Modal Truncation Augmentation and Moment Matching:

two faces of the same coin.
Preliminary remarks

1. Reduction methods for i/o transfers VS. reduction for entire model

   *In structural dynamics typically interested in finding locations of maximum stresses or global mode shapes. So the aim is different than in control for instance.*

2. A reduction method must be

   - Robust \( \checkmark \)
   - Automatic \(?\)

   *Tuning free if possible, but not a black box!

   *If you know something about the specific problem you solve, use it!*

1. Modal Truncation, Mode acceleration correction, Truncation augmentation

2. Substructuring in Structural Dynamics

3. Craig-Bampton method

4. Craig-Bampton with Modal Truncation Augmentation
Mode Superposition

Linear dynamics

\[ M\ddot{q} + C\dot{q} + Kq = f \]

Free vibration modes

\[ x_r^T M x_s = \delta_{rs} \]
\[ x_r^T K x_s = \delta_{rs} \omega_s^2 \]

Small damping

\[ \beta_s = x_s^T C x_s \]

N decoupled oscillators

\[ \ddot{\eta}_s + \beta_s \dot{\eta}_s + \omega_s^2 \eta_s = x_s^T f(t) \]

Modal truncation

In practice, \( N \gg \) one can compute \( k \ll N \) modes

\[ q = \sum_{s=1}^{N} x_s \eta_s(t) + \sum_{r=k+1}^{N} x_r \eta_r(t) \]

Mode displacement approximation

- **Spatial** convergence: \( x_r^T f(t) \sim 0 \)
- **Spectrale** convergence: \( \omega_{\text{excit}} \ll \omega_r \)
Modal acceleration correction

Include contribution of higher modes to quasi-static solution $K^{-1} f(t)$

$$q = \sum_{s=1}^{k} x_s \eta_s(t) + \sum_{r=k+1}^{N} x_r \eta_r(t)$$

$$\ddot{\eta}_s + \beta_s \dot{\eta}_s + \omega_s^2 \eta_s = x_s^T f(t)$$

$$\omega_r^2 \eta_r = x_r^T f(t)$$

Mode Acceleration approximation

[Rayleigh (1877), *Theory of Sound*, chap V, § 100]
[Williams (1945)]

Better convergence than MDM if $\omega_{excit} \sim 0$
Modal acceleration correction (higher orders)

Other interpretation of the MAM:

\[ \mathbf{M} \ddot{\mathbf{q}} + \mathbf{K} \mathbf{q} = f \]

Relative solution

\[ \mathbf{q} = \mathbf{K}^{-1} f(t) + \mathbf{y} \]

\[ \mathbf{M} \ddot{\mathbf{y}} + \mathbf{K} \mathbf{y} = -\mathbf{M} \mathbf{K}^{-1} \ddot{f}(t) = f_y \]

Modal displacement on

\[ \mathbf{y} = \sum_{s=1}^{k} \mathbf{x}_s \alpha_s(t) \]
Modal acceleration correction (higher orders)

Define high order corrections

\[ \mathbf{x}_{\text{cor},j} = \left( \mathbf{K}^{-1} - \sum_{r=1}^{k} \frac{\mathbf{x}(r)\mathbf{x}^T(r)}{\omega_r^2} \right) \left( -\mathbf{M}\mathbf{K}^{-1} \right)^{j-1} \frac{\partial^2 f - 2 f}{\partial t^2} \]

\[ \mathbf{q} = \sum_{s=1}^{k} \mathbf{x}_s \eta_s(t) + \sum_{j=0}^{n-1} \mathbf{x}_{\text{cor},j} \]

Generalized Mode Accelerations

[ Leung (83): Force Derivative Method]
[ Akgün (93), [Liu et al. (94)]
[ Rixen (00)]

Modal truncation augmentation

Generalized MAM:

\[ \mathbf{q} = \sum_{s=1}^{k} \mathbf{x}_s \eta_s(t) + \sum_{j=0}^{n-1} \mathbf{x}_{\text{cor},j} \]

