		1180 KB
WRITING TIME	60 d.	600 d.
READING TIME	2 d.	$\frac{1}{10}$ d.
		2 SEC.

F. WIEDIJK IN 21ST CENTURY

AS POWERFUL AS 1990 PC's

BURROUGH's ALGOL COMPUTER 1970 - 1975

THEORY OF REAL NUMBERS ETC.

JUTTING
1971 - 1975
OFF AND ON

1180 kB
600 d.

HONEST LANDAU TRANSLATION

NAT. NRS
+ - x

INTEGERS
+ - x

RATIONALS
+ - x :

REALS
+ - x :

COMPLEX NRS.

MATHEMATICIAN

ZUCKER
1974 - 1975

AUT_TI
WITH

SYNTACTIC ABBREVIATIONS

&

TELESCOPES

(NEVER CHECKED).

NO SCRAP PAPER

$$e^x = \sum_0^\infty \frac{x^k}{k!}$$

LOGICIAN

UDDING
1979
ONE MAN SHOW

620 KB
300 d. ALL IN

DIRECT DEFINITION OF REALS

10011 . 0100111010 ...

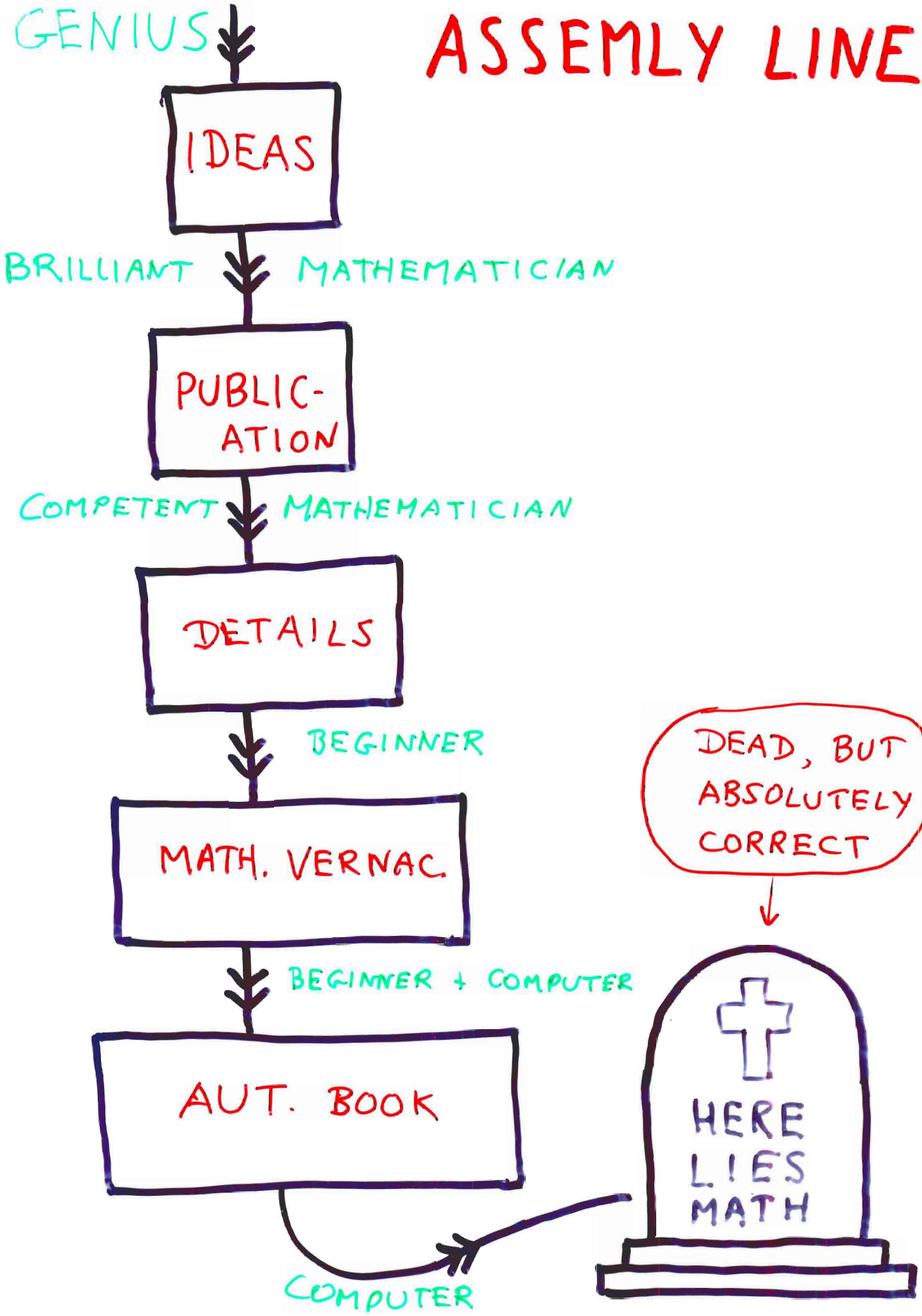
+ - x

INFIMUM

DIVISION

COMPUTER SCIENTIST

ASSEMLY LINE



The AUTOMATH RESTAURANT

In one and the same restaurant

one can eat fish or meat,

eat vegetarian or kosher

or just have a few drinks,

smoking or non-smoking.

In one and the same Automath book:

classical or intuitionistic mathematics,

constructive or non-constructive,

with or without Cantor's paradise.

Protected against clash of

conflicting axioms.

THE RESTAURANT'S FLEXIBLE UNIVERSE

Once we admit the type $\mathcal{Z}$ we have admitted

$$\mathcal{Z}^{\mathcal{Z}},\ \mathcal{Z}^{\mathcal{Z}^{\mathcal{Z}}},\ \mathcal{Z}^{\mathcal{Z}^{\mathcal{Z}^{\mathcal{Z}}}},\ldots$$

but not their union.

We can always create types, saying: " Let X be a type" (without demanding it to fit in the above universe), so we can, if we insist, introduce BIG types by means of axioms.

SOME LATER DEVELOPMENTS

1974 AUT4.

1974 Framework for description of generalizations of Automath.

1978 AUT-QE-NTI (weaker than AUT, leaving as much as possible to the user).

1978 The segment calculus $\Lambda\Sigma$.

1980 Formalized mathematical vernacular (MV).

1980 J.T. Udding's book on the Landau material.

1983 Program semantics in space and time.

1984 Geometric constructions.

1986 H. Balsters' theory on $\Lambda\Sigma$.

1987 $\Lambda\Delta$ (a very simple vehicle for all systems with internalized definitions).

Before about 1990 it was hard to publish such material. Some of it appeared in 1994 in: Selected Papers on Automath. Editors R.P. Nederpelt, J.H. Geuvers and R.C. de Vrijer. Studies in Logic, Vol. 133. North-Holland 1994.

The Automath project generated a large number of reports, electronic copies of which are being made now (January 2004).

End of sequence of transparencies of the talk
"Memories of the Automath project"
by N.G. de Bruijn