Complex Representation in Two-Dimensional Theory of Elasticity

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Literature:

- Muskhelishvili: Some Basic Problems of the Mathematical Theory of Elasticity, Chapter 5

- Prof. ir. C. de Pater: Collegediktaat Elasticiteitstheory, Technische Hogeschool Twente (afdeling der werktuigbouwkunde)
This presentation consists of the following parts:

1. Basic equations in theory of elasticity
2. The Airy stress function in 2D problems
3. Cauchy Riemann equations
4. Determination of the displacements from the stress function
5. Representation of the Airy stress function by two complex analytic functions
6. Complex representation of displacements and stresses
7. Example
8. Results
1. **Basic equations in theory of elasticity**

Solutions of problems in elasticity have to satisfy

- Conservation laws
- Compatibility conditions
- Constitutive equations

Variables:
- \( \mathbf{u} \): displacement vector
- \( \mathbf{E} \): deformation tensor
- \( \mathbf{T} \): stress tensor
• Conservation of momentum:

\[
\left( \sum_j T_{ij,j} \right) + F_i = 0
\]

• Relation between deformation and displacement:

\[
\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})
\]
or

\[
\mathcal{E} = \frac{1}{2} (\nabla u + \nabla u^T)
\]
Constitutive equations (Hooke’s law)

\[ T_{ij} = \lambda \delta_{ij} \left( \sum_k \varepsilon_{kk} \right) + 2\mu \varepsilon_{ij} \]

or

\[ T = \lambda (\text{tr}\mathcal{E}) I + 2\mu \mathcal{E} \]

where \( \lambda \) and \( \mu \) are the Lamé parameters.
Combining Balance laws and constitutive equations and assuming that there are no body forces we get 

\[ \mu \Delta u_i + (\lambda + \mu) \left( \sum_k u_{k,ki} \right) = 0. \]

These are called the equations of Navier-Cauchy.

Differentiating with respect to \( x_i \) and summing over \( i \) we get 

\[ \Delta(\text{tr} \mathcal{E}) = 0 \]
2. The Airy stress function in 2D problems

We assume that there exists a function \( U = U(x, y) \) which is called the Airy stress function such that

\[
\begin{align*}
\tau_{11} &= \frac{\partial^2 U}{\partial y^2} \\
\tau_{22} &= \frac{\partial^2 U}{\partial x^2} \\
\tau_{12} &= -\frac{\partial^2 U}{\partial x \partial y}
\end{align*}
\]

The balance equations are automatically satisfied:

\[
\frac{\partial \tau_{i1}}{\partial x} + \frac{\partial \tau_{i2}}{\partial y} = 0
\]
From

\[ \Delta(\text{tr}\mathbf{T}) = 0 \]

we get

\[ \Delta\Delta U = 0 \]

The function \( U \) is called a biharmonic function.
3. Cauchy-Riemann equations

- Let $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ be a complex function $\varphi(z) = p(x, y) + iq(x, y)$.
- $\varphi$ is differentiable at $z \in \mathbb{C}$ then there exists a $C \in \mathbb{C}$ such that for small $h \in \mathbb{C}$
  \[ \varphi(z + h) = \varphi(z) + Ch + O(|h|^2). \]
- Write $C = D + iE$ and $h = \xi + i\eta$
\begin{itemize}
  \item We get

\[ \varphi(z + h) = p(x + \xi, y + \eta) + iq(x + \xi, y + \eta) \]
\[ = p(x, y) + iq(x, y) + \xi \frac{\partial p}{\partial x} + \eta \frac{\partial p}{\partial y} + i\xi \frac{\partial q}{\partial x} + i\eta \frac{\partial q}{\partial y} + \mathcal{O}(|h|^2), \]

\[ \varphi(z) + Ch = \varphi(z) + D\xi - E\eta + i(E\xi + D\eta). \]

  \item Differentiable functions satisfy the Cauchy-Riemann equations

\[ \frac{\partial p}{\partial x} = D = \frac{\partial q}{\partial y} \]
\[ \frac{\partial p}{\partial y} = -E = -\frac{\partial q}{\partial x} \]
\end{itemize}
4. Determination of displacements from the stress function

Let \( u \) and \( v \) denote the components of the displacement vector \( \mathbf{u} \) in the two-dimensional case. From the constitutive equations we get

\[
\lambda \text{tr}\mathbf{E} + 2\mu \frac{\partial u}{\partial x} = \frac{\partial^2 U}{\partial y^2}
\]

\[
\lambda \text{tr}\mathbf{E} + 2\mu \frac{\partial v}{\partial y} = \frac{\partial^2 U}{\partial x^2}
\]

\[
\mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = -\frac{\partial^2 U}{\partial x \partial y}
\]
We can easily derive

\[ 2\mu \frac{\partial u}{\partial x} = \frac{\partial^2 U}{\partial y^2} - \frac{\lambda}{2(\lambda + \mu)} \Delta U \]

\[ 2\mu \frac{\partial v}{\partial y} = \frac{\partial^2 U}{\partial x^2} - \frac{\lambda}{2(\lambda + \mu)} \Delta U \]

because

\[ \Delta U = \text{tr} T = 2(\lambda + \mu) \text{tr} \mathcal{E}. \]
• Introduce the function $P := \Delta U$. This $P$ is harmonic:

$$\Delta P = \Delta \Delta U = 0.$$  

• We get

\[
2\mu \frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + \frac{\lambda + 2\mu}{2(\lambda + \mu)} P \\
2\mu \frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \frac{\lambda + 2\mu}{2(\lambda + \mu)} P
\]
• Define $Q$ as the harmonic conjugate of $P$:

