Maximum Principles for Parabolic Equations

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Textbooks:
Friedman, A. Partial Differential Equations of Parabolic Type;
Outline

- Review of MP for the elliptic equations;
- MP for the heat equation $L(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}$
- Weak MP for the parabolic equations;
  - Applications;
  - Comparison Principle;
  - Uniqueness Results;
- Strong MP for the parabolic equations;
Review of MP for the elliptic equations

Consider the operator

\[ Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u \tag{1} \]

in an \( n \)-dimensional domain \( \Omega \) (open and bounded).

(A) We say that \( L \) is elliptic in \( \Omega \), if there exists \( \lambda > 0 \) such that for every \( x \in \Omega \) and for any real vector \( \xi \neq 0 \),

\[ \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j > \lambda |\xi|^2 \]

(B) We assume that the coefficients in \( L \) are bounded and continuous functions in \( D \)
\[ Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u \]

\[ u \in C^2(\Omega) \cap C(\overline{\Omega}) \quad \text{and} \quad Lu \geq 0 \quad \text{in} \ \Omega \quad \Rightarrow \quad \sup_{\Omega} u = \max_{\overline{\Omega}} u = \max_{\partial \Omega} u \]

\[ u \in C^2(\Omega) \cap C(\overline{\Omega}) \quad \text{and} \quad Lu \geq 0 \quad \text{in} \ \Omega \quad \Rightarrow \quad \sup_{\Omega} u = \max_{\overline{\Omega}} u \leq \max_{\partial \Omega} u^+ \]

\[ u^+ = \max(u, 0) \]

\[ \Omega \quad \text{Open, bounded and connected, if} \quad Lu \geq 0 \quad \text{and} \quad u \text{ attains maximum at an interior point,} \quad \Rightarrow \quad u \equiv \text{constant in } \Omega \]

\[ \Omega \quad \text{Open, bounded and connected, if} \quad Lu \geq 0 \quad \text{and} \quad u \text{ attains a non-negative maximum,} \quad \Rightarrow \quad u \equiv \text{constant in } \Omega \]
MP for the Heat Equation \( L(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} \)

- Suppose \( u(x, t) \) satisfies the inequality \( L(u) > 0 \) in the rectangular region \( \Omega_T = (0, \ell) \times (0, T] \) then \( u \) cannot have a (local) maximum at any interior point.

- For at such a point \( \frac{\partial^2 u}{\partial x^2} \leq 0 \) and \( \frac{\partial u}{\partial t} = 0 \), thereby violating \( Lu > 0 \)
Suppose $u(x, t)$ satisfies in $L(u) \geq 0$ in $\Omega_T$. Then $\max_{\Omega_T} u = \max_{S_1 \cup S_2 \cup S_3} u$.

- Define $M := \max_{S_1 \cup S_2 \cup S_3} u$. Let $(x_0, t_0) \in \Omega_T$, such that $M_1 := u(x_0, t_0) > M$.
- Define $v(x) := u(x) + \frac{M_1 - M}{2\ell^2} (x - x_0)^2$, then $v(x) < M_1$ on $S_1 \cup S_2 \cup S_3$ and $v(x_0, t_0) = M_1$.
- Furthermore $L(v) = L(u) + \frac{M_1 - M}{\ell^2} > 0$ on $\Omega_T \Rightarrow v$ cannot have an interior maximum.
- At a maximum on $S_4$, $\partial^2 v / \partial x^2 \leq 0$ and therefore $\partial v / \partial t < 0$ and this contradicts with $u(x_0, t_0) = v(x_0, t_0) < M$. 
Weak MP for the Parabolic Equations

Consider the operator

\[ Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} \]  \hspace{1cm} (2)

in \( \Omega_T = \Omega \times (0, T] \), with \( T > 0 \), and \( \Omega \) domain in \( \mathbb{R}^n \), (open and bounded).

