SPECTRAL METHODS:
ORTHOGONAL POLYNOMIALS

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What are Spectral Methods?

The main components for their formulation are

- Trial functions
- Test functions

The three types of Spectral Schemes are;
Preview of Spectral Methods

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- Galerkin
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- Collocation
Preview of Spectral Methods

What are Spectral Methods?

The main components for their formulation are

- Trial functions
- Test functions

The three types of Spectral Schemes are;

- Galerkin
- Collocation
- Tau
What choice for the trial function $\omega(x)$?

- Periodic Problem: $\omega(x) \mapsto$ Trigonometric Polynomials
- Non-Periodic Problem: $\omega(x) \mapsto$ Orthogonal Polynomials
Orthogonal Polynomials

Sturm-Liouville problems (SLP)
Orthogonal Polynomials
Sturm - Liouville problems (SLP)

- A Sturm - Liouville problem is an eigenvalue problem of the form
  
  \[-(pu')' + qu = \lambda \omega u\]

  in the interval \((-1, 1)\) with boundary condition for \(u\) where \(p, q\) and \(\omega\) are given \(p \in C^1(-1, 1), q\) is bounded, \(\omega\) is the weight function
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- How is spectral accuracy guaranteed?
Properties of Orthogonal Polynomials

- Given $(-1, 1)$ and weight $\omega(x) > 0$ on $(-1, 1)$ and $\omega \in L^1(-1, 1)$. The weighted Sobolev $L^2_\omega(-1, 1)$ is defined by

$$L^2_\omega(-1, 1) = \left\{ p : \int_{-1}^{1} p^2(x)\omega(x)dx < +\infty \right\}$$

The inner product of $L^2_\omega(-1, 1)$ is given by

$$(p, g)_\omega = \int_{-1}^{1} p(x)g(x)\omega(x)dx$$

and the norm

$$\|p\|_{L^2_\omega} = (p, p)_\omega^{1/2}$$
Properties of Orthogonal Polynomials

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\|p\|_{L^2_\omega} = (p, p)^{1/2}_\omega
\]
Properties of Orthogonal Polynomials

- A system of algebraic polynomials \( \{p_k\}_{k=0,1,...} \) with degree \( k \) is said to be orthogonal in \( L^2_\omega(-1,1) \) if \( (p_k, p_m)_\omega = 0, \ m \neq k \) ie

  \[
  \int_{-1}^{1} p_k(x)p_m(x)\omega(x)dx = 0 \text{ whenever } m \neq k
  \]
Properties of Orthogonal Polynomials

1. A system of algebraic polynomials \( \{P_k\}_{k=0,1,...} \) with degree \( k \) is said to be orthogonal in \( L_\omega^2(-1,1) \) if \( (P_k, P_m)_\omega = 0, \; m \neq k \) ie

\[
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\]

2. A series of a function \( u \in L_\omega^2(-1,1) \) can be represented in terms of the system \( \{P_k\} \) by

\[
Su = \sum_{k=0}^{\infty} \hat{u}_k P_k
\]

where

\[
\hat{u}_k = \frac{1}{\|P_k\|_\omega^2} \int_{-1}^{1} u(x)P_k(x)\omega(x)dx
\]

is the polynomial transform of \( u \)
Properties of Orthogonal Polynomials

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Existence and Uniqueness of Orthogonal Polynomials

Lemma

If a sequence of polynomials \( \{p_k\}_{k=0}^{\infty} \) is orthogonal then the polynomial \( p_{N+1}(x) \) is orthogonal to any polynomial \( q \) of degree \( N \) or less.
Existence and Uniqueness of Orthogonal Polynomials

**Theorem**

For any positive function \( \omega(x) \in L^1(-1, 1) \), \( \exists \) a unique set of Monic orthogonal polynomials \( \{p_k\} \), which can be constructed as follows

\[
p_0 = 1, \quad p_1 = x - \alpha_1 \text{ with } \alpha_1 = \frac{\int_{-1}^{1} \omega(x)x\,dx}{\int_{-1}^{1} \omega(x)\,dx}
\]

and

\[
p_{k+1}(x) = (x - \alpha_{k+1})p_k(x) - \beta_{k+1}p_{k-1}(x) \quad k \geq 1
\]

where

\[
\alpha_{k+1} = \frac{\int_{-1}^{1} x\omega(x)p_k^2(x)\,dx}{\int_{-1}^{1} \omega(x)p_k^2(x)\,dx}
\]

and

\[
\beta_{k+1} = \frac{\int_{-1}^{1} x\omega(x)p_k(x)p_{k-1}(x)\,dx}{\int_{-1}^{1} \omega(x)p_{k-1}^2(x)\,dx}
\]
The Spectral Representation of Function

