Linearized theory of elasticity

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### Seminar: Continuum mechanics

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The Cauchy stress tensor

For a linear elastic solid we have the identity:

$$T_{Ji}(X, t) = \tilde{T}_{Ji}(x + u(x, t)).$$

In terms of Cartesian components, the first Piola-Kirchhoff stress tensor $T^0_{Ji}$ is related to the Cauchy stress $T_{ri}$ at the points $x = X + u$ by

$$T^0_{Ji} = \rho_0 \frac{\partial X_J}{\partial x_r} T_{ri}. \quad (1)$$

The displacement-gradient components are small compared to unity. The equations of motion in the reference state are given by

$$\frac{\partial T^0_{Ji}}{\partial X_J} + \rho_0 b_{0i} = \rho_0 \frac{d^2 x_i}{dt^2}. \quad (2)$$
### Field Equations of Linearized Isotropic Isothermal Elasticity

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<td>$\frac{\partial T_{ji}}{\partial x_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2}$</td>
<td>(3)</td>
<td>$T_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij}$</td>
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<td>$E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$</td>
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15 eqs. for 6 stresses, 6 strains, 3 displacements

The two Lamé elastic constants $\lambda$ and $\mu$, introduced by Lamé in 1852, are related to the more familiar shear modulus $G$, Young’s modulus $E$, and Poisson’s ratio $\nu$ as follows:

$$\mu = G = \frac{E}{2(1 + \nu)} \quad \text{and} \quad \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}.$$
Boundary conditions

1. **Displacement boundary conditions**, with the three components \( u_i \) prescribed on the boundary.

2. **Traction boundary conditions**, with the three traction components \( t_i = T_{ji}n_j \) prescribed at a boundary point.

3. **Mixed boundary conditions** include cases where
   1. Displacement boundary conditions are prescribed on a part of the bounding surface, while traction boundary conditions are prescribed on the remainder, or
   2. at each point of the boundary we choose local rectangular Cartesian axes \( \bar{X}_i \) and then prescribe:
      1. \( \bar{u}_1 \) or \( \bar{t}_1 \), but not both,
      2. \( \bar{u}_2 \) or \( \bar{t}_2 \), but not both, and
      3. \( \bar{u}_3 \) or \( \bar{t}_3 \), but not both.
Navier’s displacement equations

Equations of this form were given by Navier in a memoir of 1821, published in 1827, but they contained only one elastic constant because they were deduced from an inadequate molecular model. The two-constant version was given by Cauchy in 1822.

Navier Equation

\[(\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \rho \mathbf{b} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}\] (8)

Traction Boundary Condition

\[\lambda (\nabla \cdot \mathbf{u}) \mathbf{n} + \mu (\mathbf{u} \mathbf{\nabla} \cdot \mathbf{n} + \mathbf{\nabla} \mathbf{u}) \cdot \mathbf{n} = \text{prescribed function}\] (9)

Elastostatics: \[\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathbf{0}.\]
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In **plane deformation**, the assumptions $u_z = 0$ and $u_x$ and $u_y$ independent of $z$ lead to only three independent strain components $e_x$, $e_y$, and $e_{xy} = \frac{1}{2} \gamma_{xy}$, which are independent of $z$, a state of **plane strain** parallel to the $xy$-plane. The isotropic Hooke's law reduces to

$$\begin{align*}
\sigma_x &= \lambda e + 2G e_x \\
\sigma_y &= \lambda e + 2G e_y \\
\tau_{xy} &= 2G e_{xy}
\end{align*} \tag{3}$$

with, in addition,

$$\sigma_z = \lambda e = \nu(\sigma_x + \sigma_y), \tag{4}$$

where

$$e = e_x + e_y, \quad G = \frac{E}{2(1 + \nu)}, \quad \text{and} \quad \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}. \tag{5}$$
To these must be added two equations of motion

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho b_x &= \rho \frac{\partial^2 u_x}{\partial t^2} \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho b_y &= \rho \frac{\partial^2 u_y}{\partial t^2} 
\end{align*}
\]

(6)

and one compatibility equation

\[
\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2 \frac{\partial^2 e_{xy}}{\partial x \partial y},
\]

(7)

if for small displacements we ignore the difference between the material coordinates \(X, Y\) and the spatial coordinates \(x, y\).
Particular solution for body forces

The linearity can also be used to construct the solution in two parts:

\[ \sigma_{ij} = \sigma_{ij}^H + \sigma_{ij}^P, \quad e_{ij} = e_{ij}^H + e_{ij}^P. \]

The particular solution \( \sigma_{ij}^P, e_{ij}^P \) satisfies the given equations with given body-force distributions but not the boundary conditions. The distribution \( \sigma_{ij}^H, e_{ij}^H \) satisfies the homogeneous differential equations (with no body force) and suitably modified boundary conditions.

