Travelling wave solutions for the Richards equation including nonequilibrium effects in the capillary pressure

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# Introduction

## Porous media flow

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- **Prime Parameters & Model Unknowns**

\[
\phi \rightarrow \text{Porosity} \left( \frac{\text{Volume of pores}}{\text{Total volume}} \right)
\]

\[
S_j \rightarrow \text{Saturation of phase } j, \quad \left( \frac{\text{Volume occupied by the phase}}{\text{Total pore volume}} \right) \quad j = n, w
\]

\[
P_j \rightarrow \text{Pressure of phase } j, \quad j = n, w
\]

\[
K_j(S) \rightarrow \text{Permeability}, \ (\text{Diffusivity of phase } j)
\]
Equations

• Mass Balance:
  \[ \phi \frac{\partial S_j}{\partial t} + \frac{\partial q_j}{\partial z} = 0, \quad q_j \rightarrow \text{Flux for phase } j \]

• Darcy’s Law:
  \[ q_j = -\frac{k_j \kappa}{\mu_j} \cdot \left( \frac{\partial P_j}{\partial z} - \rho_j g \right) \]

  with \( k_j \rightarrow \text{Relative permeability, } \kappa \rightarrow \text{Absolute permeability, } \mu \rightarrow \text{Viscosity, } \rho_j \rightarrow \text{Density and } g \rightarrow \text{Body force term (gravitational acceleration in most cases)} \)

  If one phase is gaseous (like air) then we neglect the effect of gaseous phase as:
  \[ \rho_w \gg \rho_{\text{air}} \approx \frac{\rho_w}{1000} \text{ and } \mu_w \gg \mu_{\text{air}} \approx \frac{\mu_w}{100} \]
Richards Equation

- The resulting equation is Richards equation,
  \[ \phi \frac{\partial S}{\partial t} + \frac{\partial q}{\partial z} = 0 \]
  \[ q = -\frac{k_{rw}}{\mu} \cdot \left( \frac{\partial p}{\partial z} - \rho g \right) \]

- Which after non-dimensionalization and redefinition becomes
  \[ \frac{\partial S}{\partial t} + \frac{\partial q}{\partial x} = 0 \]
  \[ q = -K(S) \left( \frac{\partial p}{\partial x} - 1 \right) \]

- Thing to note is that there is one relation missing between pressure and saturation
Nonequilibrium Effects

- Previously the constitutive relation used was: \(-p = P_c(S)\)
- Hysteretic effects*: different curves during imbibition and drainage
- Experiments show that \(P_c\) also depends on \(\partial_t S\) as well**

* Bear, Morrow & Harris(1964), Jerauld & Salter
** DiCarlo (2004), Bottero et al
Nonequilibrium Effects

Based on this it was proposed*:

\[
P_c = P_c^+(S) - P_c^-(S) \cdot \text{sign} \left( \frac{\partial S}{\partial t} \right) - \tau L(S) \frac{\partial S}{\partial t}
\]

With

\[
P_c^+ = \frac{1}{2} (p_{cdr} + p_{imb})
\]

\[
P_c^- = \frac{1}{2} (p_{cdr} - p_{imb})
\]

* Hassanizadeh & Grey (1994), Belyaev & Hassanizadeh:
Traveling Wave Formulation

Current problem is unsaturated water flow with gravity in a homogeneous porous long column.

\[
\frac{\partial S}{\partial t} + \frac{\partial q}{\partial x} = 0, \\
q = -K(S) \left( \frac{\partial p}{\partial x} - 1 \right), \\
-p + \tau \frac{\partial S}{\partial t} = \begin{cases} 
    p_{imb} & \text{if } \partial_t S > 0, \\
    [p_{imb}, p_{drn}] & \text{if } \partial_t S = 0, \\
    p_{drn} & \text{if } \partial_t S < 0.
\end{cases}
\]

The Boundary Conditions after extension of the domain to \( \mathbb{R} \)

\[
P(-\infty, t) = P_T, \quad P(\infty, t) = P_B \quad \text{and} \]
\[
S(x, 0) = S_0(x)
\]

Assume \( \lim_{x \to -\infty} S(x, t) = S_T \) and \( \lim_{x \to \infty} S(x, t) = S_B \)
Travelling Wave Formulation