Orthogonal projections on the space of the polynomials of degree $\leq N$,

$$P_N u = \sum_{k=1}^{N} \hat{u}_k p_k$$

The completeness of $\{p_k\}$

$$\Rightarrow \|u - P_N u\| \to 0 \text{ as } N \to \infty \quad \forall \ u \in L_2^2(-1, 1)$$
How to compute the integral

\[
\int_{-1}^{1} u(x)p_k(x)\omega(x)dx = (u, p_k)_{\omega}
\]
How to compute the integral

\[ \int_{-1}^{1} u(x)p_k(x)\omega(x)dx = (u, p_k)_{\omega} \]

By Gauss Integration.

Let \( x_0 < x_1 < \ldots < x_N \) be the roots of \((N + 1) - th\) orthogonal polynomials \( p_{N+1}(x) \) and let \( \omega_0 \ldots \omega_N \) be the solution of the linear system

\[ \sum_{j=0}^{N} (x_j)^k \omega_j = \int_{-1}^{1} x^k \omega(x)dx; \quad 0 \leq k \leq N. \]

Then \( \omega_j > 0 \) for \( j = 0, 1 \ldots N \) and

\[ \int_{-1}^{1} x^k p(x)\omega(x)dx = \sum_{j=0}^{N} (x_j)^k \omega_j \quad \forall p \in p_{2N+1} \]
Consider the polynomial

\[ q(x) = p_{N+1}(x) + ap_N(x) \]

Let \(-1 = x_0 < x_1 \cdots < x_N\) be the \((N + 1)\) roots of the polynomial and \(\omega_0, \ldots, \omega_N\) be the solution of the linear system

\[
\sum_{j=0}^{N} (x_j)^k \omega_j = \int_{-1}^{1} x^k \omega(x)dx.
\]

Then

\[
\int_{-1}^{1} x^k p(x) \omega(x)dx = \sum_{j=0}^{N} (x_j)^k \omega_j \quad \forall \quad p \in p_{2N}
\]
Gauss -Lobatto Integration

The Way Out Is As Follows
Gauss Lobatto Integration

Consider the polynomial

\[ q(x) = P_{N+1}(x) + aP_N(x) + bP_{N-1}(x) \]

let \(-1 = x_0 < x_1 \cdots < x_N = 1\) be the roots of the polynomial and \(\omega_0, \ldots, \omega_N\) be the solution of the linear system

\[
\sum_{j=0}^{N} (x_j)^k \omega_j = \int_{-1}^{1} x^k \omega(x) dx
\]

Then

\[
\int_{-1}^{1} x^k p(x) \omega(x) dx = \sum_{j=0}^{N} (x_j)^k \omega_j \quad \forall p \in P_{2N-1}
\]
Gauss- Lobatto points for the Jacobi Polynomials corresponding to the weight \( \omega(x) = (1 - x)^\alpha (1 + x)^\alpha \) for \( N = 8 \) and \(-1/2 \leq \alpha \leq 1/2\).

**Figure:** The Gauss-Lobatto points for \( N = 8 \) Jacobi polynomials with the weight function \( \omega(x) = (1 - x)^{-\alpha} (1 + x)^{-\beta} \).
The Interpolating Polynomial

The interpolating polynomial associated with \( \{x_j\}_{j=0}^{N} \) is defined as a polynomial of degree less than or equal to \( N \), such that

\[
I_N u(x_j) = u(x_j) \quad j = 0, 1 \ldots N
\]

Hence

\[
I_N u = \sum_{k=0}^{N} \tilde{u}_k p_k
\]