(A) We say that \( L \) is parabolic in \( \Omega_T \), if there exists \( \lambda > 0 \) such that for every \( (x, t) \in \Omega_T \) and for any real vector \( \xi \neq 0 \),

\[ \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i \xi_j > \lambda |\xi|^2 \]

(B) We assume that the coefficients in \( L \) are bounded functions in \( \Omega_T \)
Weak MP for the Parabolic Equations (1)

\[ Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} \]

Notation:

\[ C^{(2,1)}(\Omega_T) = \{ u : \Omega_T \to \mathbb{R}; u, u_t, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in C(\Omega_T) \} \]

Define \( \partial^*\Omega_T = \partial\Omega_T \setminus \Omega \times \{ T \} \).

Theorem: Let (A), (B) hold and \( c = 0 \). If \( u \in C^{(2,1)}(\Omega_T) \cap C(\Omega_T) \) satisfies \( L(u) = A(u) - u_t \geq 0 \), then

\[
\sup_{\Omega_T} u = \max_{\Omega_T} u = \max_{\partial^*\Omega_T} u
\]
Proof.

- Suppose $L(u) > 0$ and max is attained at $(x_0, t_0) \in \Omega_T$. Therefore $\partial u/\partial x_i = \partial u/\partial t = 0$ at $(x_0, t_0)$ and $D^2u := (\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0, t_0))_{i,j}$ is negative semi-definite, therefore

\[ 0 < L(u) = (a_{ij}) : D^2(u) \leq 0, \quad \text{contradiction!!} \]

- If the max is attained at $(x_0, T)$, then $\partial u/\partial t(x_0, T) \geq 0 \Rightarrow

\[ 0 < L(u) = (a_{ij}) : D^2(u) - u_t \leq 0, \quad \text{contradiction!!} \]

- If $L(u) \geq 0$, then take $u^\epsilon = u - \epsilon t \Rightarrow

\[ L(u^\epsilon) = (A - \partial_t)(u - \epsilon t) = L(u) + \epsilon > 0 \]

This implies that $\max_{\overline{\Omega_T}} u^\epsilon = \max_{\partial^*\Omega_T} u^\epsilon$ for every $\epsilon > 0$. The assertion follows as $\epsilon \searrow 0$. 
Weak MP for the Parabolic Equations (2)

\[ Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} \]

**Theorem:** Let (A),(B) hold and \( c \leq 0 \) implies that, if \( u \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T}) \) satisfies \( L(u) = A(u) - u_t \geq 0 \), then

\[
\sup_{\Omega_T} u = \max_{\Omega_T} u \leq \max_{\partial^*\Omega_T} u^+
\]

where \( u = u^+ - u^- \), \( u^+ = \max(u, 0) \).
Proof.

- Suppose $L(u) > 0$, and that $u$ has a nonnegative maximum at $(x_0, t_0) \in \Omega_T$, then

$$0 < L(u) = (a_{ij}) : D^2(u) + c(x_0, t_0)) u \leq 0,$$

contradiction!!

- If the max is attained at $(x_0, T)$, then $\partial u / \partial t(x_0, T) \geq 0 \Rightarrow$

$$0 < L(u) = (a_{ij}) : D^2(u) - u_t + c(x_0, T)u \leq 0,$$

contradiction!!
Proof.

• If $L(u) \geq 0$. Suppose $\Omega \subset \{\|x_1\| < d\}$. Consider $u_\epsilon = u + \epsilon e^{\alpha x_1} \Rightarrow$

\[
L(u_\epsilon) = (A - \partial_t)(u + \epsilon e^{\alpha x_1})
\]

\[
= L(u) + \epsilon(\alpha^2 a_{11}(x, t) + \alpha b_1(x, t) + c(x, t))e^{\alpha x_1}
\]

\[
\geq \epsilon(\alpha^2 \lambda - \alpha\|b_1\|_\infty - \|c\|_\infty)e^{\alpha x_1}.
\]

• By choosing $\alpha$ large enough, $L(u_\epsilon) > 0$, therefore

\[
\sup_{\Omega_T} u \leq \sup_{\Omega_T} u_\epsilon \leq \max_{\Omega_T} u_\epsilon^+ = \max_{\partial^*\Omega_T} u_\epsilon^+ \leq \max_{\partial^*\Omega_T} u^+ + \epsilon e^{\alpha d}
\]

for every $\epsilon > 0$.

