Maximum Principles for Parabolic Equations

Kamyar Malakpoor

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Textbooks:
Friedman, A. Partial Differential Equations of Parabolic Type;
Protter, M. H, Weinberger, H. F, Maximum Principles in Differential Equations;

Outline

- Review of MP for the elliptic equations;
- MP for the heat equation $L(u) = \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t}$
- Weak MP for the parabolic equations;
  - Applications;
  - Comparison Principle;
  - Uniqueness Results;
- Strong MP for the parabolic equations;

Review of MP for the elliptic equations

- Consider the operator

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u \tag{1}$$

  in an n-dimensional domain Ω (open and bounded).

- (A) We say that L is elliptic in Ω, if there exists $\lambda > 0$ such that for every $x \in \Omega$ and for any real vector $\xi \neq 0$,

$$\boxed{\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > \lambda|\xi|^2}$$

- (B) We assume that the coefficients in L are bounded and continuous functions in D

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

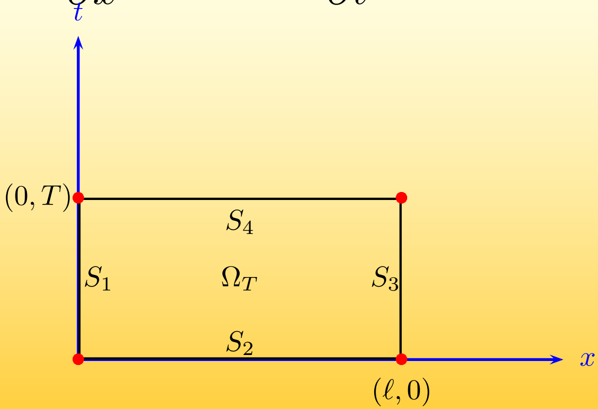
MP-elliptic

- **Weak**
  - A,B $c = 0$: $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $Lu \geq 0$ in $\Omega \Longrightarrow \sup_{\overline{\Omega}} u = \max_{\overline{\Omega}} u = \max_{\partial\Omega} u$
  - A,B $c \leq 0$: $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $Lu \geq 0$ in $\Omega \Longrightarrow \sup_{\overline{\Omega}} u = \max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+$
    $u^+ = \max(u, 0)$
- **Strong**
  - A,B $c = 0$: Ω Open, bounded and connected, if $Lu \geq 0$ and u attains maximum at an interior point, $\Longrightarrow u \equiv$ constant in Ω
  - A,B $c \leq 0$: Ω Open, bounded and connected, if $Lu \geq 0$ and u attains a non-negative maximum, $\Longrightarrow u \equiv$ constant in Ω

MP for the Heat Equation $L(u) = \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t}$

- Suppose $u(x,t)$ satisfies the inequality $L(u) > 0$ in the rectangular region $\Omega_T = (0,\ell) \times (0,T]$ then u cannot have a (local) maximum at any interior point.

- For at such a point $\dfrac{\partial^2 u}{\partial x} \leq 0$ and $\dfrac{\partial u}{\partial t} = 0$, thereby violating $Lu > 0$



- Suppose $u(x,t)$ satisfies in $L(u) \geq 0$ in Ω_T. Then $\max_{\overline{\Omega_T}} u = \max_{S_1 \cup S_2 \cup S_3} u$
  - Define $M := \max_{S_1 \cup S_2 \cup S_3} u$. Let $(x_0, t_0) \in \Omega_T$, such that $M_1 =: u(x_0, t_0) > M$.
  - Define $v(x) := u(x) + \dfrac{M_1 - M}{2\ell^2}(x - x_0)^2$, then $v(x) < M_1$ on $S_1 \cup S_2 \cup S_3$ and $v(x_0, t_0) = M_1$,
  - Furthermore $L(v) = L(u) + \dfrac{M_1 - M}{\ell^2} > 0$ on $\Omega_T \Rightarrow v$ cannot have an interior maximum.
  - At a maximum on S_4, $\partial^2 v/\partial x^2 \leq 0$ and therefore $\partial v/\partial t < 0$ and this contradicts with $u(x_0, t_0) = v(x_0, t_0) < M$.

Weak MP for the Parabolic Equations

- Consider the operator

$$Lu \equiv \underbrace{\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u}_{Au} - \frac{\partial u}{\partial t} \qquad (2)$$

  in $\Omega_T = \Omega \times (0, T]$, with $T > 0$, and Ω domain in $\mathbb{R}^n$, (open and bounded).

