Complex Analysis revisited

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What is a complex number?

All complex numbers form a field that is an extension of the real number field.

**Definition**

A complex number is an expression of the form $z = x + iy$, where $x, y \in \mathbb{R}$. Components defined as $x = \Re(z)$, $y = \Im(z)$, $i^2 = -1$

Thus, we identify the bijection from $\mathbb{R}^2$ to $\mathbb{C}$ as

$$
\begin{bmatrix}
x \\
y
\end{bmatrix} \mapsto z = x + iy
$$
The complex field \( \mathbb{C} \) is the set of pairs \((x, y)\) with addition and multiplication defined by

\[
\begin{align*}
  z + w &= (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \\
  z \ast w &= (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)
\end{align*}
\]

The following laws also holds

1. \( z + w = w + z \) and \( zw = wz \) (commutative)
2. \((z + w) + p = z + (w + p) \) and \((zw)p = z(wp)\) (associative)
3. \( z(w + p) = zw + zp \) (distributive)

The **complex conjugate** of a complex number \( z = x + iy \) is defined to be \( \bar{z} = x - iy \).

\[
\begin{align*}
  \Re(z) &= \frac{(z + \bar{z})}{2} \\
  \Im(z) &= \frac{(z - \bar{z})}{2i}
\end{align*}
\]
Complex plane

The set of complex numbers forms the complex plane $\mathbb{C}$. To each complex number $z = x + iy$ we associate the point $(x, y)$ in the Cartesian plane. Also a complex number can be represented by a vector $(r, \theta)$ in polar coordinates.

A modulus of $z$ is

$$r = \sqrt{x^2 + y^2} = |z|.$$

From $x = r \cos \theta$ and $y = r \sin \theta$ it follows

$$z = r(\cos \theta + i \sin \theta),$$

where $\theta$ is called an argument of $z$. 
A function $F : \mathbb{C} \mapsto \mathbb{C}$ is called a complex function of a complex variable.

$$F(z) = F(x + iy) = \Re(F(z)) + i\Im(F(z)) = F_1(x, y) + iF_2(x, y),$$
where $f_1(x, y), f_2(x, y)$ are two real functions of two real variables $x$ and $y$.

Also can be represent in the following way

$$F : \mathbb{R}^2 \mapsto \mathbb{R}^2 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix} = \begin{bmatrix} \Re(F(x + iy)) \\ \Im(F(x + iy)) \end{bmatrix}$$
Differentiability

Definition

A complex-valued function $F(z)$ is called a differentiable in a point $z_0$ if exist

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0}$$

Thus, the complex derivative of $F(z)$ at $z_0$ is

$$\frac{dF}{dz}(z_0) = F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{F(z_0 + \Delta z) - F(z_0)}{\Delta z}$$

The point $z_0 + \Delta z$ may approach the point $z_0$ along an arbitrary curve ending at $z_0$. The limit is the same regardless of the path along which $z_0$ is approached.
Cauchy-Riemann equations

It follows limit should exist and is the same for $z$ approaching $z_0$ through the paths parallel to the coordinate axes. First, let $z = x + iy_0$ and $x \to x_0$. Then

$$F'(z_0) = \partial_x F_1 + i\partial_y F_1$$

For $z = x_0 + iy$ and $y \to y_0$ we will have

$$F'(z_0) = \partial_x F_2 - i\partial_y F_2$$

Comparing the real and the imaginary parts of two equations, we get Cauchy-Riemann equations

$$\partial_x F_1 = \partial_y F_2 \quad \partial_y F_1 = -\partial_x F_2 \quad \text{(1)}$$
Analytic function

Equations (1) can also be rewritten as

$$\partial_x F = -i \partial_y F$$

Satisfying these equations is a necessary condition for $F(z)$ to be differentiable at point $z = z_0$, but not a sufficient condition.

**Definition**

We say that the complex function $F$ is analytic at the point $z_0$, provided there is some $\epsilon > 0$ such that $F'(z)$ exist for all $z \in D_\epsilon(z_0)$. In other words, $F$ must be differentiable not only at $z_0$, but also at all points in some $\epsilon$ neighborhood of $z_0$. If $F$ is analytic at each point in the region $D$, then we say that $F$ is analytic on $D$. 
The necessary and sufficient condition for a function $F(z) = F_1 + iF_2$ to be analytic on a region $D$ is that $F_1$ and $F_2$ have first order continuous partial derivatives on $D$ and satisfy C-R equations (1).

If $F(z)$ is analytic in a region $D$, then the derivative of $F(z)$ is also an analytic function on $D$. Hence, the second order partial derivatives of $F_1$ and $F_2$ are also continuous. Using the C-R equations, we get the Laplace equations:

$$
\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} = 0 \quad \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} = 0
$$
Thus, the real part and the imaginary part of an analytic function $F = F_1 + iF_2$ are harmonic functions.

