

Proposed abbreviation for abstraction with quantification
in Aut-II.

The following abbreviation for abstraction has been proposed by de Bruijn (T.H., Dept of Math, Nov. 15, 1972). \mathcal{A} , in a context

$$x_n : \alpha_n, \dots, x_1 : \alpha_1$$

we have an identifier p (defined w/ PN), we can denote the k -fold abstraction of p ($k \leq n$), i.e.

$$[x_k : \alpha_k] \dots [x_1 : \alpha_1] p(x_k, \dots, x_1)$$

(with context x_n, \dots, x_{k+1}) as

$$\textcircled{k} p.$$

We propose the following generalization of this.

By quantification, we mean here an operation on predicates to form propositions (e.g. by \forall or \exists), or, more generally, an operation on functions (e.g. by \prod or Σ).

We write Q (with subscript) for any such quantifier ($\forall, \exists, \prod, \Sigma, \dots$)

Then, with p as above, we denote a k -fold abstraction-with-quantification of p ($k \leq n$), i.e.

$$Q_k [x_k : \alpha_k] \dots Q_1 [x_1 : \alpha_1] p(x_k, \dots, x_1)$$

(in context x_n, \dots, x_{k+1}) as

$$\textcircled{Q_k \dots Q_2 Q_1} p.$$

Further we denote a k -fold abstraction-with-quantification of p , followed by l -fold abstraction alone ($l > 0, k+l \leq n$), i.e.

$[x_{k+l} = \alpha_{k+l}] \dots [x_{k+1} = \alpha_{k+1}] Q_k [x_k = \alpha_k] \dots Q_1 [x_1 = \alpha_1] P(x_{k+l}, \dots, x_1)$
 (in context x_n, \dots, x_{k+l+1}) as

$$\textcircled{Q_k \dots Q_2 Q_1} P$$

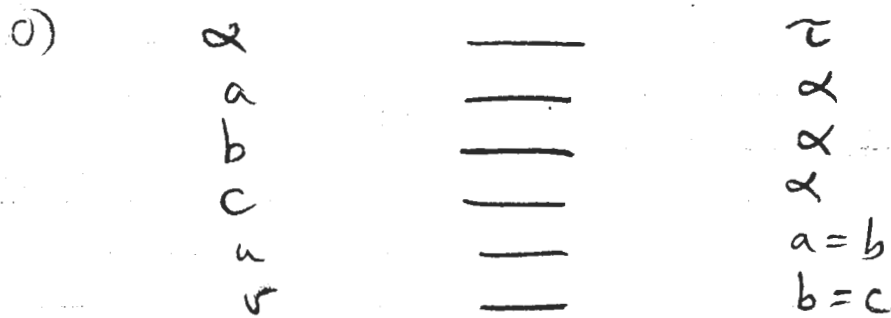
These circled expressions can be called abstrac-tion-quantifier strings or just quantifier strings.

We can use number superscripts to indicate repetitions of quantifiers, e.g.

$$\textcircled{\forall^3 \exists^2} \text{ for } \textcircled{\forall \forall \forall \exists \exists}$$

Note: on anti- \forall -superals, \forall and \exists can (in principle) be used interchangeably, and \exists is defined as $\neg \forall \neg$.

Example 1. Axiom of transitivity for equality. Consider



$p_0 := a=c$: τ

$AxTr$:= \mathcal{PN} : p_0

Now we can define the universal closure of $AxTr$ (in context α) as

$\alpha $ClAxTr$:= $(\exists)^5 AxTr$: $(\forall^5)^5 p_0$$

where $(\forall^5)^5 p_0$ stands for

$$\forall [x:\alpha] \forall [y:\alpha] \forall [z:\alpha] (x=y \rightarrow (y=z \rightarrow x=z)).$$

So (just) abstracting 5 times from $AxTr$ corresponds to abstracting-with \forall 5 times from p_0 .

Example 2. Definition of limit.

Consider a context (α, d, ass) where ass is the assumption that (α, d) is a metric space.

ass)	S	—	$N \rightarrow \alpha$
	a	—	α
	ϵ	—	\mathbb{R}
	u	—	$\epsilon > 0$
	k	—	N
	l	—	N
	v	—	$k \leq l$
	$p_1 := d(\{l\}s, a) < \epsilon : \pi$		

Now we define (in context S) the predicate of being a limit of s:

$$s) \quad \underline{\text{Lim}} := (\exists \epsilon > 0 \exists N \forall k \geq N \forall l \leq k \forall v) p_1 : \alpha \rightarrow \pi \quad (1)$$

Next, we define (in same context) the proposition "s is convergent"

$$s) \quad \underline{\text{Conv}} := \exists (\text{Lim}) : \pi$$

We could also have defined Conv straight from p_1 :

$$s) \quad \underline{\text{Conv}} := (\exists \epsilon > 0 \exists N \forall k \geq N \forall l \leq k \forall v) p_1 : \pi \quad (2)$$

i.e. the same quantifier-string as in (1), but with "E" instead of "1" at the beginning.

Note for ease of reading, we may want to distinguish in practice between the use of "A" (for universally quantifying over objects), and "E" (for other uses, e.g. univ. quant. per proof). Then (1) can be re-written: $(\forall \epsilon > 0 \exists N \forall k \geq N \forall l \leq k \forall v) p_1$ and similarly for (2).