The assertion follows as $\epsilon \searrow 0$. 
\[ Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} \]

**Weak**

- \( A,B \quad c = 0 \)
- \( A,B \quad c \leq 0 \)

**Strong**

- \( A,B \quad c = 0 \)
- \( A,B \quad c \leq 0 \)

\[ u \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T}) \quad \text{and} \quad Lu \geq 0 \quad \text{in} \quad \Omega_T \implies \sup_{\Omega_T} u = \max_{\Omega_T} u = \max_{\partial^+ \Omega_T} u \]

\[ u \in C^2(\Omega_T) \cap C(\overline{\Omega_T}) \quad \text{and} \quad Lu \geq 0 \quad \text{in} \quad \Omega_T \implies \sup_{\Omega_T} u = \max_{\Omega_T} u \leq \max_{\partial^+ \Omega_T} u^+ \]

\[ u^+ = \max(u, 0) \]
Applications

In this section we derive bounds on solution $u$ of the equation $L(u) = f$ in $\Omega_T$.

(i). Let (A) and (B) hold and $c(x, t) \leq 0$. If $L(u) = 0$ in $\Omega_T$, then

$$\max_{\Omega_T} |u| \leq \max_{\partial^*\Omega_T} |u|$$

(apply the weak MP to $u$ and to $-u$).

(ii). Let (A) and (B) hold and $c(x, t) \leq \eta$. If $L(u) = 0$ in $\Omega_T$, then

$$\max_{\Omega_T} |u| \leq e^{\eta T} \max_{\partial^*\Omega_T} |u|$$

(apply (i) to $v := ue^{-\eta t}$. Indeed, $(A-\partial_t)(ue^{-\eta t}) = e^{-\eta t}(A(u)-\partial_t u+\eta u)$.
Applications (Continue)

(3). Let (A) and (B) hold and $c(x, t) \leq 0$. Also assume that $\Omega \subset \{ \| x_1 \| < d \}$ and $a_{11}\lambda^2 + b_1\lambda \geq 1$ in $\Omega_T$, for some positive constant $\lambda$. If $L(u) = f$ in $\Omega_T$, then

$$\max_{\Omega_T} |u| \leq \max_{\partial^*\Omega_T} |u| + (e^{\lambda d} - 1)\max_{\Omega_T} |f|$$

define $w := \pm u - \max_{\partial^*\Omega} |u| - (1 - e^{\lambda x_1})e^{\lambda d}\max_{\Omega_T} |f|$, then $L(w) \geq 0$ in $\Omega_T$, therefore $w \leq 0$ on $\partial^*\Omega_T$, and this results the above inequality.
Applications (Continue)

(4). If in (3) the assumption $c(x, t) \leq 0$ replaced by $c(x, t) \leq \eta$, then

$$\max_{\Omega_T} |u| \leq e^{\eta T} \left[ \max_{\partial^* \Omega_T} |u| + (e^{\lambda d} - 1) \max_{\overline{\Omega_T}} |f| \right]$$

This follows by applying (3) to $v := ue^{nt}$. 
Comparison Principle

**Theorem.** Let (A) and (B) hold. Let $c \leq 0$ and suppose that $f(x, t, u)$ is a continuous function of variables $x, t$ and $u$ and satisfies the one-sided uniform Lipschitz condition in $u$

$$f(x, t, v) - f(x, t, u) \leq k(v - u), \quad \forall x, t, u, v, \quad v > u,$$

If $u, v \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy $Lu + f(x, t, u) \geq 0$ and $Lv + f(x, t, v) \leq 0$ in $\Omega_T$, and $u \leq v$ in $\partial \Omega_T$, then

$$u \leq v, \quad \text{in } \Omega_T.$$

**Proof.** $0 \leq L(u - v) + f(x, t, u) - f(x, t, v) \leq (L + k)(u - v)$, therefore

$$\max\left(\frac{u - v}{\Omega_T}\right) \leq e^{(k + \|c\|_\infty T)}\max\left(\frac{u - v}{\partial\Omega_T}\right) \leq 0$$
Uniqueness Results

The First initial boundary value problem consists of solving the differential equation

\[
\begin{cases}
Lu(x, t) = f(x, t), & \text{in } \Omega_T; \\
u(x, 0) = \varphi(x), & \text{on } \Omega \times \{0\}; \\
u(x, t) = g(x, t), & \text{on } \partial\Omega \times (0, T].
\end{cases}
\]

Theorem. Let (A) and (B) hold. Then there exists at most one solution to the above problem.