IDEA: Corrections = Ritz vectors

Compute correction for unit \( \frac{\partial^2 f - 2 f}{\partial t^2} \)

Amplitudes of \( \mathbf{x}_{\text{cor},j} \) are d.o.f

\[ \mathbf{x}_{\text{cor},j} = \left( \mathbf{K}^{-1} - \sum_{r=1}^{k} \frac{\mathbf{x}(r)\mathbf{x}^T(r)}{\omega_r^2} \right) \left( -\mathbf{M}\mathbf{K}^{-1} \right)^{j-1} f(t) \]
Modal truncation augmentation

\[ x_{cor,j} = K^{-1} - \sum_{r=1}^{k} \frac{x_r x_r^T}{\omega_r^2} \left( -MK^{-1} \right)^{j-1} f(t) \]

\[ q = \sum_{s=1}^{k} x_s \eta_s(t) + \sum_{j=0}^{n-1} x_{cor,j} \zeta_j(t) \]

**Augmentation by modal truncation**

[Dickens-Wilson (80)]
[Rixen(00)]

- Clear similarity with moments (around 0)
- Can be obtained “for free”
  if eigenmodes computed by a Krylov method, starting with the static solution

### Summary

**Mode Displacements MDM**

\[ q = \sum_{s=1}^{k} x_s \eta_s(t) \]

**Generalized MAM**

\[ q = \sum_{s=1}^{k} x_s \eta_s(t) + \sum_{j=0}^{n-1} x_{cor,j} \]

**Generalized MTA**

\[ q = \sum_{s=1}^{k} x_s \eta_s(t) + \sum_{j=0}^{n-1} x_{cor,j} \zeta_j(t) \]
Remarks

1. Modal Truncation, Mode acceleration correction, Truncation augmentation

- Modal truncation yields exact static response (and derivatives) matching moments at 0
- Can easily be generalized for quasi-static corrections around specific frequencies matching moments at other frequencies
- Significantly improves the truncated mode superposition (especially around expansion frequencies and in particular for the stress evaluation).
- Orthogonalization of the corrections (to improve numerical robustness):

\[
\begin{align*}
\left( X^T_{\text{cor}} K X_{\text{cor}} \right) Z &= \left( X^T_{\text{cor}} M X_{\text{cor}} \right) Z \Lambda \\
\text{then} \quad X_{\text{cor}} &= X_{\text{cor}} Z
\end{align*}
\]

\( \Lambda \) give an indication of the upper frequency range covered by the corrections

Fully decouples the reduced problem (i.e. correction modes can be handled as true modes)

Remarks

1. Modal Truncation, Mode acceleration correction, Truncation augmentation

- Why not replace all eigenmodes in the approximation basis by correction modes? Because we want the approximate solution to have the correct resonances!
- Modal Truncation Augmentation is useful for “a posteriori” reduction (i.e. if one can compute the modes of the full model and use the reduced model for transient/harmonic dynamic simulations)
- But how to reuse the same good ingredients for “a priori” reduction (i.e. to reduce the system without having to compute the eigenmodes and corrections on the full system)?
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Preliminary remarks

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2. Substructuring in Structural Dynamics
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Large models and how they are made

• Many elements needed to properly represent geometry
• Many elements used because model also for static analysis
• Different groups (companies) create model of different parts
• Companies (or departments in a company) do not want to share the full model of their subpart!

