LINEAR DISPERSIVE WAVES

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Outline

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   - Dispersion Relations
   - Definition of Dispersive Waves

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   - Asymptotic Behaviour

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   - Energy Propagation

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Dispersion Relation

Ansatz:

\[ \varphi(x, t) = Ae^{(i\kappa x - i\omega t)}, \]

\( \kappa, \omega, A \) are constants.

For \( \kappa, \omega, \)

\[ G(\omega, \kappa) = 0 \]

is the dispersive relation.

\[ \omega = W(\kappa), \]

are called modes.

\[ \text{Re}(\varphi) = |A| \cos(\kappa x - \omega t + \eta), \eta = \arg(A), \]

\( \theta = \kappa x - \omega t \) determines \( \text{Re}(\varphi). \)
Dispersion Relation

Hence,

\[ \theta_x = \kappa, \quad -\theta_t = \omega, \quad \Rightarrow \lambda = \frac{2\pi}{\kappa}, \quad \tau = \frac{2\pi}{\omega}. \]

The phase velocity:

\[ c = \frac{\omega}{\kappa}, \quad \frac{\omega}{\kappa} \]

normal to \( \kappa \).

We need

\[ \text{determinant} \left| \frac{\partial^2 W}{\partial \kappa_i \partial \kappa_j} \right| \neq 0, \]

for \( W(\kappa) \) real. In 1D

\[ W''(\kappa) \neq 0. \]
Dispersion Relation

Examples:
Klein-Gordon (quantum theory):
\[ \varphi_{tt} - \alpha^2 \nabla^2 \varphi + \beta^2 \varphi = 0, \quad \omega = \pm \sqrt{\alpha^2 \kappa^2 + \beta^2}; \]

Korteweg-de Vries equation (Long water waves):
\[ \varphi_t + \alpha \varphi_x + \beta \varphi_{xxx} = 0, \quad \omega = \alpha \kappa - \beta \kappa^3; \]

Boussinesq equation (Longitudinal waves for elasticity):
\[ \varphi_{tt} - \alpha^2 \nabla^2 \varphi = \beta^2 \nabla^2 \varphi_{tt}, \quad \omega = \pm \frac{\alpha \kappa}{\sqrt{1 + \beta^2 \kappa^2}}; \]
Dispersive Relations

The Beam equation

\[ \varphi_{tt} + \gamma^2 \varphi_{xxxx} = 0, \quad \omega^2 - \gamma^2 \kappa^4 = 0, \]

with modes \( \omega = \pm \gamma \kappa^2 \).

Generally,

\[ p \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) = 0, \]

and \( P \) a polynomial. Hence,

\[ \frac{\partial}{\partial x_j} \rightarrow i \kappa_j, \quad \frac{\partial}{\partial t} \rightarrow -i \omega. \]

and

\[ P(-i \omega, i \kappa_1, i \kappa_2, i \kappa_3) = 0, \quad \Rightarrow \quad \frac{\partial}{\partial t} \leftrightarrow -i \omega, \quad \frac{\partial}{\partial x_j} \leftrightarrow i \kappa_j. \]
Definition of Dispersive Waves

A linear dispersive wave satisfies

\[ \varphi(x, t) = Ae^{i\kappa x - i\omega t}, \quad \omega = W(\kappa), \]

and

\[ \text{determinant } \left| \frac{\partial^2 W}{\partial \kappa_i \partial \kappa_j} \right| \neq 0. \]

Hence when oscillations in space are coupled with oscillations in time through a dispersion relation, we expect typical effects of Dispersive Waves.
The Beam Equation

Consider

\[ \varphi_{tt} + \gamma^2 \varphi_{xxxx} = 0, \]

recall

\[ \omega^2 - \gamma^2 \kappa^4 = 0, \quad \Rightarrow \quad \omega = \pm \gamma \kappa^2, \]

