

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics

Memorandum 78-08.

Issued August 1978.

A note on weak diamond properties.

by

N.G. de Bruijn.

Eindhoven University of Technology
Department of Mathematics
P.O.Box 513
5600 MB Eindhoven
The Netherlands.

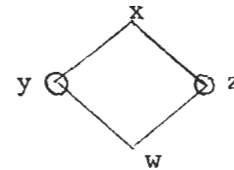
A note on weak diamond properties.

1. Introduction. Let S be a set with a binary relation $>$. We assume it to satisfy $x > x$ for all $x \in S$. We are interested in establishing a property CR (named after its relevance for the Church-Rosser theorem of lambda calculus, cf. [1]). We say that $x \sim y$ if $x > y$ or $y > x$. We say that $x >^* y$ if there is a finite sequence x_1, \dots, x_n with $x = x_1 > x_2 > \dots > x_n = y$, and also if $x = y$. We say that $(S, >)$ satisfies CR if for any sequence x_1, \dots, x_n with

$$x_1 \sim x_2 \sim \dots \sim x_n$$

there exist an element $x \in S$ with both $x_1 >^* x$ and $x_n >^* x$.

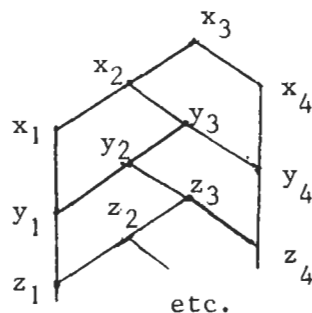
It is usual to say that $(S, >)$ has the diamond property (DP) if for all x, y, z with $x > y, x > z$ there exists a w with $y > w, z > w$. This is depicted in the following diagram:



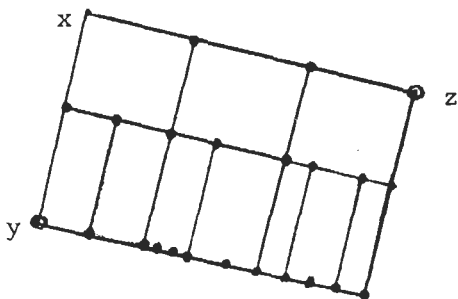
where $x > y$ is indicated by a line from x downwards to y , etc. The little circles around y and z illustrate the logical situation: the diagram $y \overset{x}{\wedge} z$ can be closed by $y \underset{w}{\vee} z$.

It is not hard to show that DP implies CR. A simple way to present a proof is by counting "inversions" in sequences like $x_1 > x_2 < x_3 < x_4 > x_5 < x_6 > x_7$: if $i < j$ and $x_i < x_{i+1}, x_j > x_{j+1}$, then we say that the pair (i, j) forms an inversion. Applications of DP, like replacing $x_3 < x_4 > x_5$ by $x_3 > x_4^* < x_5$, decrease the number of inversions. Once all inversions are gone, we have established CR.

The following property WDP_1 is weaker than DP. It says: "if $x > y$ and $x > z$ then w exists such that $y >^* w$ and $z >^* w$ ". It is very frustrating in attempts to prove the Church-Rosser theorem for various systems, that WDP_1 does not imply CR. A counterexample can be obtained by means of the following picture (cf. [2] p. 49):



This example also shows that CR neither follows from WDP_2 where WDP_2 is slightly stronger than WDP_1 and says: "if $x > y$ and $x > z$ then w exists such that $y >^* w$ and $z >^* w$ and at least one of $y > w$ and $z > w$ ". Stronger again is WDP_3 , expressing: "if $x > y$ and $x > z$ then w exists such that $y >^* w$ and $z > w$ ". This WDP_3 does imply CR. Actually WDP_3 implies WDP_4 , which says: "if $x >^* y$ and $x >^* z$ then w exists such that both $y >^* w$ and $z >^* w$ ". This WDP_4 is the DP for $(S, >^*)$, and therefore implies CR for $(S, >^*)$, and that is the same thing as CR for $(S, >)$. The derivation of WDP_4 from WDP_3 is illustrated by the following picture (cf. [2] p. 59) which speaks for itself:



In this note we go considerably further. Instead of having just one relation $>$ we consider a set of relations $>_m$ where m is taken from an index set M . The idea behind this is that in the Church-Rosser theorem the relations represent lambda calculus reductions; there may be reductions of various types, and diamond properties may depend on these types. It is our purpose to establish weak diamond properties which guarantee CR (where CR has to be interpreted as in section 4).

