Enhancement of Crossing Elongated Structures in Images

Erik Franken

Eindhoven University of Technology
Department of Biomedical Engineering
and Department of Mathematics and Computer Science

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Many elongated structures in medical images
Our goal

- Develop image processing methods for *enhancing elongated structures* in *noisy* images
- Appropriately handle *crossing* elongated structures
Image processing

- A 2D image is a function where the domain corresponds to spatial positions and the codomain corresponds to grey-values.
- Note: a single grey-value \( f(x,y) \) does not say anything about elongated structures.
- How to obtain information about orientation? → apply an oriented filter (i.e., a convolution).

- No a priori knowledge on what orientations occur → so image should be filtered for all orientations.
Orientation Scores (OS)

- From 2D image to orientation score with position \((x,y)\) and orientation \(\theta\)

Example:
- Image can be reconstructed from OS (for specific choices of the kernel)
Our approach: image processing using OS

We consider orientation score as

\[ \mathcal{W}_\psi \]

where

\[ \gamma \]

\[ \mathcal{W}_\psi^* \]

\[ \Phi \]
Why processing via this “detour”? 

Crossing lines are “torn apart”

→ makes it possible to handle singular points

(a) crossing  (b) bifurcation  (c) T-junction  (d) end-point  (e) discontinuity in slope
Outline

• Invertible Orientation Scores
• Operations in Orientation Scores (OS)
• Measuring Image Features in OS
• Coherence-Enhancing Diffusion in OS
• 3D Orientation Scores
• Conclusions
Invertible orientation score transformation

Image to orientation score

\[ U_f(x, \theta) = \mathcal{W}_\psi[f](x, \theta) = \int_{\mathbb{R}^2} \psi(R_\theta^{-1}(x' - x)) f(x') \, dx' \]

Orientation score to image:

\[ \tilde{f}(x) = \mathcal{W}_\psi^*[U_f](x) = \int_0^{2\pi} \int_{\mathbb{R}^2} \psi(R_\theta^{-1}(x' - x)) U_f(x') \, dx' \]

Stable reconstruction requires

\[ 0 < M_\psi(\omega) = \int_0^{2\pi} |\mathcal{F}[R_\theta[\psi]]|^2 \, d\theta < \infty \]

i.e. “fill up the entire Fourier spectrum”.
Invertible Orientation Score Transformation

Design considerations: reconstruction, directional, spatial localization, quadrature, discrete number of orientations
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Orientation score is a function on SE(2)

Properties of SE(2)

• Group element
  \[ g = (b, \theta), \quad b \in \mathbb{R}^2 \text{ and } e^{i\theta} \in \mathbb{T} \]

  translation  rotation

• Group product
  \[ g_1 g_2 = (b_1 + R_{\theta_1} b_2, \theta_1 + \theta_2) \]

• Group inverse

Important representations
We need *left-invariant* operators

- The effective operator on the image $\gamma$ should be Euclidean invariant $U_g(\gamma(f)) = \gamma(U_g(f))$
- This implies that the operator $\Phi$ on the orientation score must be *left-invariant*, i.e.

$$\mathcal{L}_g(\Phi(U)) = \Phi(\mathcal{L}_g(U))$$
Taking Derivatives of Orientation Scores

“normal”

\[ \frac{\partial}{\partial y} U \]
Taking Derivatives of Orientation Scores

Orientation score layers

“normal”
\[ \frac{\partial}{\partial y} U \]

Left-invariant:
\[ \frac{\partial}{\partial \eta} U = (- \sin \theta \, \partial_x + \cos \theta \, \partial_y) U \]
Left-invariant Derivative Operators

\[
\partial_\xi = \cos \theta \partial_x + \sin \theta \partial_y \\
\partial_\eta = -\sin \theta \partial_x + \cos \theta \partial_y \\
\partial_\theta = \partial_\theta
\]

These are left-invariant derivatives, so

Not all left-invariant derivatives on SE(2) commute!