When the body force is simply the weight, say \( b_x = 0, \ b_y = -g \), then a possible particular solution is

\[ \sigma_y^P = \rho gy - C, \quad \sigma_x^P = \tau_{xy}^P = 0, \]

where \(-C\) is the value of \( \sigma_y^P \) at \( y = 0 \).
Airy stress function

For plane strain with no body forces, the equilibrium equations are identically satisfied if the stresses are related to a scalar function $\phi(x, y)$, called Airy’s stress function, by the equations

$$
\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}.
$$

(9)

The compatibility equation then becomes the biharmonic equation

$$
\nabla_1^2 (\nabla_1^2) \phi = 0
$$

(10)

or

$$
\nabla_1^4 \phi = 0,
$$

where $\nabla_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the 2D Laplace operator.
Airy stress function useful for boundary conditions for tractions. Since $t_i = \sigma_{ji} n_j$, we have $t_x = \sigma_x n_x + \tau_{xy} n_y$ and $t_y = \tau_{xy} n_x + \sigma_y n_y$ or

$$
\begin{align*}
  t_x &= \frac{\partial^2 \phi}{\partial y^2} \frac{dy}{ds} + \frac{\partial^2 \phi}{\partial x \partial y} \frac{dx}{ds} = \frac{d}{ds} \left( \frac{\partial \phi}{\partial y} \right) \\
  t_y &= - \frac{\partial^2 \phi}{\partial x \partial y} \frac{dy}{ds} \frac{\partial^2 \phi}{\partial x^2} \frac{dx}{ds} = - \frac{d}{ds} \left( \frac{\partial \phi}{\partial x} \right).
\end{align*}
\tag{11}
$$

Hence, integrating along the boundary, we obtain $\frac{\partial \phi}{\partial x} = - \int_C t_y ds + C_1$ and $\frac{\partial \phi}{\partial y} = \int_C t_x ds + C_2$. Now we can calculate $\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x} \frac{dx}{ds} + \frac{\partial \phi}{\partial y} \frac{dy}{ds}$, $\frac{d\phi}{dn} = \frac{\partial \phi}{\partial x} \frac{dy}{ds} - \frac{\partial \phi}{\partial y} \frac{dx}{ds}$ and $\phi = \int_C \frac{d\phi}{ds} ds + C_3$ at the boundary.
Elasticity model in cylindrical coordinates

In cylindrical coordinates, we have

\[ \mathbf{u} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z. \]

Then the displacement-gradient tensor \( \vec{\nabla} \mathbf{u} \) is derived by differentiating the variable unit vectors \( \hat{e}_r, \hat{e}_\theta \) as well as the coefficients of the three unit vectors. Small-strain components in cylindrical coordinates are

\[
\begin{align*}
E_{rr} &= \frac{\partial u_r}{\partial r} \\
E_{\theta \theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
E_{zz} &= \frac{\partial u_z}{\partial z} \\
E_{r \theta} &= \frac{1}{2} \left[ \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right] \\
E_{\theta z} &= \frac{1}{2} \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \\
E_{z r} &= \frac{1}{2} \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right). 
\end{align*}
\] (12)
Plane stress equations in polar coordinates

Plane-strain components will be denoted for

\[
\begin{align*}
\mathbf{u} &= \mathbf{u}_r + \mathbf{u}_\theta \\
\mathbf{e}_r &= \frac{\partial u}{\partial r} \\
\mathbf{e}_\theta &= \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \\
\mathbf{e}_{r\theta} &= \frac{1}{2} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right)
\end{align*}
\]  

