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A note on one-round beta reduction

by

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1. Introduction. We shall express this note in terms of typed lambda calculus, but the ideas apply to untyped lambda calculus as well. We use AUTOMATH notation for application and abstraction.

One-sweep beta reduction (to be denoted by the operator Σ) has the following intuitive meaning: Apply beta reduction wherever this is possible, ~~to~~ but do not do it with new redices. A new redex is a redex that arises when an applicational knot directly in front of a variable turns into an AT pair, because by substitution and possible further beta reductions an abstraction turns up directly after it. On the other hand, redices that appear in the formula just by transportation (i.e. by substitution on behalf of beta reduction) are not to be considered as new, unless they were new already at the place where they came from.

2. Recursive definition. As an auxiliary operator we define $\theta(\alpha)$ for any lambda formula α as follows. If

α has the form $\langle\varepsilon\rangle[x:\gamma]\delta$ then

$$\theta(\alpha) = S_x^{\varepsilon}\,\delta$$

(i.e. the effect of substituting ε for all x's in δ, in other words just exterior beta reduction applied to α). On the other hand, if α is not of that form $\langle\varepsilon\rangle[x:\gamma]\delta$ then we just define $\theta(\alpha)=\alpha$.

The one-round beta result of a formula α will be denoted by $\xi(\alpha)$. It is defined recursively by

(i) $\xi(\alpha) = \alpha$ if α is a variable or a constant

(ii) $\xi([x:\gamma]\delta) = [x: \xi(\gamma)]\ \xi(\delta)$

(iii) $\xi(\langle\varepsilon\rangle\eta) = \theta(\langle\xi(\varepsilon)\rangle\ \xi(\eta))$.

We remark that ξ is just one definite operator. In that respect it differs from beta reduction: if a beta-reduction is internal it has to be indicated at what subexpression it has to be applied.

3. <u>The use of ξ</u>. As yet we have no suggestion for the use of the one-round beta reduction ξ rather than the

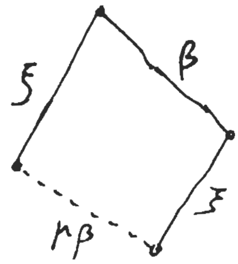
ordinary beta reduction. We just vaguely state that it might have advantages to deal with a single operator rather than with the many beta's, especially for theoretical purposes. It is inconvenient to talk about beta reducti[ons] at places that have not even arisen yet, but it is very easy to say "apply ξ k times". There is no trouble about commutativity of products of ξ's either.

In a way we can reduce with ξ's as effectively as with β's. This can be seen from the properties in the next sections, in particular section 6.

4. Reducibility rules. Let us write $\delta \geq_\xi \gamma$ if δ reduces to γ by means of single-round beta reduction. As usual we write $\delta \geq_\beta \gamma$ if δ reduces to γ by (external or internal) beta reduction, and $\delta \geq_{\boldsymbol{\beta}} \gamma$ if $\delta = \gamma$ or if δ reduces to γ by a finite sequence of beta reductions.

Rule 1. If $\delta \geq_\xi \eta$, $\delta \geq_\beta \zeta$, $\zeta \geq_\xi \gamma$ then $\eta \geq_{\boldsymbol{\beta}} \gamma$

We can express this rule by the following diagram:



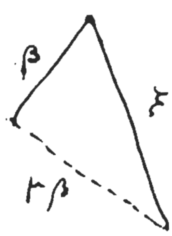
The drawn lines refer to the left-hand side of the implication, the dotted line to the right-hand side. And ξ, β, μβ refer to $\underset{\xi}{>}$, $\underset{\beta}{>}$, $\underset{\beta}{\geqslant}$, respectively. The higher end-point of a line refers to the left-hand side of the inequality sign.

It is not very hard to prove Rule 1 by means of the recursive definition of section 2.

Rule 2

Rule 3 If $\delta \underset{\beta}{>} \eta$, $\delta \underset{\xi}{>} \gamma$ then $\eta \underset{\beta}{\geqslant} \gamma$.

This has the diagram



~~This rule is very easy to prove by means of the recursive definition of section 2.~~

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The proof of rule 3 requires more effort than those of rules 1 and 2. We present it in some detail. (Throughout this proof we abbreviate $\geq_\beta$ to $\geq$ and $>_\beta$ to $>$.) We have to prove that if $\eta > \zeta$ then $\zeta \geq \xi(\eta)$. The proof goes by induction with respect to the length of η. The only case where the induction step is not entirely trivial is the one where $\eta = \langle\gamma\rangle\delta$.

First we look at the sub-case where the passage from η to ζ is internal beta reduction. So $\zeta = \langle\gamma'\rangle\delta'$ where $\gamma \geq \gamma'$, $\delta \geq \delta'$. By induction hypothesis $\gamma' \geq \xi(\gamma)$, $\delta' \geq \xi(\delta)$, whence $\zeta \geq \langle\xi(\gamma)\rangle\xi(\delta)$. By the definition of ξ we have $\xi(\langle\gamma\rangle\delta) = \theta(\langle\xi(\gamma)\rangle\xi(\delta))$ and by the definition of θ we have $\langle\xi(\gamma)\rangle\xi(\delta) \geq \theta(\langle\xi(\gamma)\rangle\xi(\delta))$. Therefore $\zeta \geq \xi(\eta)$.

