Fourier Transformation and Sobolev Spaces

kamyar Malakpoor

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Outline

- Introduction
- Hilbert Space
- The Space of Test Functions
- Schwartz Space and Fourier Transformation
- Temperate Distributions
- Extension of Fourier Transformation to Temperate Spaces
- Sobolev Spaces
- Differential Operators
Introduction

The central notation is that of Fourier transformation.

For each $u$ defined on $\mathbb{R}^n$, (with a controlled growth at infinity), one can defined its Fourier transform $\hat{u}$ on $\mathbb{R}^n$ with the following properties:

- Differentiations on $u$ correspond to multiplication by polynomials on $\hat{u}$.
- One can recover $u$ from $\hat{u}$ essentially by achieving the same transformation a second time.
- The Fourier transformation of an $L_2$ function is an $L_2$ function.
Thus in order to study the properties of this transformation it is convenient to work in a spaces that are closed under operations of differentiation and multiplication. (Schwartz space).

Since there is correspondence between differentiation of \( u \) and multiplication of \( \hat{u} \) by polynomials, there is a correspondence between the smoothness of \( u \) and the growth of \( \hat{u} \) at infinity.

This fact is used to define the so-called Sobolev spaces.
Hilbert Space

A Hilbert Space is a vector space \( V \) with an inner product which is complete as a normed space. i.e., is a Banach space.

Any closed and convex subspace \( C \) of a Hilbert space \( H \) has a unique element of smallest norm, i.e., there exists \( u \in C \) such that

\[
\|u\| = \inf \{ \|v\|; v \in C \}.
\]

A fundamental Fact on Hilbert Space. For any continuous linear functional \( L : H \to \mathbb{C} \) there is a unique element \( v \in H \) such that

\[
Lu = (u, v) \ \forall u \in H.
\]

and

\[
\|L\| = \sup_{\|u\|=1, u \neq 0} |L(u)| = \|v\|
\]
The Space of Test Functions

Partial differentiation of functions on $\mathbb{R}^n$ is indicated by

$$\frac{\partial}{\partial x_j} = \partial_j \text{ or } \frac{1}{i} \frac{\partial}{\partial x_j} = D_j.$$ 

In more complicated expressions we use multi-index notation: When $\alpha \in \mathbb{N}_0^n$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, then for $|\alpha| = \alpha_1 + \cdots + \alpha_n$

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = (-i)^{|\alpha|} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n},$$

Leibniz formula and the for $T^\alpha = \partial^\alpha$, $D^\alpha$

$$T^\alpha(uv) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} T^\beta u T^{\alpha-\beta} v, \quad |\alpha| \leq N,$$
Let $\Omega$ be an open subset of $\mathbb{R}^n$. The space $C^\infty_0$, consisting of the $C^\infty$ functions on $\Omega$ with compact support in $\Omega$, is called the space of test functions.

Does such a function exist?

- Define $f(t) = e^{-1/t}$, for $t > 0$ and $f(t) = 0$ elsewhere. $f$ is a $C^\infty$ function on $\mathbb{R}$.

- For a given $R > r > 0$, define $f_1(t) = f(t - r)f(R - t)$ and $f_2(t) = \int_t^\infty f_1(s)ds$. We see that $f_2 \geq 0$ for all $x$ and equals $0$ for $x \geq R$ and equals $C = \int_r^R f_1(s)ds > 0$ for $t \leq r$.

- Now the function $\chi_{r,R}(x) = \frac{1}{C}f_2(|x|)$, $x \in \mathbb{R}^n$, is in $C^\infty_0(\mathbb{R}^n)$ and equals $1$ for $|x| \leq r$, $\chi_{r,R}(x) \in [0, 1]$ for $r \leq |x| \leq R$ and equals $0$
Schwartz Space and Fourier Transformation

By introducing the function $<x> = (1 + |x|^2)^{1/2}$, we will see that this function is of the same order of magnitude as $|x|$, but has an advantage of being positive $C^\infty$ function on $\mathbb{R}^n$.

**Schwartz Space.** The vector space $\mathcal{S}(\mathbb{R}^n)$ is defined as the space of $C^\infty$ functions $\varphi(x)$ on $\mathbb{R}^n$ such that $x^\alpha D^\beta \varphi(x)$ is bounded for all multi-indices $\alpha$ and $\beta \in \mathbb{N}_0^n$, or

$$\mathcal{S}(\mathbb{R}^n) = \{ u : \mathbb{R}^n \rightarrow \mathbb{C}; u \in <x>^{-l} C^k_0(\mathbb{R}^n) \text{ for all } k \text{ and } l \in \mathbb{N} \}.$$

$e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$.

