Decentralized control with input saturation

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Outline of the talk

- Systems and control - short introduction
- Decentralized control systems - conditions for stabilization
- Systems with input saturation - conditions for stabilization
- Decentralized control systems with input saturation - semi-global stabilization
  - necessary conditions
  - sufficient conditions for distinct eigenvalues on the imaginary axis
- Multiple eigenvalues on the imaginary axis case
- Conclusions
Control systems - description

● Structure:

\[
\Sigma_{cl} \quad \Sigma \quad K
\]

\[\text{y} \quad \text{u}\]

● Characteristics:
  - \(\Sigma\): system to be controlled (open loop system); input \(u\), output \(y\);
  - \(K\): controller; input \(y\), output \(u\);
  - \(\Sigma_{cl}\): interconnection between \(\Sigma\) and \(K\) (closed-loop system).
Open loop and closed loop

• Open loop dynamics:

\[ \Sigma : \begin{cases} 
\dot{x} = F(x, u) \\
y = G(x) 
\end{cases} \]

• Feedback controller:

\[ K : \begin{cases} 
\dot{z}(t) = f(z, y) \\
u(t) = g(z, y) 
\end{cases} \]

• Closed loop dynamics:

\[ \Sigma_{cl} : \dot{x}_c = \tilde{F}(x_c) \]
Linear time-invariant case

- Open loop dynamics:
  \[ \Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \]

- Feedback controller:
  \[ K : \begin{cases} \dot{z} = Kz + Ly \\ u = Mz + Ny \end{cases} \]

- Closed loop dynamics:
  \[ \Sigma_{cl} : \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A + BNC & BM \\ LC & K \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \]
The role of feedback is to adjust the system behavior according to certain desired goals: asymptotic stability, disturbance rejection, optimal control, etc.

In the linear case, the (asymptotic) behavior of a system is determined by the nature of the system eigenvalues.

Changing the system behavior reduces to changing system eigenvalues, which can be done if certain conditions are satisfied.
Controllability

\[ \Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \]

- An eigenvalue is said to be controllable if it can be moved to any desired location by an appropriate state feedback.

- Example:

\[ \dot{x} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \]

\( \lambda_1 \) controllable, \( \lambda_2 \) uncontrollable.

- Hautus test:

\[ \lambda \text{ controllable} \iff \text{rank} \left( \begin{pmatrix} \lambda I - A | B \end{pmatrix} \right) = n. \]
**Observability**

\[ \Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \]

- An eigenvalue is said to be observable if its corresponding dynamics appears in the measurements.
- Example:

\[ \begin{cases} \dot{x} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x \\ y = \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{cases} \]

\( \lambda_1 \) observable, \( \lambda_2 \) unobservable.
- Hautus test:

\[ \lambda \text{ observable} \iff \text{rank} \left( \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} \right) = n \]
A system is \textit{asymptotically stable} if all its eigenvalues are in the open left-half plane (so they have negative real part).

The pair \((A, B)\) is \textit{stabilizable} if all the uncontrollable eigenvalues are in the open left-half plane.

The pair \((C, A)\) is \textit{detectable} if all the unobservable eigenvalues are in the open left-half plane.

Stabilizability and detectability are necessary and sufficient conditions for the existence of a stabilizing feedback.
Decentralized control systems

● Structure:

![Decentralized control system diagram]

● Characteristics:
  – No centralized controller
  – At each station $i$, the input $u_i$ depends only on the output $y_i$
  – No explicit communication between control stations
Linear time-invariant case

open-loop system $\Sigma$:
\[
\begin{align*}
\dot{x} &= Ax + \sum_{i=1}^{\nu} B_i u_i \\
y_i &= C_i x, \quad i = 1, \ldots, \nu
\end{align*}
\]

controllers $\mathcal{K}_i$:
\[
\begin{align*}
\dot{z}_i &= K_i z_i + L_i y_i \\
u_i &= M_i z_i + N_i y_i
\end{align*}
\]
i = 1, \ldots, \nu

closed-loop system $\Sigma_{cl}$:
\[
\begin{pmatrix}
\dot{x} \\
\dot{z}
\end{pmatrix} =
\begin{pmatrix}
A + B N C & B M \\
L C & K
\end{pmatrix}
\begin{pmatrix}
x \\
z
\end{pmatrix}
\]

static feedbacks case $\Sigma_{cl}$:
\[
\dot{x} = (A + B N C)x
\]
Decentralized fixed modes