The discrete polynomial coefficients of \( u \) and its inverse relationship is

\[
\tilde{u}_k = \frac{1}{\gamma_k} \sum_{j=0}^{N} u(x_j) p_k(x_j) \omega_j \quad k = 0, \ldots N
\]
The discrete polynomial coefficient $\tilde{u}_k$ can also be expressed in terms of continuous coefficients $\hat{u}_k$ as

$$\tilde{u}_k = \hat{u}_k + \frac{1}{\gamma_k} \sum_{\ell > N} (p_\ell, p_k)_N \hat{u}_\ell \quad k = 0 \ldots N$$

$$\implies I_N u = p_N u + R_N u \quad \text{where}$$

$$R_N u = \sum_{k=0}^{N} \left\{ \frac{1}{\gamma_k} \sum_{\ell > N} (p_\ell, p_k)_N \hat{u}_\ell \right\} p_k$$

is known as the aliasing error (as a result of the interpolation)
Jacobi Polynomials

The Jacobi Polynomials is denoted by $J_{n}^{\alpha,\beta}(x)$ with

$$\omega(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$$

for $\alpha, \beta > -1$ on $(-1, 1)$.
Jacobi Polynomials

- The Jacobi Polynomials is denoted by $J_{n}^{\alpha,\beta}(x)$ with
  $\omega(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$ for $\alpha, \beta > -1$ on $(-1, 1)$.
- Also normalized by
  \[
  J_{n}^{\alpha,\beta}(1) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\beta + 1)}
  \]

  where $\Gamma(x)$ is a gamma function.
Jacobi Polynomials

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- Also normalized by
  $$J_n^{\alpha,\beta}(1) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\beta + 1)}$$

  where $\Gamma(x)$ is a gamma function.
- Satisfies the orthogonality condition
  $$\int_{-1}^{1} J_n^{\alpha,\beta}(x)J_m^{\alpha,\beta}(x)(1 - x)^\alpha(1 + x)^\beta \, dx = 0 \quad \forall \quad n \neq m$$
Jacobi Polynomials

- The Jacobi Polynomials is denoted by $J_{n,\beta}^{\alpha}(x)$ with
  $\omega(x) = (1 - x)^\alpha (1 + x)^\beta$ for $\alpha, \beta > -1$ on $(-1, 1)$.
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  $$J_{n,\beta}^{\alpha}(1) = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\beta + 1)}$$
  where $\Gamma(x)$ is a gamma function.
- Satisfies the orthogonality condition
  $$\int_{-1}^{1} J_{n,\beta}^{\alpha}(x) J_{m,\beta}^{\alpha}(x) (1 - x)^\alpha (1 + x)^\beta \, dx = 0 \quad \forall \quad n \neq m$$
- satisfies the singular Sturm-Liouville Problem
  $$(1 - x)^{-\alpha} (1 + x)^{-\beta} \frac{d}{dx} \left\{ (1 - x)^{\alpha+1} (1 + x)^{\beta+1} \right\} \frac{d}{dx} J_{n,\beta}^{\alpha}(x)$$
  $$+ n(n + 1\alpha + \beta) J_{n,\beta}^{\alpha}(x) = 0$$
Legendre polynomials

Denoted by $L_k(x)$, $k = 0, 1 \ldots$ are eigenfunctions of SLP

$$((1 - x^2)L'_k(x))' + k(k + 1)L_k(x) = 0$$

with

$$p(x) = 1 - x^2, \quad q(x) = 0, \quad \omega(x) = 1$$

Normalized by

$$L_k(x) = \frac{1}{2^k} \sum_{\ell=0}^{\lfloor k/2 \rfloor} (-1)^\ell \binom{k}{\ell} \binom{2k - 2\ell}{k} x^{k-2\ell}$$
Legendre Polynomials

Properties of Legendre polynomials

\[ |L_k(x)| \leq 1, \quad -1 \leq x \leq 1, \]
\[ L_k(\pm 1) = (\pm 1)^k, \]
\[ |L'_k(x)| \leq \frac{1}{2}(k + 1), \quad -1 \leq x \leq 1, \]
\[ \int_{-1}^{1} L^2_k(x) \, dx = \left( k + \frac{1}{2} \right)^{-1}. \]

The expansion of any \( u \in L^2_\omega(-1, 1) \) in terms of the \( L_k' \)s is

\[ u(x) = \sum_{k=0}^{\infty} \hat{u}_k L_k(x), \quad \hat{u}_k = (k + \frac{1}{2}) \int_{-1}^{1} u(x)L_k(x) \, dx \]
Discrete Legendre Series

The explicit formulas for the quadrature points and weights are

- Legendre Gauss (LG)
- Legendre Gauss - Radau (LGR)
- Legendre Gauss - Lobatto (LGL)
Legendre Polynomials