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

• Define $f : G \subseteq \mathbb{C} \rightarrow \mathbb{C}$ by

$$f(x + iy) = P(x, y) + iQ(x, y)$$

• $f : G \rightarrow \mathbb{C}$ is **analytic**. This means that for all $a \in G$ there exist $A_k$’s such that in a neighborhood of $a$

$$f(z) = \sum_{k=0}^{\infty} A_k(z - a)^k.$$
• Define $\varphi$ by

$$\varphi(z) := \frac{1}{4} \int_0^z f(\zeta) d\zeta.$$  

and define $p$ and $q$

$$p(z) := \operatorname{Re} \varphi(z), \quad q(z) := \operatorname{Im} \varphi(z).$$

• We have

$$\varphi'(z) = \frac{\partial p}{\partial x} + i \frac{\partial q}{\partial x} = \frac{1}{4}(P + iQ).$$

By the Cauchy-Riemann equations we get

$$P = 4 \frac{\partial p}{\partial x} = 4 \frac{\partial q}{\partial y}$$

$$Q = -4 \frac{\partial p}{\partial y} = 4 \frac{\partial q}{\partial x}$$
• We can write

\[
2\mu \frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} \frac{\partial p}{\partial x},
\]

\[
2\mu \frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} \frac{\partial q}{\partial y}.
\]

• After integration we get

\[
2\mu u = -\frac{\partial U}{\partial x} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} p + f_1(y),
\]

\[
2\mu v = -\frac{\partial U}{\partial y} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} q + f_2(x).
\]
From the Cauchy Riemann equations and

\[ \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = -\frac{\partial^2 U}{\partial x \partial y} \]

we get

\[ f_1'(y) + f_2'(x) = 0. \]

The general form must be

\[ f_1(y) = 2\mu(-\varepsilon y + \alpha) \]
\[ f_2(x) = 2\mu(\varepsilon x + \beta). \]

These terms \( f_1 \) and \( f_2 \), which only give rigid body displacement, are often omitted.
After omitting $f_1$ and $f_2$

\[ 2\mu u = -\frac{\partial U}{\partial x} + \frac{2(\lambda + 2\mu)}{\lambda + \mu}p \]

\[ 2\mu v = -\frac{\partial U}{\partial y} + \frac{2(\lambda + 2\mu)}{\lambda + \mu}q. \]
5. Representation of the biharmonic function by two complex analytic functions

- Theorem:
  \[ \Delta(U - px - qy) = 0 \]
  where \( \varphi(z) = p(z) + iq(z) \) as introduced earlier.

- Write
  \[ U = px + qy + p_1 \]
  for a harmonic function \( p_1 \).

- Let \( q_1 \) be the harmonic conjugate of \( p_1 \) and define the analytic function \( \chi = \chi(z) \) by
  \[ \chi(z) = p_1(z) + iq_1(z). \]
• One may write

\[ U = \text{Re}(\bar{z}\varphi(z) + \chi(z)) \]

or

\[ U = \frac{1}{2} \left( \bar{z}\varphi(z) + z\bar{\varphi}(z) + \chi(z) + \bar{\chi}(z) \right). \]

• Every \( U \) of this form is biharmonic if \( \varphi \) and \( \chi \) are analytic.
• The method of deduction by Goursat.

Consider the biharmonic equation

\[ \Delta \Delta U = 0. \]

Introduce new variables \( \xi = x + iy = z \) and \( \eta = x - iy = \bar{z} \). Then the preceding equation takes the form

\[ \frac{\partial^4 U}{\partial \xi^2 \partial \eta^2} = 0, \]

whence it follows directly that

\[ U = \varphi_1(z) + \varphi_2(\bar{z}) + \bar{z}\chi_1(z) + z\chi_2(\bar{z}), \]

where \( \varphi_1, \varphi_2, \chi_1 \) and \( \chi_2 \) are "arbitrary" functions. If \( U \) is real then it is easily seen that one must put \( \varphi_2(\bar{z}) = \overline{\varphi_1(z)} \) and \( \chi_2(\bar{z}) = \overline{\chi_1(z)} \).
It is easily found that

\[ 2 \frac{\partial U}{\partial x} = \varphi(z) + \bar{z} \varphi'(z) + \overline{\varphi(z)} + z \varphi'(z) + \chi'(z) + \chi'(\bar{z}) \]

and

\[ 2 \frac{\partial U}{\partial y} = i \left( -\varphi(z) + \bar{z} \varphi'(z) + \overline{\varphi(z)} - z \varphi'(z) + \chi'(z) - \chi'(\bar{z}) \right). \]