2. The index set. $(M, <)$ is a well-ordered set. That is, the order $<$ is total (i.e. it is transitive, and for all m_1, m_2 we have exactly one of $m_1 = m_2$, $m_1 < m_2$, $m_2 < m_1$, and there are no infinite descending chains $m_1 > m_2 > m_3 > \dots$. Note that in M we do not have $m < m$, in contrast to what will be assumed in S .

There might be use for cases with more general $(M, <)$, like partial order with descending chain condition. We shall not study such extensions in this note.

3. The set S with its order relation. S is a set, and for each $m \in M$ there is a relation $>_m$ on S . The only thing we require is that $x >_m x$ for all $x \in S$ and all $m \in M$.

For all $m \in M$ we now introduce two further relations $>_{m+}$ and $>_{m-}$. We say that $x >_{m+} y$ if there is a finite chain $x = x_0, x_1, \dots, x_n = y$ (possibly $n=0$ $x=y$) and elements $k_1 \leq m, \dots, k_n \leq m$ such that $x_0 >_{k_1} x_1 >_{k_2} \dots >_{k_n} x_n$. And we say that

$x >_{m^-} y$ if there is a chain $x=x_0 >_{k_1} x_1 >_{k_2} \dots >_{k_n} x_n = y$ with $k_1 < m, \dots, k_n < m$. Again this includes the case that $n=0, x=y$, even in the case that m is the minimal element of M and no k with $k < m$ exist. Note that if $x >_{m^-} y$ then $(x=y) \vee \exists_{k \in M} | k < m \ x >_{k^+} y$.

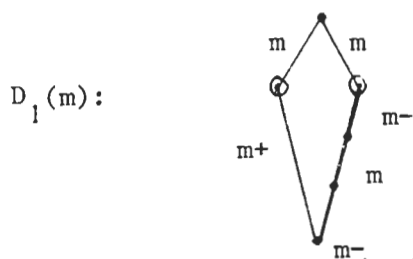
4. The properties CR(m). We write $x \sim_m y$ if $k \in M$ exists with $k \leq m$ and $x >_k y$ or $y >_k x$. We write $x \sim y$ if $m \in M$ exists with $x \sim_m y$. In other words, $x \sim y$ means that $k \in M$ exists such that $x >_k y$ or $y >_k x$.

Let $m \in M$. We say that CR(m) holds if for every finite sequence $x_1 \sim_m x_2 \sim_m \dots \sim_m x_n$ there exists z such that both $x_1 >_{m^+} z$ and $x_n >_{m^+} z$.

We say that CR holds if for every finite sequence $x_1 \sim x_2 \sim \dots \sim x_n$ there exists z such that both $x_1 >_{m^+} z$ and $x_n >_{m^+} z$ for some m .

Obviously CR is equivalent to $\bigvee_{m \in M} CR(m)$.

5. The basic diamond properties. If $m \in M$, the diamond property $D_1(m)$ is defined by the following diagram.



This has to be read as follows (and further diagrams have to be interpreted analogously: If x, y, z are such that $x >_m y, x >_m z$, then u, v, w exist such that

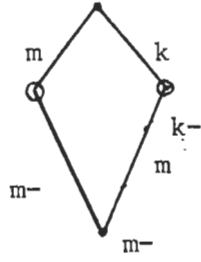
$$y >_{m^+} w, z >_{m^-} u >_m v >_{m^-} w.$$

(so on the left we have a chain from y to w with all links $\leq m$; on the right we have a chain from z to w with all links $\leq m$ but with at most one $= m$).

Note that $D_1(m)$ is a generalization of WDP_3 and not of WDP_2 (see section 1). We get WDP_3 as a special case of $D_1(m)$ if m is the minimal element of M and if $>_m$ is just written as $>$.

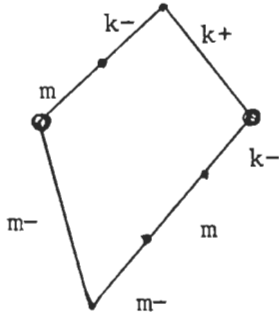
The second diamond property to be considered depends on two elements m, k of M , with $k < m$. Its diagram is

$D_2(m,k):$

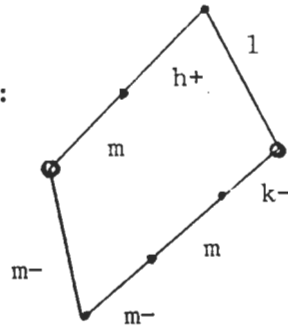


6. Some auxiliary diamond properties. We intend to show that $D_1(m)$ and $D_2(m,k)$ (for all m,k with $k < m$) lead to CR. In order to achieve this we formulate a number of diamond properties that will play a rôle in the proof.