Proof.

• The assumption (B) implies that \( c(x, t) \) is bounded, \( c(x, t) \leq \eta \). Define \( v := u e^{\eta t} \). This transformation carries \( Lu = 0 \) into \( \tilde{L}v := Lv - \eta v = 0 \). Now the assertion of the theorem follows from the weak MP for \( v \) and \(-v\).
Nonlinear Parabolic Equations

Consider the nonlinear differential operator

\[ L u \equiv F(x, t, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}) - \frac{\partial u}{\partial t}, \]

where \( F \) is a nonlinear function of its arguments.

We say that \( F \) is parabolic at a point \((x_0, t_0)\) if for any \( p, p_1, \cdots, p_n, p_{11}, \cdots, p_{nn} \), the matrix

\[ \left( \frac{\partial F(x_0, t_0, p, p_i, p_{ii})}{\partial p_{hk}} \right) \]

is positive definite.

If \( Lu^1 = Lu^2 \) in the domain \( \Omega_T \) then, by the mean value theorem,
\[
\frac{\partial (u^1 - u^2)}{\partial t} = F(x, t, u^1, \frac{\partial u^1}{\partial x_i}, \frac{\partial^2 u^1}{\partial x_i \partial x_j}) - F(x, t, u^2, \frac{\partial u^2}{\partial x_i}, \frac{\partial^2 u^2}{\partial x_i \partial x_j})
\]

\[
= \sum a_{hk} \frac{\partial^2 (u^1 - u^2)}{\partial x_h \partial x_k} + \sum b_h \frac{\partial (u^1 - u^2)}{\partial x_h} + c(u^1 - u^2),
\]

where \(a_{hk}, b_h, c\) are continuous functions provided \(\frac{\partial F}{\partial p}, \frac{\partial F}{\partial p_h}, \frac{\partial F}{\partial p_{hk}}\) are continuous functions.

\(\mathbf{(a_{hk})}\) is positive definite matrix.

Applying the previous theorem, we conclude that there exists at most one solution to \(Lu = 0\).
Theorem. Let $\Omega$ be open, bounded, and connected in $\mathbb{R}^n$. Let (A) and (B) hold. Let $u \in C^{(1,2)}(\Omega_T) \cap C(\overline{\Omega_T})$ with $Lu = Au - \partial_t u \geq 0$, then

- If $c \equiv 0$, then $u$ cannot have a global maximum in $\Omega_T$, unless $u$ is constant.
- If $c \leq 0$, then $u$ cannot have a global nonnegative maximum in $\Omega_T$, unless $u$ is constant.
\[ Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} \]

**Weak**

- **A,B**
  - \( c = 0 \)
  - \( c \leq 0 \)

**MP-parabolic**

- **A,B**
  - \( c = 0 \)
  - \( c \leq 0 \)

**Strong**

- **A,B**
  - \( c = 0 \)
  - \( c \leq 0 \)

\( \Omega_T \) Open, bounded and connected,

if \( Lu \geq 0 \) and \( u \) attains a non-negative maximum,

\[ \Longrightarrow u \equiv \text{constant in } \Omega_T \]

**Weak**

- \( u \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T}) \) and \( Lu \geq 0 \) in \( \Omega_T \) \[ \Longrightarrow \sup_{\Omega_T} u = \max u = \max_{\partial^*\Omega_T} u \]

\( u \in C^2(\Omega_T) \cap C(\overline{\Omega_T}) \) and \( Lu \geq 0 \) in \( \Omega_T \) \[ \Longrightarrow \sup_{\Omega_T} u = \max u \leq \max_{\partial^*\Omega_T} u^+ \]

\[ u^+ = \max(u, 0) \]

\( \Omega_T \) Open, bounded and connected,

if \( Lu \geq 0 \) and \( u \) attains maximum at an interior point,

\[ \Longrightarrow u \equiv \text{constant in } \Omega_T \]