=Tangent to line structures
= orthogonal to line structures
Example of a left-invariant Gaussian derivative jet

orientations
In general: all linear & left-invariant operators are \(SE(2)\)-convolutions

Normal 3D convolution – versus \(SE(2)\)-convolution on OS

\[
[f \ast g](x) = \int_{\mathbb{R}^n} f(x - y) g(y) \, dy
\]

\[
[K \ast_G U_f](x, \theta) = \int_{\mathbb{R}^2} \int_0^{2\pi} K(R_{\theta'}^{-1}(x - x'), \theta - \theta') U_f(x', \theta') \, d\theta' \, dx'
\]

Complexity of \textit{Steerable} \(G\)-convolution \(\mathcal{O}(MS^2 \log S) + \mathcal{O}(M^2S^2)\), with \(M\) is \# orientations, \(S\) is \# pixels in \(x\) and \(y\) dimension.
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Measuring image features in orientation scores

Increasing orientation confidence

curvature
Curvature estimation in OS

$\mathbb{R}^2$-curvature can be obtained from exp-curve ("tangent spiral") in OS.

→ What is curvature of the locally best fitting exponential curve?

→ Minimize the change of left-invariant gradient over the exp-curve

$$\min_{\mathbf{c}} \left\{ \left\| \frac{d}{ds} \left( \nabla U (\gamma_{\mathbf{c}}(s)) \right) \right\|_2^{2} \left\| \mathbf{c} \right\|_{\beta} = 1 \right\}$$

with $\nabla U = \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta} \right) U$ (components of a covector)

Vector norm: $\left\| \mathbf{c} \right\|_{\beta} = (c_\theta)^2 + \beta^2(c_\xi)^2 + \beta^2(c_\eta)^2$

"Covector norm" $\left\| \tilde{\mathbf{b}} \right\|_{\beta} = (b_\theta)^2 + \beta^{-2}(b_\xi)^2 + \beta^{-2}(b_\eta)^2$
Solving the Minimization Problem

To minimize \( \min_c \left\{ \left\| \frac{d}{ds}(\nabla U(\gamma c(s))) \right\|_{s=0}^2 \left\| c \right\|_{\beta} = 1 \right\} \)

Euler-Lagrange gives \( \nabla c \| (HU)c \|_{\beta}^2 - \lambda (1 - \nabla c \| c \|_{\beta}) = 0 \)

with \( HU = \nabla (\nabla U) = \begin{pmatrix} \frac{\partial^2 U}{\partial \theta^2} & \frac{\partial \xi}{\partial \theta} \frac{\partial U}{\partial \theta} & \frac{\partial \eta}{\partial \theta} \frac{\partial U}{\partial \theta} \\ \frac{\partial \theta}{\partial \xi} \frac{\partial U}{\partial \xi} & \frac{\partial^2 U}{\partial \xi^2} & \frac{\partial \eta}{\partial \xi} \frac{\partial U}{\partial \xi} \\ \frac{\partial \theta}{\partial \eta} \frac{\partial U}{\partial \eta} & \frac{\partial \xi}{\partial \eta} \frac{\partial U}{\partial \eta} & \frac{\partial^2 U}{\partial \eta^2} \end{pmatrix} \)

\( \rightarrow (M_\beta HUM_\beta)^T (M_\beta HUM_\beta) \bar{c}^* = \lambda \bar{c}^* \)

with \( \bar{c}^* = M^{-1}_\beta c^* \) and \( M_\beta = \text{diag}\{1, 1/\beta, 1/\beta\} \)

\( \rightarrow \) Eigensystem analysis on 3x3 matrix, pick right eigenvector \( c = (c_\theta, c_\xi, c_\eta) \)

\( \kappa_{est} = \frac{c_\theta \text{sign}(c_\xi)}{\sqrt{c_\xi^2 + c_\eta^2}} \)

Orientation confidence can be found from the eigenvalues.
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Inspiration from Image Diffusion Techniques

\[
\begin{cases}
\frac{\partial}{\partial t} u = \nabla \cdot D \nabla u \\
u(x; 0) = f(x)
\end{cases}
\]

