Model Order Reduction for Parameter-Varying Systems

Xingang Cao

Promotors:
Wil Schilders
Siep Weiland

Supervisor:
Joseph Maubach

Eindhoven University of Technology

CASA Day

November 1, 2016
Overview

1. Introduction
2. Model Order Reduction
3. Model Order Reduction for Linear Parameter-Varying Systems
4. Conclusions and Future Work
Self Introduction

Biography:

1990 born in Baotou

2009 - 2013
BE in Electrical Engineering
Tongji University, Shanghai

2013-2015
MSc. In Systems and Control
TU Eindhoven

Mongolia
Russia
Introduction

Modeling & Control of Thermal Effects in Printing Systems
Model Order Reduction in a Broad Field

- Scientific computing, numerical MOR, Krylov methods, linear algebra, tensors
  - Pade-via-Lanczos and PRIMA
  - Passivity preserving
  - Structure preserving
  - Linear and nonlinear problems
  - Parameterized methods
  - Tensor analysis

- Systems and control, Lyapunov, Truncated Balanced Realization
  - Large-scale Lyapunov systems
  - Balanced realizations
  - Observability and controllability Gramians
  - Port-Hamiltonian systems
  - Hankel singular values
  - Projection methods

- Mathematical modeling, behavioral modeling, models via data
  - MOR at operator level
  - Reduced basis methods
  - Karhunen-Loeve expansions
  - Neural networks
  - Vector fitting
  - Behavioral models

- Xingang Cao (CASA TU/e)
Overview

1 Introduction

2 Model Order Reduction

3 Model Order Reduction for Linear Parameter-Varying Systems

4 Conclusions and Future Work
What is Model Order Reduction?
Abstract Problem Formulation

Given a dynamical system

\[ \Sigma : \begin{cases} \dot{x} &= f(x, u) \\ y &= g(x, u) \end{cases} \]

with \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \) and \( y(t) \in \mathbb{R}^p \). Find another dynamical system

\[ \hat{\Sigma} : \begin{cases} \dot{\hat{x}} &= \hat{f}(\hat{x}, u) \\ \hat{y} &= \hat{g}(\hat{x}, u) \end{cases} \]

with \( \hat{x}(t) \in \mathbb{R}^r \) and

\[ r \ll n. \]

Constraints:

- Approximation error small, i.e., \( \| \Sigma - \hat{\Sigma} \| \leq \epsilon \);
- Preserve characteristics, e.g., stability, dissipativity, etc;
- Computational efficient for simulation and control design.
Why Model Order Reduction?

Realistic models have high complexity \((n \geq 10^2 - 10^9)\), but we want to

- perform simulations fast and with reliable outcomes;
- enable model based control design;
- cope with limited memory storage capacity;
- perform online tasks (optimization, prediction) efficiently;
- enable implementation on embedded systems.
How to do Model Order Reduction?

Model Reduction by Projection:
Signal projection: \( V : \mathbb{R}^n \rightarrow \mathbb{R}^r \)

\[
x \approx V \hat{x}
\]

Residual:
\[
\mathcal{R} = V \dot{\hat{x}} - f(V \hat{x}, u)
\]

Residual projection:
\[
W^T \cdot \mathcal{R} = W^T V \dot{\hat{x}} - W^T f(V \hat{x}, u)
\approx 0
\]

- Galerkin projection: \( W = V \) & \( V^T V = I_r \).
- Petrov–Galerkin projection: \( W \neq V \) & \( W^T V = I_r \).
How to Find the Projections?

Consider a linear time-invariant (LTI) system

\[ \Sigma : \begin{cases} \dot{E}x = Ax + Bu \\ y = Cx + Du \end{cases} \]

Take the Laplace transformation

\[ \mathcal{L} : \dot{x}(t) \rightarrow sX(s), \ u(t) \rightarrow U(s) \text{ and } y(t) \rightarrow Y(s) \]

\( \Sigma \) defines an input-output mapping

\[ G(s) = C(sE - A)^{-1}B + D. \]

Expand \( G(s) = G(s_0 + \sigma) \) around a point \( s_0 \),

\[ G(\sigma) = \sum_{i=0}^{\infty} \frac{C (s_0 E - A)^{-1} E (s_0 E - A)^{-1} B \sigma^i}{M \tilde{B}} \]
How to Find Projections (Cont’d)?

**Theorem**

*If* $W$ and $V$ *span the left and right principal subspaces of* 

$$M = (s_0E - A)^{-1}E,$$

*the approximation error is small.*

Compute $W$ and $V$ by Krylov subspace methods

$$W := \text{span}\{W\} = \text{span}\{C^T, \ M^T C^T, \ldots, (M^T)^{q_1} C^T\}$$

$$V := \text{span}\{V\} = \text{span}\{\tilde{B}, \ M\tilde{B}, \ldots, M^{q_2} \tilde{B}\}$$

$\dim(W) = \dim(V) = r$.

