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Proposed abbreviation for abstraction with quantification in Aut-Π.

The following abbreviation for abstraction has been proposed by de Bruijn (T.H., Dept of Math, Nov. 15, 1972). If, in a context

$$x_n : \alpha_n, \ldots, x_1 : \alpha_1$$

we have an identifier p (defined or PN), we can denote the k-fold abstraction of p $(k \leq n)$, i.e.

$$[x_k : \alpha_k] \cdots [x_1 : \alpha_1]\, p(x_k, \ldots, x_1)$$

(with context $x_n, \ldots, x_{k+1}$) as

$$(k)\, p.$$

We propose the following generalization of this.

By quantification, we mean here an operation on predicates to form propositions (e.g. by $\forall$ or $\exists$), or, more generally, an operation on functions (e.g. by Π or Σ).

We write Q (with subscript) for any such quantifier ($\forall, \exists, \Pi, \Sigma, \ldots$)

Then, with p as above, we denote a k-fold abstraction-with-quantification of p $(k \leq n)$, i.e.

$$Q_k[x_k : \alpha_k] \cdots Q_1[x_1 : \alpha_1]\, p(x_k, \ldots, x_1)$$

(in context $x_n, \ldots, x_{k+1}$) as

$$(Q_k \cdots Q_2 Q_1)\, p.$$

Further we denote a k-fold abstraction-with-quantification of p, followed by l-fold abstraction alone $(l > 0,\ k+l \leq n)$, i.e.

$$[x_{k+l}:\alpha_{k+l}]\cdots[x_{k+1}:\alpha_{k+1}]Q_k[x_k:\alpha_k]\cdots Q_1[x_1:\alpha_1]p(x_{k+l},\ldots,x_1)$$

(in context $x_n,\ldots,x_{k+l+1}$) as

$$(l,\ Q_k\ldots Q_2 Q_1)p$$

These circled expressions can be called abstraction-quantifier strings or just quantifier strings.

We can use number superscripts to indicate repetitions of quantifiers, e.g.

$$(\Pi^3\exists^2) \text{ for } (\Pi\Pi\Pi\exists\exists)$$

Note. In Aut-Π-superteal, Π and $\forall$ can (in principle) be used interchangeably, and $\exists$ is defined as $\neg\forall\neg$.

Example 1. Axiom of transitivity for equality. Consider

$$
\begin{array}{lll}
0) & \alpha & \underline{\quad\quad} \quad \tau \\
 & a & \underline{\quad\quad} \quad \alpha \\
 & b & \underline{\quad\quad} \quad \alpha \\
 & c & \underline{\quad\quad} \quad \alpha \\
 & u & \underline{\quad\quad} \quad a = b \\
 & v & \underline{\quad\quad} \quad b = c
\end{array}
$$

$$p_0 := a = c \quad : \Pi$$

$$\underline{AxTr} := PN \quad : p_0$$

Now we can define the universal closure of AxTr (in context α) as

$$\alpha) \quad \underline{ClAxTr} := (5)\, AxTr \quad : (\Pi^5)\, p_0$$

where $(\Pi^5)\, p_0$ stands for

$$\forall [x : \alpha] \forall [y : \alpha] \forall [z : \alpha] (x = y \to (y = z \to x = z)).$$

So (just) abstracting 5 times from AxTr corresponds to abstracting-with-Π 5 times from p_0.

Example 2. Definition of limit.

Consider a context (α, d, ass) where ass is the assumption that (α, d) is a metric space.

ass)	s	—	$N \to \alpha$
	a	—	α
	ε	—	Rl
	u	—	$\varepsilon > 0$
	k	—	N
	l	—	N
	v	—	$k \leq l$

$$p_1 := d(\{l\}s, a) < \varepsilon \quad : \pi .$$

Now we define (in context s) the predicate of being a limit of s:

$$s) \quad \underline{Lim} := (1, \pi^2 \exists \pi^2) p_1 : \alpha \to \pi \qquad (1)$$

Next, we define (in same context) the proposition "s is convergent"

$$s) \quad \underline{Conv} := \exists(Lim) \quad : \pi$$

We could also have defined Conv straight from p_1:

$$s) \quad \underline{Conv} := (\exists \pi^2 \exists \pi^2) p_1 \quad : \pi \qquad (2)$$

i.e. the same quantifier-string as in (1), but with "$\exists$" instead of "1" at the beginning.

Note For ease of reading, we may want to distinguish in practice between the use of "$\forall$", (for universally quantifying over objects), and "π" (for other uses, e.g. univ. quant. over proofs). Then (1) can be re-written: $(1, \forall \pi \exists \forall \pi) p_1$; and similarly for (2).