Obviously, $\mathcal{S}$ is **closed** under the operations of differentiation and multiplication by polynomials.

It is clear to see that test functions, $C^\infty_0(\mathbb{R}^n)$ are automatically in $\mathcal{S}$. 
Observe that as a linear space

\[ S(\mathbb{R}^n) = \bigcap_k < x >^{-k} C^k_0(\mathbb{R}^n). \]

For \( u \in S \), set \( \| u \|_{(k)} = \| < x >^k u \|_{C^k} \). We can see this norm is equivalent with \( \sum_{|\alpha|,|\beta| \leq k} \| x^{\alpha} D^\beta u \|_\infty \).

For \( p \geq 1 \),

- \( S \) is continuously injected in \( L_p(\mathbb{R}^n) \).
- \( S \) is dense in \( L_p(\mathbb{R}^n) \).

A linear functional \( \Lambda : S \rightarrow \mathbb{C} \) is continuous iff there exists \( C, k \) such that

\[ \Lambda \varphi \leq C \sum \sup_{|\alpha|,|\beta| \leq k} \| x^{\alpha} D^\beta u \|_\infty \]
Clearly, \( \partial^{\alpha} \) and \( D^{\alpha} \) are continuous operators in \( S(\mathbb{R}^n) \).

When \( f \in L_1(\mathbb{R}^n) \), the Fourier transformed function \((\mathcal{F}f)(\xi)\) is defined by the formula

\[
(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x) dx
\]

- The Fourier transform \( \mathcal{F} \) is a continuous linear map of \( L_1(\mathbb{R}^n) \) into \( C_{L_\infty}(\mathbb{R}^n) \), such that when \( f \in L_1(\mathbb{R}^n) \), then

\[
\|\hat{f}_{L_\infty}\| \leq \|f\|_{L_1}, \quad \hat{f}(\xi) \to 0, \quad |\xi| \to \infty
\]

- The Fourier transform is a continuous linear map of \( S(\mathbb{R}^n) \) into \( S(\mathbb{R}^n) \) and one has for \( f \in S(\mathbb{R}^n), \xi \in \mathbb{R}^n \),

\[
\mathcal{F}[x^{\alpha} D_x^{\beta} f(x)](\xi) = (-D_\xi)^{\alpha}(\xi^{\beta}\hat{f}(\xi)),
\]
• Defining the co-Fourier transform $\tilde{F}$ by $\tilde{F} = \int e^{ix \cdot \xi} f(x) dx$, one has for $f \in S$ with $\hat{f} = \mathcal{F} f$,

$$f(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} \hat{f}(\xi)$$

• The Fourier transform $\mathcal{F}$ on $S$ extends in a unique way to an isometric isomorphism $\mathcal{F}_2$ of $L_2(\mathbb{R}^n, dx)$ onto $L_2(\mathbb{R}^n, (2\pi)^{-n} dx)$. For $f, g \in L_2$ the Parseval equations hold:

$$\int f(x)g(x) dx = (2\pi)^{-n} \int \mathcal{F}_2 f(\xi) \mathcal{F}_2 g(\xi) d\xi$$

$$\int |f(x)|^2 dx = (2\pi)^{-n} \int |\mathcal{F}_2 f(\xi)|^2 d\xi$$

• One has that $\mathcal{F}_2 f = \mathcal{F} f$ for $f \in L_2 \cap L_1$. 
Temperate Distributions

The space $S'(\mathbb{R}^n)$ is called temperate distributions.

We shall show how the gaps between $C_0^\infty(\Omega)$ and $L_2(\Omega)$, and between $L_2(\Omega)$ and $S'(\Omega)$, are filled out by Sobolev spaces.

$$C_0^\infty(\Omega) \subset L_2(\Omega) \subset S'(\Omega).$$
Let’s notice that if for \( u \in S' \) we define \( x_j u(\psi) = u(x_j \psi) \), \( \forall \psi \in S \) and \( D_j u(\psi) = -u(D_j \psi) \), then \( x_j u, D_j u \in S' \). Iterating these definition we find that \( D^\alpha \), for any multi-index \( \alpha \), defines a linear map

\[
D^\alpha : S' \rightarrow S'.
\]

All functions \( v \in L_{1,loc} \) with \( |v(x)| \leq C < x >^N \) for some \( N \) are in \( S' \). Note that these functions include the polynomials, but they need not to be differentiable.