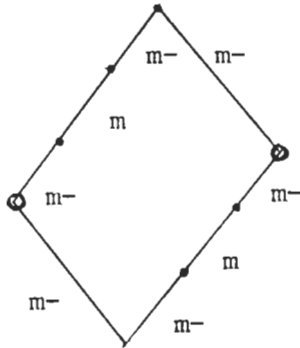
$D_3(m,k):$



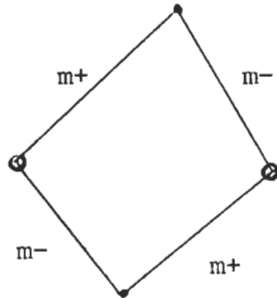
$D_4(m,k,1,h):$



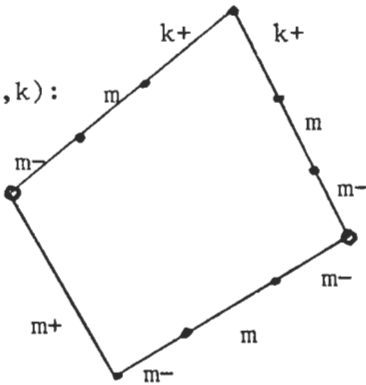
$D_5(m):$



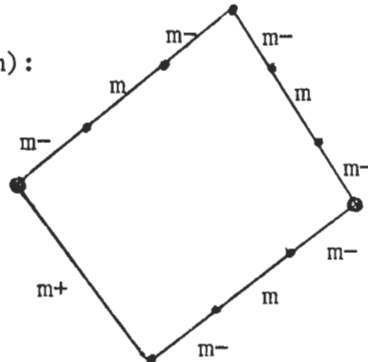
$D_6(m):$



$D_7(m,k):$



$D_8(m):$



The diagrams D_3 and D_7 will play their rôle only if $k < m$, and D_4 only if $h < k < m, 1 \leq m$.

7. Derivation of CR from the basic diamond properties. Throughout this section we assume that for all $m \in M$ we have $D_1(m)$, and for all $k \in M$, $m \in M$ with $k < m$ we have $D_2(m,k)$.

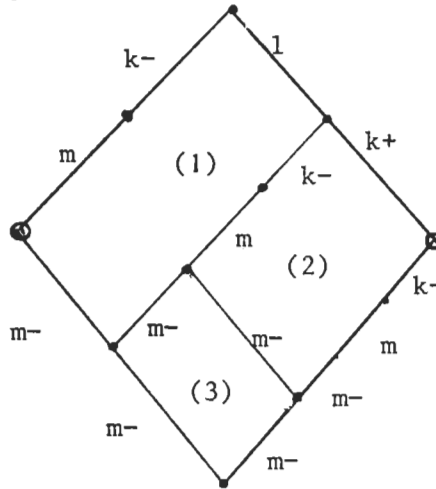
For brevity we introduce $CR^*(m)$ by

$$CR^*(m) = \forall_{k \in M | k < m} CR(k).$$

Lemma 1. Let $m, k \in M$, $k < m$. Then we have

$$(CR^*(m) \wedge \forall_{l \in M | l \leq k} \forall_{h \in M | h < k} D_4(m,k,l,h)) \Rightarrow D_3(m,k).$$

Proof. In the diagram for D_3 we see a branch $k+$. It consists of a number of steps $x_0 >_{l_1} x_1 >_{l_2} x_2 \dots >_{l_n} x_n$ with all l_i 's $\leq k$. The case $n=0$ is trivial. We prove the general case by induction with respect to n . The branch $k+$ can be split in a branch l (with $l \leq k$) and a branch $k+$ (of $n-1$ steps). The following diagram now produces the proof:



Inside this diagram we have given reference to applied properties.

$$(1) \quad \forall_{l \in M | l \leq k} \forall_{h \in M | h < k} D_4(m,k,l,h)$$