\( f = \text{image} \)

\( u = \text{scale space of image} \)

\( D = \text{diffusion tensor} \)

Linear diffusion

\[ D = I \]

Perona&Malik

\[ D(x) = g(|\nabla u(x)|) I \]

Coherence-enhancing diff.

\[ D(x) = s(x) v(x) v^T(x) + \alpha I \]
Diffusion Equation in Orientation Scores

\[ \partial_t u = \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta} \right) \begin{pmatrix} D'_{11} + D_{22} \kappa^2 & D_{22} \kappa & 0 \\ D_{22} \kappa & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \begin{pmatrix} \partial_{\theta} \\ \partial_{\xi} \\ \partial_{\eta} \end{pmatrix} u \]

- **Left-invariant derivatives**
- **Diffusion orthogonal to oriented structures**
- **Diffusion tangent to oriented structures**
- **Diffusion in orientation**
- **Evolving orientation score**

**Gaussian derivatives:** \( D_{22} = D_{33}, \kappa = 0 \)

**CED-OS:** non-linear adaptive diffusion tensor coefficients

- Oriented regions: \( D'_{11} \) and \( D_{33} \) small, \( D_{22} \) large and \( \kappa \) according to estimate
- Non-oriented regions: \( D'_{11} \) large, \( D_{22} = D_{33} \) large, \( \kappa = 0 \)
Example diffusion kernels

\[ \partial_t u = \begin{pmatrix} \partial_\theta & \partial_\xi & \partial_\eta \end{pmatrix} \begin{pmatrix} D'_{11} + D_{22} \kappa^2 & D_{22} \kappa & 0 \\ D_{22} \kappa & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \begin{pmatrix} \partial_\theta \\ \partial_\xi \\ \partial_\eta \end{pmatrix} u \]
Example of CED-OS

Size: 128 x 128 x 64

CED-OS

CED
Biomedical Example: Muscle Cell

Original

CED-OS
Biomedical Example: bone structure

Original

CED-OS
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3D Orientation Scores

- Relevant for medical applications: high angular resolution diffusion imaging (HARDI).
- On each spatial position \((x, y, z)\) we have a function on the sphere \(S^2\).
- So a 3D orientation score is a 5D space, parameterized (e.g.) by \((x, y, z, \beta, \gamma)\).
3D Orientation Scores

Example visualization of a 3D orientation score
3D Orientation scores and $\alpha$-right-invariance

- 3D Orientation score is a function on rather than 3D Euclidean motion group
- The unit sphere $S^2$ is isomorphic to the left cosets $SO(3)/SO(2)$ given by

\[
R_3 o S^2 \equiv SE(3)
\]

- So: we consider 3D OS as an $\alpha$-right-invariant function which should fulfill
- Operators on 3D OS should preserve $\alpha$-right-invariance
Operations on 3D Orientation Scores

Similar to the 3 left-invariant derivatives in 2D, we now have 6 left-inv. Derivatives

Left-invariant diffusion on SE(3) is given by

Constant diffusion: only
Adaptive diffusion: fine as long as the adaptivity is calculated in left-invariant way.
Results of linear diffusion on 3D OS

Input, no noise

Result, isotropic diffusion

Result, anisotropic diffusion

Input, no noise

Result, isotropic diffusion

Result, anisotropic diffusion
Results of linear diffusion on 3D OS

Input, no noise
Result, isotropic diffusion
Result, anisotropic diffusion

Input, noisy
Result, isotropic diffusion
Result, anisotropic diffusion
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Conclusions

• We developed a framework for image processing / analysis via orientation scores of 2D and 3D images
  Important notion: An orientation score is a function on $SE(n) \rightarrow$ use group theory.
• Developed nonlinear diffusion on orientation scores to enhance crossing line structures in 2D images
• All methods also apply to 3D orientation scores

Future work:
• Detection of elongated structures via OS
• Further develop 3D orientation scores, apply to HARDI