**Remark**

*For* $\tilde{B}, C^T \in \mathbb{R}^{n \times 1}$ * (single-input single-output system),* $r = q_1 = q_2$. 
Overview

1. Introduction

2. Model Order Reduction

3. Model Order Reduction for Linear Parameter-Varying Systems

4. Conclusions and Future Work
Linear parameter-varying (LPV) system is

\[ \Sigma(p) : \begin{cases} 
E(p)x' = A(p)x + B(p)u \\
y = C(p)x + D(p)u 
\end{cases} \]

where \( p(t) : \mathbb{R}_0^+ \rightarrow \mathbb{P} \subseteq \mathbb{R}^\mu \).
Track the Parameterized Projection Subspaces

Problem

Track/Estimate the principal subspaces of

\[ M(p) = (s_0 E(p) - A(p))^{-1} E(p) \]

with starting vectors \( C(p) \) and \( \tilde{B}(p) \).
Geodesics on Grassmann Manifold

Definition

Let \( \gamma(t) : I \to S \) be a smooth, injective curve and \( \gamma'(t) \neq 0 \) for all \( t \in \mathbb{R} \). There exists a smooth vector field \( X : S \to TS \) such that \( \gamma'(t) = X(\gamma(t)) \) for all \( t \in I \). If

\[
\nabla_{\gamma'} \gamma := (\nabla X)(\gamma(t)) = 0,
\]

then \( \gamma(t) \) is a geodesic. On Grassmann manifold, it satisfies

\[
\ddot{\gamma}(t) + \gamma(t)(\dot{\gamma}(t)^\top \dot{\gamma}(t)) = 0, \quad t \in [0, 1].
\]

Given \( \gamma(0) = S_0 \) and \( \gamma'(0) = H_0 \),

\[
\gamma(t) = (S_0 Z_0 \quad U_0) \begin{pmatrix} \cos(\Sigma_0 t) \\ \sin(\Sigma_0 t) \end{pmatrix} Z_0^\top, \quad H_0 \xrightarrow{\text{thin svd}} U_0 \Sigma_0 V_0^\top, \quad t \in [0, 1];
\]

\( S_1 := \gamma(1) \).
Data-Driven Subspace Tracking

Step 1. Sample parameter $p$ and get $\{p_i\}_{i=1}^N$;

Step 2. Compute the projections $\{W\}_{i=1}^N$ and $\{V\}_{i=1}^N$;

Step 3. Define $\psi_i = \text{vec}(W_iW_i^\top)$ and $\phi_i = \text{vec}(V_iV_i^\top)$;

Step 4. Assume that $\psi_i$ and $\phi_i$ satisfy

$$
\sum_\psi : \psi_{i+1} = A_\psi \psi_i + B_\psi p_{i+1}, \quad \sum_\phi : \phi_{i+1} = A_\phi \psi_i + B_\phi p_{i+1};
$$
A Small-scale Example

Example

Consider the subspaces

\[ \mathcal{W}(p(t)) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \cos(p(t)) \\ \sin(p(t)) \end{bmatrix} \right\}, \]

\[ \mathcal{V}(p(t)) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \cos(p(t) + \frac{\pi}{32}) \\ \sin(p(t) + \frac{\pi}{32}) \end{bmatrix} \right\}. \]

where \( e_i, i = 1, 2, 3 \in \mathbb{R}^3 \) and \( p(t) \in \left[ \frac{1}{10}, \frac{\pi}{4} - \frac{\pi}{32} \right] \).

Take 20 random examples and apply the data-driven method to obtain a 3rd-order model. When the parameter is varying linearly from \( \frac{1}{10} \) to \( \frac{\pi}{4} - \frac{\pi}{32} \), compare the approximated and the real subspace representations.
Approximation of $W(p)$
Approximation of $V(p)$

$V_{11} = 1$

$V_{12} = 0$

$V_{13} = 0$

$V_{21} = 0$

$V_{22} = \cos(p + \pi/32)$

$V_{23} = \sin(p + \pi/32)$
Overview

1. Introduction

2. Model Order Reduction

3. Model Order Reduction for Linear Parameter-Varying Systems

4. Conclusions and Future Work
Conclusions

1. Data-driven method: allows to track the parameter-varying subspaces;
2. Large-scale problems: off-line cost for data-driven method is high;
3. Data-driven method: faster at online phase than direct method.
Future Work

Formulate the LPV model order reduction problem as

$$\{ W(p(t)), V(p(t)) \} = \arg\max_{W, V \in Gr(r, \mathbb{R}^n)} \text{Trace}(W^T M(p(t)) V)$$

subject to:

$$W^T V = I \text{ or } W^T W = I, V^T V = I,$$

$$W \text{ and } V \text{ starts with specific vector } \tilde{B} \text{ and } C.$$

where $$Gr(r, \mathbb{R}^n) := \{ \text{span}(A) | A \in \mathbb{R}^{n \times r}, \text{rank}(A) = r \}.$$ Solve it by optimization on matrix manifold methods.
Questions?
References