It turns out to be convenient to work with

\[ \frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} = \varphi(z) + z \varphi'(z) + \overline{\psi(z)} \]

where

\[ \psi = \chi'. \]

We have

\[ \Delta U = 4 \text{Re} \varphi'(z). \]
6. Complex representation of displacements and stresses

- We already found the following relation between the displacement vector and $U$:

$$2\mu(u + iv) = -\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right) + \frac{2(\lambda + 2\mu)}{\lambda + \mu}\varphi(z).$$

- A more convenient formula is

$$2\mu(u + iv) = \kappa\varphi(z) - z\varphi'(z) - \overline{\psi(z)},$$

where

$$\kappa = \frac{\lambda + 3\mu}{\lambda + \mu}.$$
In order to find a representation of the stress components by means of the functions $\varphi$ and $\psi$ we will find an expression for the forces acting on an arc $AB$ lying in the plane $Oxy$.

- $t$: tangential vector
- $n$: normal vector.
The force \((T_{xn}ds, T_{yn}ds)\) acting on an element \(ds\) of the arc \(AB\), will be understood to be the force exerted on the side of the positive normal:

\[
T_{xn} = T_{xx}(n, e_x) + T_{xy}(n, e_y) = \frac{\partial^2 U}{\partial y^2}(n, e_x) - \frac{\partial^2 U}{\partial x \partial y}(n, e_y),
\]

\[
T_{yn} = T_{xy}(n, e_x) + T_{yy}(n, e_y) = -\frac{\partial^2 U}{\partial x \partial y}(n, e_x) + \frac{\partial^2 U}{\partial x^2}(n, e_y).
\]
• We have

\[(n, e_x) = (t, e_y) = \frac{dy}{ds},\]
\[(n, e_y) = -(t, e_x) = -\frac{dx}{ds}.\]

• Introducing these values into the preceding formulae, one finds

\[\mathcal{T}_{xn} = \frac{d}{ds} \left( \frac{\partial U}{\partial y} \right),\]
\[\mathcal{T}_{yn} = -\frac{d}{ds} \left( \frac{\partial U}{\partial x} \right).\]
We get

\[ T_{x_n} + i T_{y_n} = \frac{d}{ds} \left( \frac{\partial U}{\partial y} - i \frac{\partial U}{\partial x} \right) \]

\[ = -i \frac{d}{ds} \left( \frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} \right) \]

\[ = -i \frac{d}{ds} \left( \varphi(z) + z \varphi'(z) + \psi(z) \right) \]
• Let $ds$ have the direction of $Oy$, then $T_{xn} = T_{xx}$ and $T_{yn} = T_{xy}$ and

$$T_{xx} + iT_{xy} = \varphi'(z) + \overline{\varphi'(z)} - z\varphi''(z) - \psi'(z).$$

• Let $ds$ have the direction of $Ox$, then $T_{xn} = -T_{xy}$ and $T_{yn} = -T_{yy}$ and

$$T_{yy} - iT_{xy} = \varphi'(z) + \overline{\varphi'(z)} + z\varphi''(z) + \psi'(z).$$
7. Example

Use the Airy stress function to calculate displacements.
Use a stress function of the form

\[ U(x, y) = \alpha xy + \beta xy^3, \]

such that \( T_{11} = 6\beta xy \), \( T_{22} = 0 \) and \( T_{12} = \alpha + 3\beta y^2 \). The top side and the bottom side are stress free and

\[ \int_{-C}^{C} T_{xy} \, dy = -P. \]
We get the stress function

\[ U(x, y) = \frac{3P}{4C}xy - \frac{P}{4C^3}xy^3, \]

the stress tensor

\[ \mathbf{T} = \begin{pmatrix} -\frac{P}{4C^3}xy & -\frac{3P}{4C} + \frac{3P}{4C^3}y^2 \\ -\frac{3P}{4C} + \frac{3P}{4C^3}y^2 & 0 \end{pmatrix} \]

and the deformations can be calculated from

\[ \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \end{pmatrix} = \left( \begin{pmatrix} \lambda + \mu \\ \lambda \\ \lambda + \mu \end{pmatrix} \right)^{-1} \begin{pmatrix} \mathbf{T}_{11} \\ \mathbf{T}_{22} \end{pmatrix} \]

and

\[ \varepsilon_{12} = \frac{\mathbf{T}_{12}}{2\mu}. \]
After introducing some other boundary conditions, the displacement can be calculated.
8. Results

- Introduction of the Airy stress function
- Determination of displacements from the stress function
- Representation of the stress function in terms of two complex analytic functions
- Complex representation of displacements and stresses