- (A) We say that L is parabolic in Ω_T, if there exists $\lambda > 0$ such that for every $(x,t) \in \Omega_T$ and for any real vector $\xi \neq 0$,

$$\boxed{\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j > \lambda|\xi|^2}$$

- (B) We assume that the coefficients in L are bounded functions in Ω_T

Weak MP for the Parabolic Equations(1)

- $$\boxed{Lu \equiv \underbrace{\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u}_{Au} - \frac{\partial u}{\partial t}}$$

- Notation:

$$C^{(2,1)}(\Omega_T) = \{u : \Omega_T \to \mathbb{R};\ u, u_t, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in C(\Omega_T)\}$$

- Define $\partial^* \Omega_T = \partial \Omega_T \diagdown \Omega \times \{T\}$.

- Theorem: Let (A),(B) hold and $c = 0$. If $u \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies $L(u) = A(u) - u_t \geq 0$, then

$$\sup_{\Omega_T} u = \max_{\overline{\Omega_T}} u = \max_{\partial^* \Omega_T} u$$

☛ Proof.

- Suppose $L(u) > 0$ and $\max$ is attained at $(x_0, t_0) \in \Omega_T$. Therefore $\partial u/\partial x_i = \partial u/\partial t = 0$ at (x_0, t_0) and $D^2 u := (\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0, t_0))_{i,j}$ is negative semi-definite, therefore

$$0 < L(u) = (a_{ij}) : D^2(u) \leq 0, \qquad \text{contradiction!!}$$

- If the $\max$ is attained at (x_0, T), then $\partial u/\partial t(x_0, T) \geq 0 \Rightarrow$

$$0 < L(u) = (a_{ij}) : D^2(u) - u_t \leq 0, \qquad \text{contradiction!!}$$

- If $L(u) \geq 0$, then take $u^\epsilon = u - \epsilon t \;\Rightarrow$

$$L(u^\epsilon) = (A - \partial_t)(u - \epsilon t) = L(u) + \epsilon > 0$$

  This implies that $\max\limits_{\overline{\Omega_T}} u^\epsilon = \max\limits_{\partial^* \Omega_T} u^\epsilon$ for every $\epsilon > 0$.
  The assertion follows as $\epsilon \searrow 0$.

Weak MP for the Parabolic Equations(2)

- $$Lu \equiv \underbrace{\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u}_{Au} - \frac{\partial u}{\partial t}$$

- Theorem: Let (A),(B) hold and $c \le 0$ implies that, if $u \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies $L(u) = A(u) - u_t \ge 0$, then

$$\sup_{\Omega_T} u = \max_{\overline{\Omega_T}} u \le \max_{\partial^* \Omega_T} u^+$$

where $u = u^+ - u^-$, $u^+ = \max(u, 0)$.

☞ Proof.

- Suppose $L(u) > 0$, and that u has a nonnegative maximum at $(x_0, t_0) \in \Omega_T$, then

$$0 < L(u) = (\underbrace{(a_{ij}) : D^2(u)}_{\leq 0} + \underbrace{c(x_0, t_0)}_{\leq 0})\underbrace{u}_{\geq 0} \leq 0, \qquad \text{contradiction!!}$$

- If the max is attained at (x_0, T), then $\partial u/\partial t(x_0, T) \geq 0 \Rightarrow$

$$0 < L(u) = \underbrace{(a_{ij}) : D^2(u)}_{\leq 0} \underbrace{-u_t}_{\leq 0} + \underbrace{c(x_0, T)u}_{\leq 0} \leq 0, \qquad \text{contradiction!!}$$

☛ Proof.

- If $L(u) \geq 0$. Suppose $\Omega \subset \{\|x_1\| < d\}$. Consider $u_\epsilon = u + \epsilon e^{\alpha x_1} \Rightarrow$

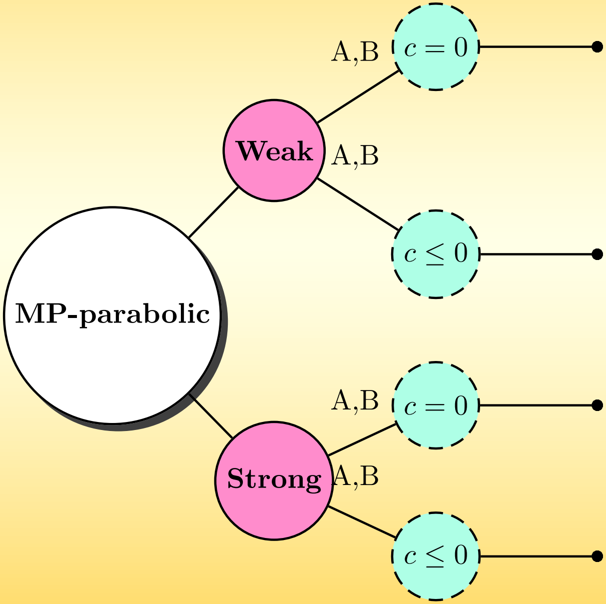
$$\begin{aligned} L(u_\epsilon) &= (A - \partial_t)(u + \epsilon e^{\alpha x_1}) \\ &= L(u) + \epsilon(\alpha^2 a_{11}(x,t) + \alpha b_1(x,t) + c(x,t))e^{\alpha x_1} \\ &\geq \epsilon(\alpha^2 \lambda - \alpha\|b_1\|_\infty - \|c\|_\infty)e^{\alpha x_1}. \end{aligned}$$