Hooke's law in polar coordinates

\[
\begin{align*}
\sigma_r &= \lambda \mathbf{e} + 2G \mathbf{e}_r \\
\sigma_\theta &= \lambda \mathbf{e} + 2G \mathbf{e}_\theta \\
\tau_{r\theta} &= 2G \mathbf{e}_{r\theta} \\
\sigma_z &= \nu(\sigma_r + \sigma_\theta)
\end{align*}
\]

where \( e = e_r + e_\theta \)
Lamé solution for cylindrical tube (1)

Consider a tube long in the $z$-direction, loaded by internal pressure $p_i$ and external pressure $p_o$, with negligible body forces, and assume plane deformation with radial symmetry, independent of $z$ and $\theta$ in the plane region $a \leq r \leq b$. The Navier equations then become, with $u_\theta = u_z = 0$, $u_r = u$, simply

$$ (\lambda + 2G) \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (ru) \right] = 0. \quad (15) $$

Two integrations with respect to $r$ then give

$$ u = Ar + \frac{B}{r}. \quad (16) $$

In polar coordinates we get

$$ e_r = \frac{\partial u_r}{\partial r} = A - \frac{B}{r^2} \quad e_\theta = \frac{u_r}{r} = A + \frac{B}{r^2} \quad e_{r\theta} = 0 \quad (17) $$
Lamé solution for cylindrical tube (2)

Hooke’s law gives

\[
\begin{align*}
\sigma_r &= 2A\lambda + 2GA - \frac{2GB}{r^2} \\
\sigma_\theta &= 2A\lambda + 2GA + \frac{2GB}{r^2} \\
\tau_{r\theta} &= 0 \\
\sigma_z &= 4A\nu(\lambda + G).
\end{align*}
\]

(18)

We apply the boundary conditions at \(r = a\) and \(r = b\).

\[
\begin{align*}
 r &= a & -p_i &= 2A(\lambda + G) - \frac{2GB}{a^2} \\
 r &= b & -p_o &= 2A(\lambda + G) - \frac{2GB}{b^2}
\end{align*}
\]

(19)

whence

\[
\begin{align*}
2GB &= \frac{(p_i - p_o)a^2b^2}{b^2 - a^2} \\
2(\lambda + G)A &= \frac{p_i a^2 - p_o b^2}{b^2 - a^2}
\end{align*}
\]

(20)

so that the stress distribution is

\[
\begin{align*}
\sigma_r &= \frac{p_i a^2 - p_o b^2}{b^2 - a^2} - \frac{p_i - p_o}{b^2 - a^2} \frac{a^2 b^2}{r^2} \\
\sigma_\theta &= \frac{p_i a^2 - p_o b^2}{b^2 - a^2} + \frac{p_i - p_o}{b^2 - a^2} \frac{a^2 b^2}{r^2} \\
\sigma_z &= 2\nu(p_i a^2 - p_o b^2) \\
\tau_{r\theta} &= 0.
\end{align*}
\]

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Direct solution of Navier equations

\[(\lambda + G) \frac{\partial e}{\partial X_k} + G \nabla^2 u_k + \rho b_k = \rho \frac{\partial^2 u_k}{\partial t^2}\]

where \[e = \frac{\partial u_k}{\partial X_k} \].

Already in 1969, for elastostatics, with \[\frac{\partial^2 u_k}{\partial t^2} = 0\], the solution of such a set of three equations in three dimensions by finite-difference or finite-element methods is beginning to be a possibility.

An advantage of direct solving the 3D problem instead of the less complex 2D problem is that strains can be obtained in terms of the first partial derivatives of the displacement field.
The Helmholtz representation

Each continuously differentiable vector field \( \mathbf{u} \) can be represented as

\[
\mathbf{u} = \nabla \phi + \nabla \times \psi.
\]  

(22)

For definiteness there is also the requirement

\[
\nabla \cdot \psi \equiv 0.
\]

Equations of motion in terms of the potentials:

\[
(\lambda + 2G)(\nabla^2 \phi),_k + Ge_{krs}(\nabla^2 \psi_s),_r = \rho\left(\frac{\partial^2 \phi}{\partial t^2}\right),_k + \rho e_{krs}\left(\frac{\partial^2 \psi}{\partial t^2}\right),_r.
\]

Specific solutions also satisfy the wave equations.