Definition: eigenvalues of the system matrix $A$ that cannot be moved by any LTI decentralized feedback

$$\Lambda(C, A, B, \mathbf{N}) = \bigcap_{N \in \mathbf{N}} \sigma(A + BNC)$$

where $\mathbf{N} = \{N|N = \text{diag}(N_1, \ldots, N_\nu), \ N_i \in \mathbb{R}^{m_i \times p_i}, \ i = 1, \ldots, \nu\}$

- fixed modes are a linear time-invariant concept.
- fixed modes w.r.t. static feedback are also fixed w.r.t. dynamic feedback.
- in centralized control, fixed modes $\equiv$ uncontrollable and/or unobservable modes.
- certain eigenvalues can be fixed w.r.t. decentralized feedbacks, but not fixed w.r.t. centralized feedback.
Theorem 1. An eigenvalue $\lambda$ is a decentralized fixed mode if and only if at least one of the following three conditions is satisfied:

- $\lambda$ is an uncontrollable eigenvalue of $(A, B)$
- $\lambda$ is an unobservable eigenvalue of $(C, A)$
- There exists a partition of the integers $\{1, \ldots, \nu\}$ into two disjoint sets $\{i_1, \ldots, i_m\}$ and $\{j_1, \ldots, j_{\nu-m}\}$ for which we have

\[
\begin{bmatrix}
\lambda I - A & B_{i_1} & \cdots & B_{i_m} \\
C_{j_1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C_{j_{\nu-m}} & 0 & \cdots & 0
\end{bmatrix}
\]

\[
\text{rank} \begin{bmatrix}
\lambda I - A & B_{i_1} & \cdots & B_{i_m} \\
C_{j_1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C_{j_{\nu-m}} & 0 & \cdots & 0
\end{bmatrix} < n.
\]
Communication between channels

- Digraph \((V, E)\) associated to a decentralized control system:
  \[ V = \{1, \ldots, \nu\}; \quad E = \{(i, j) | C_j(sI - A)^{-1}B_i \neq 0\} \]

- Example:

- Some channels can work together for stabilizing part of the dynamics
Quotient fixed modes

• Quotient system:

\[ \Sigma^* : \begin{cases} 
\dot{x} = Ax + \sum_{i=1}^{\nu^*} B_i^* u_i^* \\
y_i^* = C_i^* x 
\end{cases} \]

where \( \nu^* \) is the number of strongly connected components of the associated digraph.

• Quotient fixed modes \( \equiv \) fixed modes of the quotient system.

• \( \Lambda(C^*, A, B^*, N^*) \subset \Lambda(C, A, B, N) \)
Stabilization result

- Classical result of Wang and Davison (1973) for LTI case:

  Asymptotic stabilization is possible if and only if all the decentralized fixed modes are in the open left-half plane:

  \[ \Lambda(C, A, B, N) \subset \mathbb{C}^- \]

- Some fixed modes can be moved by non-LTI controllers, but not the so-called quotient fixed modes
Systems with input saturation

Structure of the open-loop system:

\[
\begin{cases}
\dot{x} = Ax + B\sigma(u) \\
y = Cx
\end{cases}
\]

where \(\sigma\) is the standard saturation function given by

\[
\sigma(u) = \begin{cases} 
1 & \text{if } 1 < u, \\
u & \text{if } -1 \leq u \leq 1, \\
-1 & \text{if } u < -1.
\end{cases}
\]
Semi-global stabilization result

- Necessary and sufficient conditions for semi-global stabilization are:
  - $(A, B)$ stabilizable
  - $(C, A)$ detectable
  - All eigenvalues of $A$ are in the closed left-half plane

- Global stabilization requires in general non-linear controllers

- Dynamics corresponding to eigenvalues with positive real part cannot be stabilized with saturated input even in a semi-global setting.
Observer based measurement feedback