Discrete Legendre Series
LG, LGR, LGL

\[ x_j (j = 0 \ldots N) \text{ zeros of } L_{N+1} \]

\[ \omega_j = \frac{2}{(1 - x_j^2)[L'_{N+1}(x_j)]^2} \quad j = 0 \ldots N \]
Legendre Polynomials

Discrete Legendre Series
LG, LGR, LGL

\[ x_j (j = 0 \ldots N) \text{ zeros of } L_{N+1} \]

\[ \omega_j = \frac{2}{(1 - x_j^2)[L'_{N+1}(x_j)]^2} \quad j = 0 \ldots N \]

\[ x_j (j = 0 \ldots N), \text{ zeros of } L_N + L_{N+1} \]

\[ \omega_0 = \frac{2}{(N + 1)^2}, \quad \omega_j = \frac{1}{(N + 1)^2} \left[ \frac{1 - x_j}{L_{N+1}(x_j)} \right]^2, \quad j = 1, \ldots, N. \]
Legendre Polynomials

Discrete Legendre Series
LG, LGR, LGL

\[ x_j (j = 0 \ldots N) \text{ zeros of } L_{N+1} \]

\[ \omega_j = \frac{2}{(1 - x_j^2)[L_{N+1}^\prime(x_j)]^2} \quad j = 0 \ldots N \]

\[ x_j (j = 0 \ldots N), \text{ zeros of } L_N + L_{N+1} \]

\[ \omega_0 = \frac{2}{(N + 1)^2}, \quad \omega_j = \frac{1}{(N + 1)^2} \frac{1 - x_j}{[L_{N+1}(x_j)]^2}, \quad j = 1, \ldots, N. \]

\[ x_0 = -1, \ x_N = 1, \ x_j (j = 0 \ldots N - 1), \text{ zeros of } L_N^\prime \]

\[ \omega_j = \frac{2}{(N + 1)} \frac{1}{[L_N(x_j)]^2} \text{ for all } j = 0 \ldots N \]
The normalization factor is given by

$$\gamma_k = (k + \frac{1}{2})^{-1} \text{ for } k < N$$

$$\gamma_N = \begin{cases} 
(N + \frac{1}{2}) & \text{for Gauss and Gauss-Radau formulas,} \\
2/N & \text{for the Gauss-Lobatto formula.}
\end{cases}$$
Legendre Polynomials

Differentiation of Legendre Polynomials

That is if \( u = \sum_{k=0}^{\infty} \hat{u}_k L_k \) then \( u' \) can be represented as

\[
u' = \sum_{k=0}^{\infty} \hat{u}^{(1)}_k L_k
\]

where

\[
\hat{u}^{(1)}_k = (2k + 1) \sum_{p=\text{k+1}}^{\infty} \hat{u}_p k \geq 0
\]

The recursion relation is given by

\[
u'(x) = \sum_{k=1}^{\infty} \left[ \frac{\hat{u}^{(1)}_{k-1}}{2k - 1} - \frac{\hat{u}^{(1)}_{k+1}}{2k + 3} \right] L'_k(x)
\]
Chebyshev Polynomials

- The Chebyshev Polynomial of the first kind is denoted by $T_k(x)$, $k = 0, 1 \ldots$ are the eigenfunctions of SLP

\[
\left( \sqrt{1 - x^2} T'_k(x) \right)' + \frac{k^2}{\sqrt{1 - x^2}} T_k(x) = 0
\]

with \( p(x) = (1 - x^2)^{\frac{1}{2}} \), \( q(x) = 0 \) and \( \omega(x) = (1 - x^2)^{-\frac{1}{2}} \)