We have that

$$x = \frac{1}{2}(z + \bar{z}), \quad y = -\frac{1}{2}i(z - \bar{z})$$

(2)

By the rules of derivative, we have

$$\frac{\partial F}{\partial z} = \frac{1}{2} \left( \frac{\partial F}{\partial x} - i \frac{\partial F}{\partial y} \right), \quad \frac{\partial F}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right)$$

This implies that, a function is analytic if and only if $\partial F / \partial \bar{z} = 0$. 
Laplace and Euler operators

Multiplying the last two relations we can easily derive the Laplace operator of function $F$

$$\Delta F = 4 \frac{\partial F}{\partial z} \frac{\partial F}{\partial \bar{z}}$$

The following properties of Laplace operator holds

$$\Delta \Re = \Re \Delta = 4 \Re \partial_z \partial_{\bar{z}} \quad \Delta \Im = \Im \Delta = 4 \Im \partial_z \partial_{\bar{z}}$$

The complex representation of the Euler operator is the following

$$\mathbf{x} \cdot \nabla F(\mathbf{x}) = z \frac{\partial F}{\partial z} + \bar{z} \frac{\partial F}{\partial \bar{z}}$$
Anti-analytic function

Definition

An anti-analytic function is a function $F$ satisfying the condition

$$\frac{\partial F}{\partial z} = 0$$

Using the result

$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial F}{\partial y} \frac{\partial F}{\partial y}$$

gives the anti-analytic version of the Cauchy-Riemann equations as

$$\frac{\partial F_1}{\partial x} = -\frac{\partial F_2}{\partial y} \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$
Harmonic conjugate

Definition

If the function $F_1(x, y)$ is harmonic in a domain $D$, we can associate with it another function $F_2(x, y)$ by means of C-R equations. The function $F_2(x, y)$ defined by this equations is harmonic in $D$ and is called harmonic conjugate of $F_1(x, y)$.

It is clear that harmonic conjugate is unique up to the constant.

If on a simply connected domain $G$, with $0 \in G$, a harmonic function $F_1(x) \in \mathbb{R}$ is given, a harmonic conjugate is constructed by

$$F_2(x) = \int_0^x (-\partial_y F_1(x(s))\dot{x} + \partial_x F_1(x(s))\dot{y})\,ds$$

The result doesn't depend on the path of integration.
\( L_2(S^1) \) is a Hilbert space on the unit circle \( S^1 \subset \mathbb{C} \). Let \( \tilde{f}_1 \in L_2(S^1) \) and its Fourier expansion is

\[
\tilde{f}_1(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\theta) - b_n \sin(n\theta))
\]

Extend \( \tilde{f}_1 \) to a harmonic function \( f_1 \) on the unit disc \( D \subset \mathbb{C} \) by solving the Dirichlet problem. Let \( f_2 \), the harmonic conjugate of \( f_1 \), be fixed by taking \( f_2(0) = 0 \). Let \( \tilde{f}_2 \) denote the limit to the boundary \( S^1 \) of \( D \). Then

\[
\tilde{f}_2(\theta) = \sum_{n=1}^{\infty} (b_n \cos(n\theta) + a_n \sin(n\theta))
\]
Harmonic conjugate

$L_2(S^1; \mathbb{R}; \perp 1)$ is the linear subspace of all $\tilde{g} \in L_2(S^1)$ with $\int_0^{2\pi} \tilde{g}(\theta) d\theta = 0$

The operator

$$J : L_2(S^1; \mathbb{R}; \perp 1) : \tilde{f}_1 \mapsto J\tilde{f}_1 = \tilde{f}_2,$$

is orthogonal and skew-symmetric

$$J^* = -J = J^{-1}, J^2 = -I.$$

Note that $J\{\Re(a_n + ib_n)e^{in\theta}\} = \Re\{-i(a_n + ib_n)e^{in\theta}\}.$
Harmonic conjugate

The operator $N : \mathbb{L}_2(S^1; \mathbb{R}; \perp 1) \to \mathbb{L}_2(S^1; \mathbb{R}; \perp 1)$ is defined by

$$Nf_1 = \sum_{n=1}^{\infty} n\{b_n \cos(n\theta) + a_n \sin(n\theta)\}.$$ 

We have $N^* = N$, $J\partial_\theta = \partial_\theta J = N$ and therefore $\partial_\theta = -NJ$. 

For analytic functions $f(z)$ on the unit disc $D$ we will consider a splitting in real series on $S$. We put

$$f(e^{i\theta}) = \sum_{n=1}^{\infty} (a_n + ib_n)e^{in\theta} = f_1(e^{i\theta}) + if_2(e^{i\theta}) = f_1(e^{i\theta}) + iJf_1(e^{i\theta}).$$
On a simply connected domain $G \subset \mathbb{R}$, with $0 \in G$ we consider a biharmonic function $x \mapsto \phi(x)$. This means $\Delta \Delta \phi = 0$. Then there exist an analytic functions $\varphi, \chi : \mathbb{C} \mapsto \mathbb{C}$, such that

$$\phi(x) = \Re(\overline{z}\varphi(z) + \chi(z)), \quad z = x + iy$$
Thank You!