From

\[ \varphi(x, t) = \int_{-\infty}^{\infty} F(\kappa) e^{i\kappa x - iW(\kappa)t} d\kappa, \]

\( F(\kappa) \) depends on \( \varphi, \varphi_t \).
General Solution by Fourier Integrals

Solution:

\[ \varphi(x, t) = \int_{-\infty}^{\infty} F_1(\kappa) e^{i\kappa x - iW(\kappa)t} + \int_{-\infty}^{\infty} F_2(\kappa) e^{i\kappa x + iW(\kappa)t} d\kappa \]

with

\[ \varphi = \varphi_0(x), \quad \varphi_t = \varphi_1(x). \]

we get

\[ \varphi_0(x) = \int_{-\infty}^{\infty} (F_1(\kappa) + F_2(\kappa)) e^{i\kappa x} d\kappa, \]

and

\[ \varphi_1(x) = -i \int_{-\infty}^{\infty} W(\kappa)(F_1(\kappa) - F_2(\kappa)) e^{i\kappa x} d\kappa. \]
Solution by Fourier Integrals

Inverse formula:

\[ F_1(\kappa) + F_2(\kappa) = \Phi_0(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_0(x)e^{-i\kappa x}dx. \]

\[ -iW(\kappa)[F_1(\kappa) - F_2(\kappa)] = \Phi_1(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_1(x)e^{-i\kappa x}dx. \]

which gives

\[ F_1(\kappa) = \frac{1}{2} \left[ \Phi_0 + \frac{i\Phi_1(\kappa)}{W(\kappa)} \right], \quad F_2(\kappa) = \frac{1}{2} \left[ \Phi_0 - \frac{i\Phi_1(\kappa)}{W(\kappa)} \right]. \]

Since \( \varphi_0(x), \varphi_1(x) \) are real \( \Phi_0(-\kappa) = \Phi_0^*(\kappa), \Phi_1(-\kappa) = \Phi_1^*(\kappa), \)

\[ F_1(-\kappa) = F_1^*(\kappa), \quad F_2(-\kappa) = F_2^*(\kappa), \quad \text{if} \quad W(-\kappa) = -W(\kappa), \]

\[ F_1(-\kappa) = F_2^*(\kappa), \quad F_2(-\kappa) = F_1^*(\kappa), \quad \text{if} \quad W(-\kappa) = W(\kappa). \]
Asymptotic Behaviour

From

\[ \varphi(x, t) = \int_{-\infty}^{\infty} F(\kappa) e^{i\kappa x - iW(\kappa)t} d\kappa, \]

\( x / t \text{ fixed }, t \to \infty, \)

\[ \varphi(x, t) = \int_{-\infty}^{\infty} F(\kappa) e^{-i\chi t} d\kappa, \chi = W(\kappa) - \kappa \frac{x}{t}. \]

around \( \kappa = k \) and

\[ \chi' (\kappa) = W' (\kappa) - \frac{x}{t} = 0. \]

By Taylor expansion of \( \chi(\kappa) \) and \( F(\kappa) \),

\[ F(\kappa) \simeq F(k), \chi(\kappa) = \chi(k) + (\kappa - k)^2 \chi'' (\kappa), \chi'' (\kappa) \neq 0. \]
Asymptotic Behaviour

We obtain

$$\varphi \sim F(k) e^{-i\chi(\kappa)t} \int_{-\infty}^{\infty} \exp \left( -\frac{i}{2}(\kappa - k)^2\chi''(\kappa)t \right) d\kappa,$$

from

$$\int_{-\infty}^{\infty} e^{-\alpha z^2} dz = \sqrt{\frac{\pi}{\alpha}},$$

(path through $\pm \pi/4$),

$$\varphi \sim F(k) \sqrt{\frac{2\pi}{t |\chi''(k)|}} \exp -i\chi(k)t - i\frac{\pi}{4},$$

for Beam equation.
Asymptotic Behaviour

From above,

\[ k(x, t) : \ W(k) = \frac{x}{t}, \quad k > 0, \ \frac{x}{t} > 0, \ \ W(k) = -W(k), \]