- We would try to look for Travelling Wave solution of the above mentioned equations.
- For travelling wave solution we assume:

\[ S(x, t) = S(\eta) \]
\[ P(x, t) = P(\eta) \]

with \( \eta = x - ct \)
Play type Hysteresis

First attempt: only hysteresis, $\tau = 0$

simplified setting: $K(S) = S^2$

\[
\begin{aligned}
-cS' + (S^2)' &= (S^2p')' \quad -\infty < \eta < \infty \\
-p &= P_c^+(S) - P_c^-(S) \cdot \text{sign}(-cS') \\
p(-\infty) &= P_T, \quad p(+\infty) = P_B \\
S(-\infty) &= S_T, \quad S(+\infty) = S_B.
\end{aligned}
\]

behaviour at infinity gives Rankine - Hugoniot wave speed $c$

\[
c = \frac{S_T^2 - S_B^2}{S_T - S_B} = S_T + S_B
\]

This gives:

\[
S^2p' = (S - S_T)(S - S_B).
\]
Regularization

\[ \text{sign}(u) \text{ is a discontinuous function. We regularise it with a smooth function} \]

\[ -p = P_c^+(S) - P_c^-(S) \cdot H_\epsilon \left( \frac{\partial S}{\partial t} \right) \]

where \( \epsilon \) is a small parameter and \( H_\epsilon \) approximates sign function:

\[ H_\epsilon(\pm \infty) = \pm 1, \quad H_\epsilon(0) = 0, \quad H_\epsilon' > 0 \]

\[ \lim_{\epsilon \to 0} H_\epsilon(u) = \text{sign}(u) \quad \text{for} \quad u \in \mathbb{R} - \{0\}. \]
Reformulated System

Because $H_\epsilon$ is monotonic we take its inverse. Also putting $u = -p$ and $\zeta = -\eta$ we get:

\[
S' = \Psi_\epsilon \left( \frac{P_c^+(S) - u}{P_c^-(S)} \right)
\]
\[
u' = -\frac{(S_T - S)(S - S_B)}{S^2}
\]

where $c = S_T + S_B$, \(\Psi_\epsilon = -\frac{1}{c}H_\epsilon^{-1} \).

We are interested to find solutions satisfying the following conditions

\[
S(-\infty) = S_T, \quad S(\infty) = S_B
\]
\[
u(-\infty) = P_c^+(S_T) = p_r,
\]
\[
u(\infty) = P_c^+(S_B) = p_l,
\]
Results: Case $S_T=1$

Propositions:

- The Area $H^-$ is invariant
- $E_B = (S_B, p_r)$ is a saddle point ($\lambda \sim \mathcal{O}(\epsilon^{1/2})$)
- $E_T = (1, p_t)$ is a sink
- The orbit converges uniformly to $P_c^+(S)$ for $\epsilon \to 0$, $S_B < S < 1$
- $\lim_{\epsilon \to 0} S^* = S_B$ and $\lim_{\epsilon \to 0} \zeta^* = -\infty$
Case $S_T < 1$

Propositions:

- All the results of the $E_B$ end remain the same
- the direction of orbits are as shown
- orbit around $E_T$ is a stable spiral for $\epsilon < \epsilon^\dagger$ with $R\epsilon(\lambda) \sim O(\epsilon)$
- $\lim_{\epsilon \to 0} L_\epsilon \to S_T$ and $\lim_{\epsilon \to 0} R_\epsilon \to S_T$
Incompatibility and Others

Discussed with numerical results
Numerical Scheme

For finding $S$ on $n^{th}$ timestep we use Forward Euler:

$$S_{n,j} = S_{n-1,j} + \Delta t \cdot \Psi_{\epsilon} \left( \frac{P^+(S_{n-1,j}) + p_{n-1,j}}{P^-(S_{n-1,j})} \right)$$

