A story about Non Uniform Rational B-Splines

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Speakers

- 09-06: B-spline curves (W. Dijkstra)
- 16-06: NURBS (E. Shcherbakov)
- 30-06: B-spline surfaces (M. Patricio)
Outline

* B-spline
* Rational B-spline
* Conic sections
Classes of problems

- basic shape is arrived at by experimental evaluation or mathematical calculation → 'fitting' technique
- other design problems depend on both aesthetic and functional requirements (ab initio design)

* B-spline
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B-Spline curve definition

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\[ P(t) = \sum_{i=1}^{n+1} B_i N_{i,k}(t) \quad t_{\text{min}} \leq t \leq t_{\text{max}}, \quad 2 \leq k \leq n + 1 \]

\[ N_{i,1} = \begin{cases} 1 & \text{if } x_i \leq t \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases} \]

\[ N_{i,k}(t) = \frac{(t - x_i) N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t) N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}} \]

Basis functions are defined by the Cox-de Boor recursion formulas
Properties of B-spline curves

| * B-spline | - sum of the B-spline basis functions for any parameter value is one |
| * Rational B-spline | - each basis function is positive or zero |
| * Conic sections | - precisely one maximum (except k=1) |
|            | - maximum order of the curve is one less of the number of control polygon vertices |
|            | - variation-diminishing properties (does not oscillate) |
|            | - curve generally follows the shape of the control polygon |
|            | - curve is transformed by transforming the control polygon vertices |
|            | - curve lies within the convex hull of its control polygon |
Convex hull

* B-spline
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Knot Vectors

| * B-spline | Rational B-spline | Conic sections |

The choice of a knot vector directly influences the resulting curve

The only requirements: monotonically increasing series of real numbers

Two types:
- periodic [0 1 2 3 4]
- open k =2 [0 0 1 2 3 4 4]

Two flavors:
- uniform
- nonuniform

Required number of knots = n + k +1
Recursion relation

For a given basis function $N_{i,k}$ this dependence forms a triangular pattern

\[
\begin{array}{cccc}
N_{i,k} & N_{i+1,k-1} & N_{i+2,k-2} \\
N_{i,k-1} & N_{i+1,k-2} & & \\
N_{i,k-2} & N_{i+1,k-3} & N_{i+2,k-2} & \\
& & & \\
N_{i,1} & N_{i+1,1} & N_{i+2,1} & N_{i+3,1} & N_{i+k-1,1}
\end{array}
\]
Examples for different knot vectors

* B-spline
* Rational B-spline
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### B-spline curve controls

| * B-spline | - changing the type of knot vector and hence basis function: periodic uniform, open uniform or nonuniform |
| * Rational B-spline | - changing the order $k$ of the basis function |
| * Conic sections | - changing the number and position of the control polygon vertices |
| | - using multiple polygon vertices |
| | - using multiple knot values in the knot vector |

- **B-spline**
- **Rational B-spline**
- **Conic sections**
Rational B-splines provide a single precise mathematical form capable of representing the common analytical shapes – lines, planes, conic curves including circles etc.

Interestingly enough, nonuniform rational B-splines have been an Initial Graphics Exchange Specification (IGES) standard since 1983 and incorporated into most of the current geometric modeling systems.
Rational B-spline basis functions for weights $h < 0$ are also valid, but are not convenient. Algorithmically, convention $0/0=0$ is adopted.

\[
P(t) = \sum_{i=1}^{n+1} B_i R_{i,k}(t)
\]

\[
R_{i,k} = \frac{h_i N_{i,k}(t)}{\sum_{i=1}^{n+1} h_i N_{i,k}(t)} \quad (h_i \geq 0)
\]
Characteristics of NURBS

Rational is generalization of nonrational; thus they carry forward all the analytic and geometric characteristics of their B-spline counterparts.

Also:
- a rational B-spline curve of order \( k \) is \( C^{k-2} \) continuous everywhere
- curve is invariant to any projective transformation (not only to affine)
- additional control capabilities due to weights
Conic sections are described by quadratic equations, it is convenient to first consider a quadratic rational B-spline (k=3) defined by three polygon vertices (n+1=3) with knot vector [0 0 0 1 1 1]

\[
P(t) = \frac{h_1 N_{1,3}(t) B_1 + h_2 N_{2,3}(t) B_2 + h_3 N_{3,3}(t) B_3}{h_1 N_{1,3}(t) + h_2 N_{2,3}(t) + h_3 N_{3,3}(t)}
\]
Conic sections

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Conic sections

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A full circle is formed by piecing together multiple segments.
Conclusions

* Currently, NURBS curves are the standard for curve description in computer graphics
* Nice smooth properties
* Several ways to control the resulting curve provide great flexibility
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