If \( W'(k) = \text{const}, \ W''(\kappa) = 0, \)
we need \( \chi'''(\kappa) \neq 0, \) and

\[ \varphi \sim F(k) e^{-i\chi(k)t} \int_{-\infty}^{\infty} \exp\left(-\frac{i}{6}\chi'''(\kappa)t(\kappa - k)^3\right) d\kappa. \]

So \( W''(\kappa) = 0, \ \kappa = \text{const}, \) equation is undefined on \( k \sim k(x/t) \rightarrow x/t = W'(k). \)
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Kinematic Derivation of Group Velocity

Generally, for \((x, t)\),

\[
\theta(x, t) = xk(x, t) - t\omega(x, t),
\]

\(k = \theta_x, \quad \omega = -\theta_t,\)

\[
\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial t} = 0,
\]

\(k(x, t)\) (density) and \(\omega\) (flux). For \(\omega = W(k)\),

\[
\frac{\partial k}{\partial t} + C(k) \frac{\partial k}{\partial x} = 0, \quad C(k) = W'(k),
\]

\(C(k)\), for some \(k\), if \(k = f(x), \quad t = 0,\)

\[
k = f(\xi), \quad x = \xi + C(\xi)t, \quad C(\xi) = C(f(\xi)).
\]

gives \(k\) from (1), and \(x = C(k)t, \quad C(k)\) is the group velocity.
Hence, for a $k_0$, $x = W'(k_0)t$,

$$W'(k) = \frac{d\omega}{dk}$$

is the group velocity.

For $\theta(x, t) = \theta_0$,

$$\theta_x \frac{dx}{dt} + \theta_t = 0 \Rightarrow \frac{dx}{dt} = -\frac{\theta_t}{\theta_x} = \frac{\omega}{k}$$

defines the phase velocity.
Example

![Wave 1](image1)

![Wave 2](image2)

![SUM](image3)
Kinematics Derivation of Group Velocity

Examples:
1) Beam Equation gives:

\[ W(k) = \gamma k^2 \Rightarrow W'(k) = 2\gamma k = \frac{x}{t}, \]

and

\[ k = \frac{x}{2\gamma t}, \quad \omega = \frac{x^2}{4\gamma t^2}, \quad \theta = \frac{x^2}{4\gamma t}. \]

\[ c_g = 2\gamma k, \quad c_p = \gamma k \Rightarrow c_g > c_p. \]

Group and phase lines:

\[ \frac{x}{2\gamma t} = \text{const}, \quad \frac{x^2}{4\gamma t} = \text{const}. \]
2) Deep water waves:

\[ W(k) = \sqrt{gk} \Rightarrow W'(k) = \sqrt{\frac{g}{4k}}, \]

\[ k = \frac{gt^2}{4x^2}, \quad \omega = \frac{gt^2}{2x}, \quad \theta = -\frac{gt^2}{4x} \]

implies \( c_g < c_p \). where

\[ c_g = \frac{1}{2} \sqrt{\frac{g}{k}}, \quad c_p = \sqrt{\frac{g}{k}}. \]

Thus,

\[ \frac{gt^2}{4x} = \text{const}, \quad \frac{x}{2\gamma t} = \text{const}. \]
Output for Group and Phase Velocities
Extension to Higher Dimensions

General solution:

\[ \varphi = \int_{\mathbb{R}^n} F(\kappa) e^{i\kappa \cdot x - iW(\kappa)t} d\kappa, \]

\[ \sim F(\kappa) \left( \frac{2\pi}{t} \right)^{\frac{n}{2}} \left( \det \left| \frac{\partial W}{\partial k_i \partial k_j} \right| \right)^{-\frac{1}{2}} e^{ikx - iW(k)t + i\zeta}, \]

where

\[ \frac{x_i}{t} = \frac{\partial W(k)}{\partial k_i}. \]