- The chebyshev polynomial can be expressed in a power series as

\[
T_k(x) = \frac{k}{2} \sum_{\ell=0}^{\lfloor k/2 \rfloor} (-1)^k \frac{(k - \ell - 1)}{\ell!(k - 2\ell)!} 2^\ell x^{k-2\ell}
\]
The trigonometric relation
\[
\cos (k + 1)\theta + \cos (k - 1)\theta = 2 \cos \theta \cos k\theta
\]
gives the recursion relation
\[
T_{k+1}(x) = 2xT_k - T_{k-1}(x)
\]
with \(T_0(x) \equiv 1\) and \(T_1(x) \equiv x\)
**Properties of Chebyshev Polynomials**

\[ |T_k(x)| \leq 1, \quad -1 \leq x \leq 1, \]
\[ T_k(\pm 1) = (\pm 1)^k, \]
\[ |T'_k(x)| \leq (k^2), \quad -1 \leq x \leq 1, \]
\[ T'_k(\pm) = (\pm)^{k+1}k^2, \]
\[ \int_{-1}^{1} T^2_k(x) \frac{dx}{\sqrt{1-x}} = c_k \frac{\pi}{2}, \]

where

\[ c_k = \begin{cases} 
2, & k = 0 \\
1, & k \geq 1.
\end{cases} \]

The Chebyshev expansion of a function \( u \in L^2_w(-1, 1) \) is

\[ u(x) = \sum_{k=0}^{\infty} \hat{u}_k L_k(x), \quad \hat{u}_k = c_k \frac{\pi}{2} \int_{-1}^{1} u(x)T_k(x)dx \]
Discrete Chebyshev Series

The explicit formulas for the quadrature points and weights are

- Chebyshev Gauss (CG)
- Chebyshev Gauss -Radau (CGR)
- Chebyshev Gauss -Lobatto (CGL)
Chebychev Polynomials

Discrete Chebyshev Series
Chebyshev Gauss (CG), Chebyshev Gauss -Radau (CGR), Chebyshev Gauss -Lobatto (CGL)

\[ x_j = \cos \left( \frac{(2j + 1)\pi}{2N + 2} \right), \quad \omega_j = \frac{\pi}{N + 1}, \quad j = 0, \ldots, N \]
Discrete Chebyshev Series
Chebyshev Gauss(CG), Chebyshev Gauss-Radau(CGR), Chebyshev Gauss-Lobatto(CGL)

\[ x_j = \cos \frac{(2j + 1)\pi}{2N + 2}, \quad \omega_j = \frac{\pi}{N + 1}, \quad j = 0, \ldots, N \]

\[ x_j = \cos \frac{2\pi j}{2N + 1}, \quad \omega_j = \begin{cases} \frac{\pi}{2N+1}, & j = 0, \\ \frac{\pi}{2N+2}, & j = 0, \ldots, N \end{cases} \]
Chebyshev Polynomials

Discrete Chebyshev Series
Chebyshev Gauss(CG), Chebyshev Gauss -Radau(CGR), Chebyshev Gauss -Lobatto(CGL)

\[ x_j = \cos \left( \frac{(2j + 1)\pi}{2N + 2} \right), \quad \omega_j = \frac{\pi}{N + 1}, \quad j = 0, \ldots, N \]

\[ x_j = \cos \left( \frac{2\pi j}{2N + 1} \right), \quad \omega_j = \begin{cases} \frac{\pi}{2N+1}, & j = 0, \\ \frac{\pi}{2N+2}, & j = 0, \ldots, N \end{cases} \]

\[ x_j = \cos \left( \frac{j\pi}{2N} \right), \quad \omega_j = \begin{cases} \frac{\pi}{2N}, & j = 0, N, \\ \frac{\pi}{N}, & j = 1, \ldots N - 1 \end{cases} \]
Chebyshev Polynomials

The Chebyshev transform space

This is given by

\[ C_k = \frac{2}{N \bar{c}_j \bar{c}_k} \cos \frac{\pi j k}{N} \]

where

\[ \bar{c}_k = \begin{cases} 2, & j = 0, N, \\ 1, & j = 1, \ldots, N - 1 \end{cases} \]

The inverse transform is represented by

\[ (C^{-1})_{jk} = \cos \frac{\pi j k}{N} \]

Both transforms can be evaluated by the Fast Fourier Transform
The normalization factors $\gamma_k$ is given by

$$\gamma_k = \frac{\pi}{2} c_k \quad \text{for} \quad k < N$$

$$\gamma_N = \begin{cases} 
\frac{\pi}{2} & \text{for Gauss and Gauss-Radau formulas,} \\
\pi & \text{for the Gauss-Lobatto formula}
\end{cases}$$

The aliasing error for the Chebyshev Gauss-Lobatto points is given by

$$\tilde{u}_k = \hat{u}_k + \sum_{j=2mN \pm k}^{j>N} \hat{u}_j$$
Differentiation of Chebyshev polynomials