- By choosing α large enough, $L(u_\epsilon) > 0$, therefore

$$\sup_{\overline{\Omega_T}} u \leq \sup_{\overline{\Omega_T}} u_\epsilon \leq \max_{\overline{\Omega_T}} u_\epsilon^+ = \max_{\partial^*\Omega_T} u_\epsilon^+ \leq \max_{\partial^*\Omega_T} u^+ + \epsilon e^{\alpha d}$$

for every $\epsilon > 0$.
The assertion follows as $\epsilon \searrow 0$.

$$Lu \equiv \underbrace{\sum_{i,j=1}^{n} a_{ij}(x,{\color{red}t})\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,{\color{red}t})\frac{\partial u}{\partial x_i} + c(x,{\color{red}t})u}_{Au} {\color{red}- \frac{\partial u}{\partial t}}$$



$$u \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T}) \text{ and } Lu \ge 0 \text{ in } \Omega_T \Longrightarrow \sup_{\overline{\Omega_T}} u = \max_{\overline{\Omega_T}} u = \max_{\partial^* \Omega_T} u$$

$$u \in C^{2}(\Omega_T) \cap C(\overline{\Omega_T}) \text{ and } Lu \ge 0 \text{ in } \Omega_T \Longrightarrow \sup_{\overline{\Omega_T}} u = \max_{\overline{\Omega_T}} u \le \max_{\partial^* \Omega_T} u^+$$

$$u^+ = \max(u,0)$$

Applications

- In this section we derive bounds on solution u of the equation $L(u) = f$ in Ω_T.
- (1). Let (A) and (B) hold and $c(x,t) \le 0$. If $L(u) = 0$ in Ω_T, then

$$\max_{\overline{\Omega_T}} |u| \le \max_{\partial^* \Omega_T} |u|$$

(apply the weak MP to u and to $-u$).

- (2). Let (A) and (B) hold and $c(x,t) \le \eta$. If $L(u) = 0$ in Ω_T, then

$$\max_{\overline{\Omega_T}} |u| \le e^{\eta T} \max_{\partial^* \Omega_T} |u|$$

(apply (1) to $v := ue^{-\eta t}$. Indeed, $(A - \partial_t)(ue^{-\eta t}) = e^{-\eta t}(A(u) - \partial_t u + \eta u)$.

Applications (Continue)

- (3). Let (A) and (B) hold and $c(x,t) \leq 0$. Also assume that $\Omega \subset \{\|x_1\| < d\}$ and $a_{11}\lambda^2 + b_1\lambda \geq 1$ in Ω_T, for some positive constant λ. If $L(u) = f$ in Ω_T, then

  $$\max_{\overline{\Omega_T}} |u| \leq \max_{\partial^*\Omega_T} |u| + (e^{\lambda d} - 1)\max_{\overline{\Omega_T}} |f|$$

  define $w := \pm u - \max_{\partial^*\Omega} |u| - (1 - e^{\lambda x_1})e^{\lambda d}\max_{\overline{\Omega_T}} |f|$, then $L(w) \geq 0$ in Ω_T, therefore $w \leq 0$ on $\partial^*\Omega_T$, and this results the above inequality.

Applications (Continue)

- (4). If in (3) the assumption $c(x,t) \leq 0$ replaced by $c(x,t) \leq \eta$, then

$$\max_{\overline{\Omega_T}} |u| \leq e^{\eta T} \left[\max_{\partial^* \Omega_T} |u| + (e^{\lambda d} - 1) \max_{\overline{\Omega_T}} |f| \right]$$

This follows by applying (3) to $v := ue^{\eta t}$.