For \( \theta(x, k, t), x = (x_1, x_2, x_3) \) and

\[ \omega = -\frac{\partial \theta}{\partial t}, \quad k_i = \frac{\partial \theta}{\partial x_i}, \quad \omega = W(k, x, t). \]

Eliminating \( \theta \),

\[ \frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0, \quad \frac{\partial k_i}{\partial x_j} - \frac{\partial k_j}{\partial x_i} = 0. \]
Extensions to Higher Dimensions

For \( \omega = W(k, x, t) \) varying,

\[
\frac{\partial k_i}{\partial t} + \frac{\partial W}{\partial k_j} \frac{\partial k_j}{\partial x_i} = -\frac{\partial W}{\partial x_i}.
\]

\( \partial k_j / \partial x_i = \partial k_i / \partial x_j \),

\[
\frac{\partial k_i}{\partial t} + \frac{\partial k_i}{\partial x_j} = -\frac{\partial W}{\partial x_i}, \quad C_j(k, x, t) = \frac{\partial W}{\partial k_j},
\]

\( C \) group velocity for some \( k_i \),

\[
\frac{dk_i}{dt} = \frac{\partial W}{\partial x_i} \quad \text{on} \quad \frac{dx_i}{dt} = \frac{\partial W}{\partial k_i}
\]

called the characteristics form.

From above, we obtain the Hamiltonian-Jacobi equation

\[
\frac{\partial \theta}{\partial t} + W\left( \frac{\partial \theta}{\partial x}, x, t \right) = 0.
\]
Energy Propagation

From

\[
\varphi_{tt} - \alpha^2 \varphi_{xx} + \beta^2 \varphi = 0,
\]

\(\alpha\) and \(\beta\) constant,

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \varphi_t^2 + \frac{1}{2} \alpha^2 \varphi_x^2 + \frac{1}{2} \beta^2 \varphi^2 \right) + \frac{\partial}{\partial x} (-\alpha^2 \varphi_t \varphi_x) = 0,
\]

and a slow wavetrain

\[
\varphi \sim \text{Re}(Ae^{i\theta}) = a \cos(\theta + \eta), \ a = |A|, \ \eta = \text{arg}A,
\]

since

\[
\frac{1}{2} \varphi_t^2 = \frac{1}{2} \omega^2 a^2 \sin^2(\theta + \eta),
\]

Energy density

\[
\frac{1}{2} (\omega^2 + \alpha^2 k^2) a^2 \sin^2(\theta + \eta) + \beta^2 a^2 \cos^2(\theta + \eta),
\]
Energy Propagation

Flux

\[ \alpha^2 \omega x a^2 \sin^2(\theta + \eta), \]

with slow varying \( \omega, k \). We obtain

\[ E_1 = \frac{1}{4} (\omega^2 + \alpha^2 k^2 + \beta^2) a^2, \quad E_2 = \frac{1}{2} \alpha^2 \omega k a^2, \]

\[ \omega = \sqrt{\alpha^2 k^2 + \beta^2}, \quad E_1 = \frac{1}{2} (\alpha^2 k^2 + \beta^2) a^2, \quad E_2 = \frac{1}{2} \alpha^2 k \sqrt{\alpha^2 k^2 \beta^2} a^2 \]

with \( C(k) \) and \( E \) equation,

\[ C(k) = \frac{\alpha^2 k}{\sqrt{\alpha^2 k^2 + \beta^2}}, \quad \Rightarrow \quad E_2 = C(k) E_1, \]

\[ \frac{\partial E_1}{\partial t} + \frac{\partial (CE_1)}{\partial t} = 0, \]

gives the differential form of energy.
Conclusion

- Basic notions and definition of Linear dispersive waves.
- Different modes propagate with different wave speeds.
- General solution and asymptotic analysis.
- Group and phase velocities.
- Energy propagation.
THANK YOU.