The remaining case is the one where the passage from η to ζ is by external beta reduction. So δ has the form $[x:\mu]\sigma$. Hence $\eta = \langle\gamma\rangle[x:\mu]\sigma$ and $\zeta = S_x^{\gamma}\sigma$. Furthermore $\xi(\eta) = \theta(\langle\xi(\gamma)\rangle\xi(\delta)) = \theta(\langle\xi(\gamma)\rangle[x:\xi(\mu)]\xi(\sigma)) = S_x^{\xi(\gamma)}\xi(\sigma)$. By rule 2 we have we have $\gamma \geq \xi(\gamma)$, $\sigma \geq \xi(\sigma)$. So $\zeta \geq \xi(\eta)$ follows from the observation that $\gamma \geq \gamma'$, $\sigma \geq \sigma'$ imply

X $S_x^{\gamma}\sigma \geq S_x^{\gamma'}\sigma'$

<u>Rule 3.</u> If $\delta \underset{\xi}{\geq} \gamma$ then $\delta \underset{\beta}{\geq} \gamma$.

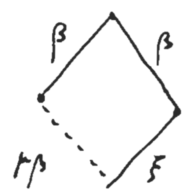
This has the diagram 

This is easy to prove by means of the recursive definition of ξ.
~~In order to prove rule 3 it suffices to show that if $\delta \underset{\xi}{\geq} \gamma$ and $\delta \neq \gamma$ then δ admits some beta reduction. Rule 2 does the rest.~~

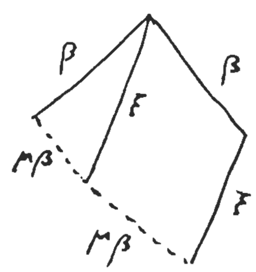
5. Derived reducibility rules.

We present some further rules as consequences of rules 1, 2, 3. For these further rules we need not go into the nature of ξ-reduction. We indicate them by their diagrams.

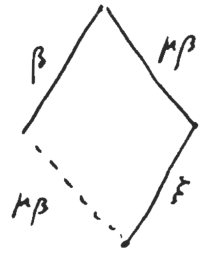
<u>Rule 4.</u>



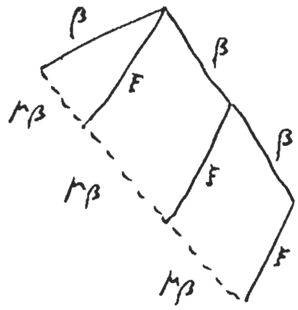
This is obvious from rules 3 and 1:



<u>Rule 5.</u>



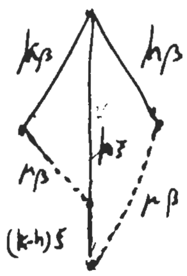
This is again obvious from rules 3 and 1, by induction with respect to the number of beta-steps in the $\mu\beta$. We show it for two steps:



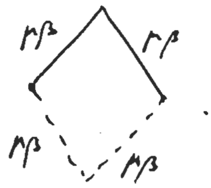
Rule 6. If $k\beta$ denotes k beta steps, and $k\xi$ denotes k-fold application of ξ ($k = 1, 2, \dots$) then

The case $k=1$ is rule 3, and the induction step is provided by rule 5.

From rules 6 and 2 we get the diamond property for $\mu\beta$. If $k \leq h$:



by rule 6, so by rule 2:



From this diamond property we of course get at once to the Church-Rosser theorem.

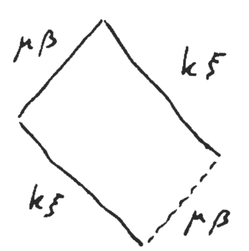
6. Normalization.

If a formula δ can be transformed into a normal form (i.e. a formula that admits no β-reduction) in k ~~beta~~ steps, then this normal form is equal to $\xi^k(\delta)$. This follows at once from rule 6.

The term *normalization depth* of δ is defined as the minimal value of k such that $\xi^k(\delta)$ is normal, ~~[illegible]~~. Notation $nd(\delta)$. If δ is not normalizable we put $nd(\delta) = \infty$.

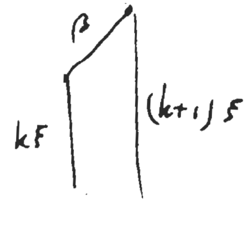
~~By repeated application of rule 1 we get~~

Rule 7 If $k \geq 1$:



The case $k=1$ follows by repeated application of rule 1. The general case follows by induction.

Rule 8 If $k \geq 0$



The case $k=0$ is rule 1 From this we

~~From rule 7 we get~~:

Theorem. If $\delta >_\beta \gamma$ then $nd(\delta) \geq nd(\gamma) \geq nd(\delta) - 1$.

Proof: By rule 7 we have $\xi^k(\delta) \geq_\beta \xi^k(\gamma)$.

If $k = nd(\delta)$ then $\xi^k(\delta)$ is normal, whence $\xi^k(\delta) = \xi^k(\gamma)$,

and therefore $\xi^k(\gamma)$ is normal. Therefore $nd(\gamma) \leq k$.

The proof of $nd(\gamma) \geq nd(\delta) - 1$ is similar, by rule 8.

The effect of ξ on nd is trivial: we have

$$nd(\xi(\delta)) = \max(nd(\delta) - 1, 0)$$