\[
\begin{align*}
\nabla^2 \phi &= \frac{1}{c_1^2} \frac{\partial^2 \phi}{\partial t^2} \quad &\nabla^2 \psi_k &= \frac{1}{c_2^2} \frac{\partial^2 \psi_k}{\partial t^2}.
\end{align*}
\]
**Papkovich-Neuber potentials**

When the Helmholtz representation is substituted into the elastostatic Navier Eq., we get

$$\nabla^2 [\alpha \nabla \phi + \nabla \times \psi] = -\frac{\rho b}{G},$$

where $\alpha = \frac{2(1-\nu)}{1-2\nu}$. The **Papkovich-Neuber potentials** are $\phi_0$ and $\Phi$, which is defined by

$$\Phi = \alpha \nabla \phi + \nabla \times \psi.$$

Then we get **Poisson’s equations**

$$\begin{cases} 
\nabla^2 \phi &= -\frac{\rho b}{G} \\
\nabla^2 \phi_0 &= \frac{\rho b \cdot r}{G}
\end{cases}$$

The solution can be found by Green’s formula. In general, the potentials satisfy complicated boundary conditions.
Green’s formula (1)

Two sufficiently differentiable functions $f$ and $g$ in volume $V$ bounded by $S$ satisfy **Green’s Second Identity**:

$$\int_S \left[ f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right] dS = \int_V \left[ f \nabla^2 g - g \nabla^2 f \right] dV. \quad (23)$$

Let $P$ be an arbitrary field point and $Q$ a variable source point. Then **Green’s formula** expresses the value $f_P$ as follows:

$$4\pi f_P = \int_S \left[ \frac{1}{r_1} \frac{\partial f}{\partial n} - f \frac{\partial}{\partial n} \left( \frac{1}{r_1} \right) \right] dS - \int_V \frac{1}{r_1} \nabla^2 f dV.$$
Green’s formula for the half-space

In potential theory the Green’s function $G(P, Q)$ for a region is a symmetric function of the form

$$G(P, Q) = \frac{1}{r_1} + g(P, Q) \quad \nabla^2 g = 0.$$ 

where $r_1 = \|P - Q\|$. We obtain the formula for $f$ in terms of Green’s function:

$$4\pi f_P = -\int_S f \frac{\partial G}{\partial n} dS - \int_V G \nabla^2 f dV.$$

Let $Q_2$ be the image point of $Q$ in the XY-plane and let $r_2 = \|P - Q_2\|$. For the half-space, it is possible to determine $G(P, Q)$:

$$G(P, Q) = \frac{1}{r_1} - \frac{1}{r_2}.$$ 

$$f(P) = -\frac{1}{2\pi} \frac{\partial}{\partial Z} \int_S \frac{f}{r_0} dS - \frac{1}{4\pi} \int_V G \nabla^2 f dV.$$
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Normal traction problem

Given on part $S_1$ of the boundary $S (z = 0)$ of the half-space a distributed normal pressure of intensity $q$, the traction boundary conditions are

$$T_{zz} = -q \quad T_{zx} = T_{zy} = 0 \quad \text{on} \ S_1.$$  

with zero tractions on the remainder of the boundary $z = 0$. We seek the Papkovich-Neubich potentials, assuming that $\phi_1 \equiv \phi_2 \equiv 0$, such that

$$u = \phi_3 \hat{e}_z - \frac{1}{4(1-\nu)} \nabla(\phi_0 + z\phi_3) \quad \text{and} \quad \nabla^2 \phi_0 = \nabla^2 \phi_3 \equiv 0.$$  

They can be found by using **Green’s formula for the half-space**.
Boussinesq problem of concentrated normal force on boundary of half-space

This problem is solved by taking limit $S_1 \to O$ and $q \to \infty$ such that $\lim_{S_1 \to O} \int_{S_1} q dS = P$, which is a finite concentrated load at $O$ in the positive $z$-direction.
Solution of Boussinesq problem

We obtain

\[ \phi_2(x, y, z) = \frac{(1 - \nu)P}{\pi GR} \]

\[ \frac{\partial}{\partial Z} [\phi_0(x, y, z)] = \frac{(1 - \nu)(1 - 2\nu)P}{\pi GR} \]

\[ \phi_0(x, y, z) = \frac{(1 - \nu)(1 - 2\nu)P}{\pi G} \log(R + z) \]

Now also the displacements and tractions can be computed.
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