The derivative of a function $u$ expanded in Chebyshev polynomial is given by

$$u' = \sum_{k=0}^{\infty} \hat{u}_k^{(1)} T_k$$

where

$$\hat{u}_k^{(1)} = \frac{2}{c_k} \sum_{p=k+1 \atop p+k \text{ odd}}^{\infty} p \hat{u}_p \quad k \geq 0$$

The above expression is a consequence of the relation

$$2T_k(x) = \frac{1}{k+1} T'_{k+1}(x) - \frac{1}{k-1} T'_{k-1}(x)$$

and finally we obtain

$$2k \hat{u}_k = c_{k-1} \hat{u}_{k-1}^{(1)} - \hat{u}_{k+1}^{(1)}$$
The recursion relation is given by

\[ c_k \hat{u}^{(1)} = \hat{u}^{(1)}_{k+2} + 2(k + 1)\hat{u}^{(1)}_{k+1}, \quad 0 \leq k \leq N - 1 \]

The generalization relation is given by

\[ c_k \hat{u}^{(q)} = \hat{u}^{(q)}_{k+2} + 2(k + 1)\hat{u}^{(q-1)}_{k+1}, \quad k \geq 0 \]

The coefficients of the second derivative are

\[ \hat{u}^{(2)} = \frac{1}{c_k} \sum_{\substack{p=k+2 \text{ even} \\ p+k \text{ even}}}^{\infty} p(p^2 - k^2)\hat{u}_p, \quad k \geq 0 \]
A simple Differential equation with boundary conditions

Let consider the 1−D second-order linear PDE

\[
\frac{d^2u}{dx^2} - 4 \frac{du}{dx} + 4u = e^x + C, \quad x \in [-1, 1]
\]

with the Dirichlet boundary conditions

\[
u(-1) = 0 \quad \text{and} \quad u(1) = 0
\]

and where \(C\) is a constant: \(C = -4e/(1 + e^2)\). The exact solution of the system is

\[
u(x) = e^x - \frac{\sinh 1}{\sinh 2}e^{2x} + \frac{C}{4}
\]
Chebyshev Polynomials

Solving by Chebyshev spectral method

Look for a numerical solution by the first Chebyshev polynomials: $T_0(x), T_1(x), T_2(x), T_3(x)$ and $T_4(x), N = 4$. Expand the source $u(x) = e^x + C$ onto the Chebyshev Polynomials

$$p_4u(x) = \sum_{n=0}^{4} \tilde{u}_n T_n(x)$$

and

$$I_4 u(x) = \sum_{n=0}^{4} \hat{u}_n T_n(x)$$

with

$$\tilde{u}_n = \frac{2}{\pi (1 + \delta_{0n})} \int_{-1}^{1} T_n(x)u(x)\frac{dx}{\sqrt{1 - x^2}}$$

and

$$\hat{u}_n = \frac{2}{\pi (1 + \delta_{0n})} \sum_{n=0}^{4} w_i T_n(x_i)u(x_i)$$
where the \( x_i \)'s being the 5 Gauss-Lobatto quadrature points for the weight \( w = (1 - x^2)^{-1/2} \):

\[
x_i = \{-\cos(i\pi/4), 0 \leq i \leq 4\} = \{-1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\}
\]

The continuous coefficient is obtained as follows:

<table>
<thead>
<tr>
<th>( \hat{u}_N )</th>
<th>( \tilde{u}_N )</th>
<th>( \hat{u}_N - \tilde{u}_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.03004</td>
<td>-0.300402</td>
<td>2.010(^{-7})</td>
</tr>
<tr>
<td>1.130</td>
<td>1.1299968</td>
<td>3.210(^{-6})</td>
</tr>
<tr>
<td>0.2715</td>
<td>0.271455</td>
<td>4.510(^{-5})</td>
</tr>
<tr>
<td>0.04488</td>
<td>0.04434</td>
<td>5.410(^{-4})</td>
</tr>
<tr>
<td>0.005474</td>
<td>0.005473999999</td>
<td>1.010(^{-12})</td>
</tr>
</tbody>
</table>
Chebyshev Polynomials

The source and its Chebyshev interpolant

![Graph showing the source and its Chebyshev interpolant.](image-url)
Chebychev Polynomials

Interpolation error and the aliasing error

Figure: $N = 4$ (5 Chebyshev polynomials)
———Thank you———