- For solving $p$ we would need inner iterations.
- Predefine $L_j, K_j = K_{rw}(S_{n,j}), P_j^\pm = P^\pm(S_{n,j})$ and $K_j = K(S_{n,j})$
- A modified version of $L$-scheme with Finite Differences ($O(\Delta x^2, \Delta t)$)

$$
\left( L_j + \frac{K_{j+1} + 2K_j + K_{j-1}}{2\Delta x^2} \right) p_{n,j}^k - \frac{K_{j-1} + K_j}{2\Delta x^2} p_{n,j-1}^k - \frac{K_{j+1} + K_j}{2\Delta x^2} p_{n,j+1}^k
$$

$$= L_j p_{n,j}^{k-1} - \Psi_{\epsilon} \left( \frac{P_j^+ + p_{n,j}^{k-1}}{P_j^-} \right) - \frac{K_{j+1} - K_{j-1}}{2\Delta x}$$

- Also used Upwind scheme for discretizing $K_j$

- Proposition: If $L_j > \sup \left( \frac{\Psi_{\epsilon}'(\zeta)}{P^-} \right)$ the iteration is convergent
Numerical Results

$\epsilon = 10^{-2}$
Numerical Results

\[ \epsilon = 10^{-2} \]
Numerical Results

\[ \epsilon = 10^{-2} \]
Numerical Results

$\epsilon = 10^{-3}$
Numerical Results

\[ \epsilon = 10^{-3} \]
Incompatible Boundary Conditions

\[ p_l > P_c^+(S_T) \]
Incompatible Boundary Conditions

\[ p_l > P_c^+(S_T) \]
Incompatible Boundary Conditions

\[ p_r < P_c^+(S_B) \]
Incompatible Boundary Conditions

\[ p_r < P_c^+ (S_B) \]
Dynamic Capillarity

- This is when $\tau > 0$ and $P_c^-(S') = 0$

- Only mathematical analysis done so far

The governing equations are:

\[
S' = \frac{P_c^+(S') - u}{\tau c L(S)}
\]

\[
u' = 1 - \frac{c(S - S_B) + K(S_B)}{K(S)}
\]

From the boundary conditions: $c = \frac{K(S_T) - K(S_B)}{S_T - S_B}$

We are interested to find a solution satisfying:

$S(-\infty) = S_T$, $S(\infty) = S_B$, $u(-\infty) = P_c^+(S_T) = p_r$, $u(\infty) = P_c^+(S_B) = p_l$, 
Sneak peek

\[ 0 < \tau < \tau^\dagger \]

\[ u \quad S \]

\[ P_c^+(S) \]

\[ S_B \quad S_T \]
Sneak peek

\[ \tau^\dagger < \tau < \tau^* \]

\[ u \]

\[ S \]

\[ P_c^+(S) \]

\[ S_B \]

\[ S_T \]
if \( \alpha = \int_{S_B}^{1} \left( 1 - \frac{c(S - S_B) + K(S_B)}{K(S)} \right) > 0 \)
Sneak peek

$$\alpha = \int_{S_B}^{1} \left( 1 - \frac{c(S - S_B) + K(S_B)}{K(S)} \right) < 0$$

\[\tau^* < \tau\]
Conclusion

• We have proved the existence of TW solutions for the (regularized) play-type hysteresis model
• We have analyzed the behavior of the system w.r.t. regularization parameter, and showed its convergence as $\epsilon \to 0$
• Non-monotonic waves if $S_T < 1$: oscillations around $S_T$ vanish but not necessary around $P_T$ with $\epsilon \to 0$
• During imbibition, the P - S relationship follows the primary imbibition curve, similar behaviour for drainage
• Also we have proved the existence of TW solutions for the dynamic capillarity case and analyzed the behavior of the system
Future Scope

- Numerical result for $\tau > 0$
- A TW model with both $\tau > 0$ and $P_c^-(S) > 0$
- Resolving effect of hysteresis in $K$
Other Work

“A domain decomposition scheme for solving Richards equation in heterogeneous porous media”

Joint work done with:
Prof. I.S.Pop (Hasselt University)
Prof. F.A. Radu (University of Bergen)
D. Seus (University of Stuttgart)

“Fractured media dimensionality reduction: We will take the transversal average over the fracture, as the fracture is much thinner than the block matrix”

Joint work done with:
Prof. I.S.Pop (Hasselt University)
Prof. K.Kumar (University of Bergen)
\[
\lim_{n \to \infty} (\text{THANK YOU})^n
\]