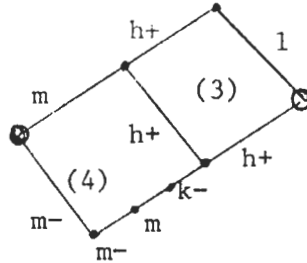
(note that if there is no $h < k$ we just apply $D_2(m,l)$),

$$(2) \quad D_3(m,k) \text{ with } n \text{ replaced by } n-1 \text{ (the induction hypothesis),}$$

$$(3) \quad CR^*(m).$$

Lemma 2. Let $m, k, l, h \in M$, $h' < k < m$, $1 \leq k$. Assume $D_3(m, t)$ for all $t \in M$ with $t' < k$, and $D_3(s, t)$ for all $s, t \in M$ with $t' < s < m$. Furthermore assume $CR^*(m)$. Then we have $D_4(m, k, l, h)$.

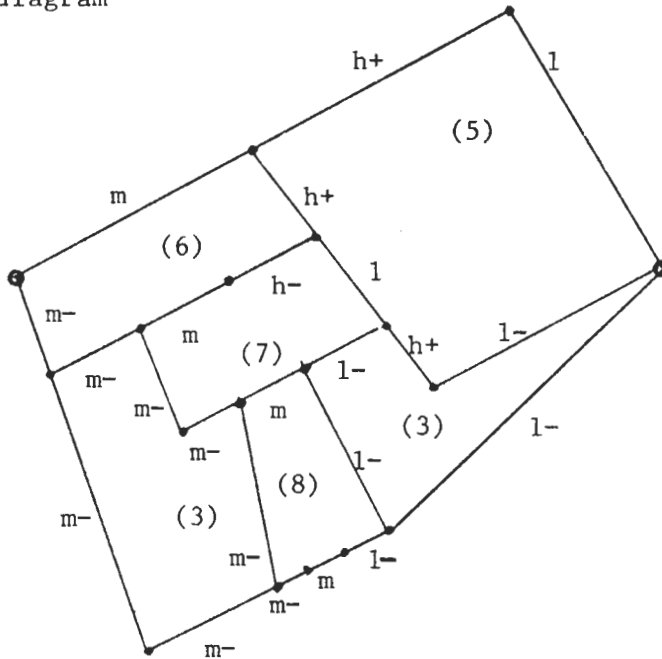
Proof. If $1 \leq h$ the argument is



with (3) as above,

(4) $D_3(m, h)$.

For $h < 1$ we proceed by induction with respect to h by means of the following diagram



where

(5) $D_3(1, h)$,

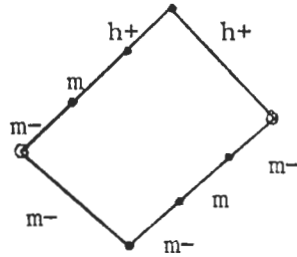
(6) $D_3(m, h)$,

(7) $(\forall t \in M | t' < h \ D_4(m, 1, 1, t)) \wedge D_2(m, 1)$,

(8) $\forall t \in M | t < 1 \ D_3(m, t)$.

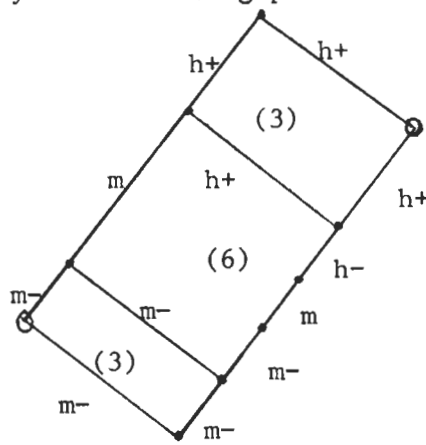
Lemma 3. Let $m \in M$. Assume $CR^*(m)$ and assume that for all $h \in M$ with $h < m$ we have $D_3(m, k)$. Then $D_5(m)$.

Proof. It suffices to prove the diagram



for all $h < m$. (Note that if m is the minimal element of M then the branches denoted by m^- are empty and there is nothing to be proved).

The proof is given by the following picture.