Comparison Principle

- Theorem. Let (A) and (B) hold. Let $c \leq 0$ and suppose that $f(x,t,u)$ is a continuous function of variables x, t and u and satisfies the one-sided uniform Lipschitz condition in u

  $$f(x,t,v) - f(x,t,u) \leq k(v-u), \quad \forall x,t,u,v, \quad v > u,$$

  If $u, v \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy $Lu + f(x,t,u) \geq 0$ and $Lv + f(x,t,v) \leq 0$ in Ω_T, and $u \leq v$ in $\partial\Omega_T$, then

  $$u \leq v, \quad \text{in } \Omega_T.$$

- Proof. $0 \leq L(u-v) + f(x,t,u) - f(x,t,v) \leq (L+k)(u-v)$, therefore

  $$\max_{\overline{\Omega_T}}(u-v) \leq e^{(k+\|c\|_\infty T)} \max_{\partial^*\Omega_T}(u-v) \leq 0$$

Uniqueness Results

- The First initial boundary value problem consists of solving the differential equation

$$\begin{cases} Lu(x,t) = f(x,t), & \text{in } \Omega_T; \\ u(x,0) = \varphi(x), & \text{on } \Omega \times \{0\}; \\ u(x,t) = g(x,t), & \text{on } \partial\Omega \times (0,T]. \end{cases}$$

- Theorem. Let (A) and (B) hold. Then there exists at most one solution to the above problem.

- Proof.

  - The assumption (B) implies that $c(x,t)$ is bounded, $c(x,t) \leq \eta$. Define $v := ue^{\eta t}$. This transformation carries $Lu = 0$ into $\tilde{L}v := Lv - \eta v = 0$. Now the assertion of the theorem follows from the weak MP for v and $-v$.

Nonlinear Parabolic Equations

- Consider the nonlinear differential operator

$$Lu \equiv F(x, t, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}) - \frac{\partial u}{\partial t},$$

  where F is a nonlinear function of its arguments.

- We say that F is parabolic at a point (x_0, t_0) if for any $p, p_1, \cdots, p_n, p_{11}, \cdots, p_{nn}$, the matrix

$$\left(\frac{\partial F(x_0, t_0, p, p_i, p_{ii})}{\partial p_{hk}} \right)$$

  is positive definite.

- If $Lu^1 = Lu^2$ in the domain Ω_T then, by the mean value theorem,

$$
\begin{aligned}
\frac{\partial(u^1-u^2)}{\partial t} &= F(x,t,u^1,\frac{\partial u^1}{\partial x_i},\frac{\partial^2 u^1}{\partial x_i\partial x_j}) - F(x,t,u^2,\frac{\partial u^2}{\partial x_i},\frac{\partial^2 u^2}{\partial x_i\partial x_j}) \\
&= \sum a_{hk}\frac{\partial^2(u^1-u^2)}{\partial x_h\partial x_k} + \sum b_h\frac{\partial(u^1-u^2)}{\partial x_h} + c(u^1-u^2),
\end{aligned}
$$

where a_{hk}, b_h, c are continuous functions provided $\partial F/\partial p, \partial F/\partial p_h, \partial F/\partial p_h k$ are continuous functions.

- (a_{hk}) is positive definite matrix.
- Applying the previous theorem, we conclude that there exists at most one solution to $Lu = 0$.

Strong MP for the Parabolic Equations

☛ Theorem. Let Ω be open, bounded, and connected in $\mathbb{R}^n$. Let (A) and (B) hold. Let $u \in C^{(1,2)}(\Omega_T) \cap C(\overline{\Omega_T})$ with $Lu = Au - \partial_t u \geq 0$, then

- If $c \equiv 0$, then u cannot have a global maximum in Ω_T, unless u is constant.
- If $c \leq 0$, then u cannot have a global nonnegative maximum in Ω_T, unless u is constant.

$$Lu \equiv \underbrace{\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u}_{Au} - \frac{\partial u}{\partial t}$$

MP-parabolic

- **Weak**
  - A,B $c=0$: $u \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T})$ and $Lu \geq 0$ in $\Omega_T \Longrightarrow \sup_{\Omega_T} u = \max_{\overline{\Omega_T}} u = \max_{\partial^* \Omega_T} u$
  - A,B $c \leq 0$: $u \in C^{2}(\Omega_T) \cap C(\overline{\Omega_T})$ and $Lu \geq 0$ in $\Omega_T \Longrightarrow \sup_{\Omega_T} u = \max_{\overline{\Omega_T}} u \leq \max_{\partial^* \Omega_T} u^+$
    $u^+ = \max(u, 0)$
- **Strong**
  - A,B $c=0$: Ω_T Open, bounded and connected, if $Lu \geq 0$ and u attains maximum at an interior point, $\Longrightarrow u \equiv$ constant in Ω_T
  - A,B $c \leq 0$: Ω_T Open, bounded and connected, if $Lu \geq 0$ and u attains a non-negative maximum, $\Longrightarrow u \equiv$ constant in Ω_T