The \( \delta \)-distribution and its derivatives \( D^\alpha \delta \) are in \( S' \).
Extension of Fourier Transformation to Tem-perate Spaces

For $u \in S'$, the prescription

$$< \mathcal{F}u, \varphi > = < u, \mathcal{F}\varphi > \quad \text{for all } \varphi \in S$$

defines a temperate distribution $\mathcal{F}u$ or $\hat{u}$ and $\mathcal{F} : u \mapsto \mathcal{F}u$ is a continuous operator on $S'$. Define $F = (2\pi)^{-n/2} \mathcal{F}$

The above operator has the following properties. For $u \in S'$ and $\varphi \in S$,

$$\mathcal{F}(D^\alpha u) = \xi^\alpha \mathcal{F}u,$$

$$\mathcal{F}(x^\alpha u) = (-D_\xi)\alpha \mathcal{F}u,$$
Sobolev Spaces

Let us define $\lambda(\xi) = (1 + |\xi|^2)^{1/2}$, defined on $\mathbb{R}^n$ and more generally we will write $\lambda^s(\xi) = (1 + |\xi|^2)^{s/2}$ for $s \in \mathbb{R}$.

For any $s \in \mathbb{R}$, we say that $u \in H^s(\mathbb{R}^n)$ (the Sobolev space of exponent $s$) if $u \in S'$ and $\lambda^s \hat{u} \in L_2$.

In other word, $u \in H^s$ if $\hat{u}$ is a function satisfying

$$\|u\|_s^2 = (2\pi)^{-n} \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty$$

Since $H^s \subset H^t$ if $s \geq t$, we will also use the notation $H^{-\infty} = \bigcup_s H^s$ and $H^\infty = \bigcap_s H^s$.

It is clear to see that for any $s > 0$,

$$C_0^\infty \subset H^\infty \subset H^s \subset L_2 \subset H^{-s} \subset H^{-\infty} \subset S'$$
For all $s \in \mathbb{R}$ one has

$$u \in H^{s+1} \iff u, D_1 u, \cdots, D_n u \in H^s$$

with the equality $\|u\|_{s+1}^2 = \|u\|_s^2 + \sum_j \|D_j u\|_s^2$.

For any $k \geq 0$, $u \in H^k \iff D^\alpha u \in L^2$ for all $|\alpha| \leq k$.

Moreover $\|u\|_k < \infty \iff \sum_{|\alpha| < k} \|D^\alpha\|_0 < \infty$

Indeed, $(\sum_{|\alpha| \geq k} x^{2\alpha} \geq 1 + |\xi|^k) \geq \sum_{|\alpha| \geq k} C_{k,\alpha} x^{2\alpha}$, and since $\mathcal{F} D^\alpha u = \xi^\alpha \hat{u}$. 
For the partial differential operator with constant coefficients, the fourier transform gives a remarkable simplification. When

\[ P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha \]

is a differential operator on \( \mathbb{R}^n \), the equation \( P(D)u = f \) with \( u, f \in S' \) is by Fourier transformation carried over to the multiplication equation

\[ p(\xi)\hat{u}(\xi) = \hat{f}(\xi) \]

where \( p(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha \) called the symbol of \( P(D) \).
as an example consider the operator $P = 1 - \Delta$. By Fourier transformation, the equation $(1 - \Delta)u = f$ is carried into the equation $(1 + |\xi|^2)\hat{u} = \hat{f}$ and now by division with $1 + |\xi|^2 = <\xi>^2$ to $\hat{u} = <\xi>^{-2} \hat{f}$. Thus the above equation has the solution
\[
 u = \mathcal{F}^{-1} \left( <\xi>^{-2} \mathcal{F} f \right).
\]

We see that for given $f \in S'$ there is one and only solution $u \in S'$.

If $f \in S$ then the solution $u \in S$.

When $f \in L_2$, we see that $(1 + |\xi|^2)\hat{u}(\xi) \in L_2$, so $u \in H^2$ and $D_j u$, $D_i D_j u \in L_2$. what we obtain is $u \in S'$ with $(1 - \Delta)u \in L^2 \Rightarrow u \in H^2$.

Conversely
\[
 u \in H^2 \Rightarrow (1 - \Delta)u \in L_2
\]
Thank you