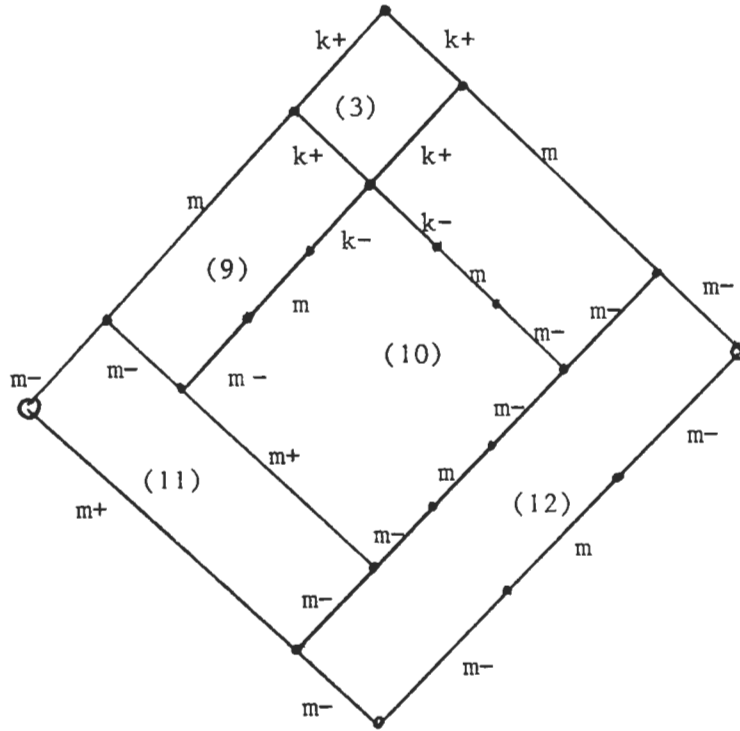
Lemma 4. If $m \in M$ then $D_5(m)$ implies $D_6(m)$.

Proof. The upper left branch m^+ can be split into a number of pieces of the form (m^-, m, m^-) (like the branches occurring in the diagram $D_5(m)$; if no $>_m$ occur in that branch we introduce one artificially, using $x >_m x$). To these pieces we apply $D_5(m)$ in succession.

Lemma 5. Let $m, k \in M$, $k < m$. Assume $CR^*(m)$, $D_5(m)$, $D_6(m)$, and $D_3(m, h)$ for all h with $h < m$. Then we have $D_7(m, k)$.

Proof. We apply induction with respect to k , assuming $D_7(m, h)$ for all h with $h < k$ (of any). The arguments for the proof (see picture) are

- (9) $D_3(m, k)$
- (10) $(\forall_{h \in M | h < k} D_7(m, h)) \wedge D_1(m),$
- (11) $D_6(m),$
- (12) $D_5(m).$



Lemma 6. Let $m \in M$ and assume $D_7(m, k)$ for all $k < m$. Then we have $CR(m)$.

Proof. Since $D_7(m, k)$ for all $k < m$, we have the diagram $D_8(m)$ (if there are no $k < m$ we apply $D_1(m)$). The branch $m+$ can be split into a number of pieces $(m-, m, m-)$, so we can interpret $D_8(m)$ as property WDP_3 for the relation $>$ which is defined as follows: $x > y$ if u and v exist such that $x >_{m-} u >_m v >_{m-} y$. By the procedure described in section 1 we now get to CR for $(S, >)$, and that means exactly the same thing as $CR(m)$.

Theorem. For all $m \in M$ we have $CR(m)$.

Proof. Assume the theorem false. Then we have an m such that $CR^*(m)$ is true, but $CR(m)$ false. We cannot have $D_3(m, h)$ for all h with $h < m$, for then we would have $D_5(m)$ by lemma 3, $D_6(m)$ by lemma 4, and then lemmas 5 and 6 would lead to $CR(m)$. So there is some k with $k < m$ and $D_3(m, k)$ false. Let n be the smallest element of M for which $j \in M$ exists with $j < n$ and $D_3(n, j)$ false. Next let i be the smallest index $< n$ with $D_3(n, i)$ false. By lemma 1 we have (note that $n \leq m$) l and h such that $1 \leq i, h < i$, $D_4(n, i, l, h)$ false. Now lemma 2 leads to a contradiction.

References.

- [1] Barendregt, H.P.: The Type Free Lambda Calculus, ch. D7 in Handbook of Mathematical Logic, ed. J.Barwise, North-Holland Publ.Comp., Amsterdam-New York - Oxford 1977.
- [2] Nederpelt, R.P.: Strong normalization in a typed lambda calculus with lambda structured types, Ph.D.Thesis, Eindhoven University of Technology, 1973.