- controller:
  \[
  \begin{aligned}
  \dot{z} &= Az + B\sigma(u) + L(Cz - y) \\
  u &= Fz
  \end{aligned}
  \]

- closed-loop:
  \[
  \begin{aligned}
  \dot{x} &= (A + BF)x - BF(x - z) + B(\sigma(Fz) - Fz) \\
  (x - z) &= (A + LC)(x - z)
  \end{aligned}
  \]

- \( L \) can be chosen such that \( A + LC \) is asymptotically stable
- In order for the inputs not to saturate, we use a low gain design technique for \( F \).
Summary

- Decentralized control stabilization requires all fixed modes asymptotically stable.

- Stabilization with saturation on inputs requires all the eigenvalues of the system matrix in the closed left-half plane.

- Semi-global stabilization with observer based measurement feedback can be achieved using low gain design.
Decentralized control systems with input saturation

\[
\begin{align*}
\dot{x} &= Ax + \sum_{i=1}^{\nu} B_i u_i \\
y_i &= C_i x, \quad i = 1, \ldots, \nu
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= Ax + B \sigma(u) \\
y &= Cx
\end{align*}
\]

- Both decentralized structure and saturation on inputs are present:

\[
\Sigma: \quad \begin{cases}
\dot{x} = Ax + \sum_{i=1}^{\nu} B_i \sigma(u_i) \\
y_i = C_i x, \quad i = 1, \ldots, \nu
\end{cases}
\]

- As stabilization conditions, we expect a combination of the previously mentioned requirements.
As expected, necessary conditions for semi-global stabilization are:

1. All eigenvalues of $A$ are in the closed left-half plane
2. All decentralized fixed modes are in the open left-half plane
   - Conditions imposed by saturation on inputs must hold also in the decentralized case
   - Although saturation may introduce non-linearity in the closed-loop, exponential stability requires all fixed modes in the open left-half plane
Sufficient conditions

- Claim: conditions 1-4 are also sufficient for solvability

- In the case of distinct eigenvalues on the imaginary axis, the claim is true

- The case of multiple eigenvalues on the imaginary axis seems more difficult

- For the case of all imaginary axis eigenvalues located in the origin, a possible approach will follow
The sufficiency proof follows an algorithmic step by step elimination of the unstable modes.

Each step consists of two phases:
- preliminary feedback phase
- observer based dynamic feedback phase

Special attention is needed to keep the state at each step inside a certain compact set.

After a finite number of steps, all unstable eigenvalues are moved.
Remarks

- Location of the fixed modes in the open left-half plane guarantees the existence of the preliminary feedback

- Low gain design is used for building the observer based feedback

- Distinct eigenvalues on the imaginary axis are required in order to keep the state bounded
Multiple eigenvalues in the origin

If all the (multiple) eigenvalues on the imaginary axis are actually in the origin, a possible approach is:

1. apply a scaled pre-feedback that makes at least one eigenvalue in the origin stabilizable and detectable w.r.t. one channel
2. separate the asymptotically stable dynamics using an appropriate basis transformation
3. stabilize (part of) the 0-dynamics using an observer-based feedback
4. repeat the previous three steps until all eigenvalues from the origin are moved to the open left-half plane
• For communication reasons, the already asymptotically stable dynamics cannot be ignored.
• As in the distinct eigenvalues approach, at each step of the stabilization algorithm the state must be kept inside a certain compact set which should not depend on the feedback parameters of that step.
• Enabling the saturation elements on certain channels for a short period of time might be required in order to avoid observer peaking.
• The main problem is how to deal with the possible Jordan blocks which might appear after the pre-feedback phase causing the corresponding state to peak and therefore the inputs to saturate.
Conclusions

- Stabilization of decentralized control systems with input saturation requires certain conditions imposed by the decentralized structure and saturation elements.
- Non-linearities introduced by the saturation elements can be avoided in a semi-global setting.
- Semi-global stabilization is possible in the case of distinct eigenvalues on the imaginary axis.
- The case of multiple eigenvalues on the imaginary axis requires more work.
Questions?