  - One starts from large (sub-)models
  - Not always possible to build the full model (because too big or because of confidentiality)

  A priori model reductions based on reduction of sub-parts
Idea of substructuring

Create “Super-Elements” or “Macro-elements”

- Easy to handle for FE codes
- Natural parallel procedure
- Hides the details of the subpart (confidentiality)
- Allows to localize modification during design
- Easy to reuse in other system where the same component is used

For a substructure, the interface DOFs are seen as inputs/outputs

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Feelings of a substructure

In each substructure \((s)\):

\[
M_{ii}^{(s)} \ddot{q}_i^{(s)} + K_{ii}^{(s)} q_i^{(s)} = -K_{ib}^{(s)} q_b^{(s)} - M_{ib}^{(s)} \dot{q}_b^{(s)}
\]

\(q_i^{(s)}\) computed by modal superposition of **fixed interface modes**:

\[
q_i^{(s)} = -K_{ii}^{(s)-1} K_{ib}^{(s)} q_b^{(s)} + \Phi^{(s)} \eta^{(s)}
\]

Craig-Bampton reduction space

\[
q = \begin{bmatrix} q_b^{(1)} \\ q_i^{(1)} \\ \vdots \\ q_i^{(N_s)} \end{bmatrix} \quad \begin{bmatrix} I & 0 & \cdots & 0 \\ \Phi^{(1)} & \Phi^{(1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi^{(N_s)} & 0 & \cdots & \Phi^{(N_s)} \end{bmatrix} \begin{bmatrix} q_b^{(1)} \\ \eta^{(1)} \\ \vdots \\ \eta^{(N_s)} \end{bmatrix}
\]

\[
\Psi^{(s)} = -K_{ii}^{(s)-1} K_{ib}^{(s)}
\]

Static modes (all)

\[
\left(K_{ii}^{(s)} - \omega_k^{(s)} M_{ii}^{(s)} \right) \phi_k^{(s)} = 0
\]

Vibration modes (fixed interface) \(k<N\)
Reduced matrices

\[
\bar{K}_{CB} = T^T_{CB} K T_{CB} = \begin{bmatrix}
S_{bb} & \Omega^{(1)^2} & 0 \\
0 & \ddots & \Omega^{(N_x)^2} \\
\end{bmatrix}
\]

\[
\bar{M}_{CB} = T^T_{CB} M T_{CB} = \begin{bmatrix}
M_{bb}^{(1)} & \ldots & M_{bb}^{(N_x)} \\
M_{b0}^{(1)} & \ldots & M_{b0}^{(N_x)} \\
0 & \ddots & I \\
\end{bmatrix}
\]

\(S_{bb} \quad M_{bb}^{+} : \text{assembly of statically condensed matrices}
\]

(Schur complement, Guyan)

Nice "diagonal" structure (Structure preserving?)

0 moment at 0 preserved, but can we do more?

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New feelings of a substructure

In each substructure $(s)$:

\[ M_{ii}^{(s)} \ddot{q}_i^{(s)} + K_{ii}^{(s)} q_i^{(s)} = -K_{ib}^{(s)} q_b^{(s)} - M_{ib}^{(s)} \ddot{q}_b^{(s)} \]

\[ q_i^{(s)} = -K_{ii}^{(s)-1} K_{ib}^{(s)} q_b^{(s)} + \tilde{q}_i^{(s)} \]

\[ M_{ii}^{(s)} \ddot{q}_i^{(s)} + K_{ii}^{(s)} q_i^{(s)} = \left( M_{ii}^{(s)} K_{ii}^{(s)-1} K_{ib}^{(s)} - M_{ib}^{(s)} \right) \ddot{q}_b^{(s)} = Y^{(s)} q_b^{(s)} \]

\[ \ddot{q}_i^{(s)} = K_{ii}^{(s)-1} Y^{(s)} q_b^{(s)} + z^{(s)} \]

4. Craig-Bampton with MTA
New feelings of a substructure

\[ q_i^{(s)} = \Psi^{(s)} q_b^{(s)} + \Phi^{(s)} \eta^{(s)} + X_{\text{cor}}^{(s)} \xi^{(s)} \]

### Static modes
- Make \( X_{\text{cor}}^{(s)} \) M- and K- orthogonal to \( \Phi^{(s)} \)

\[ x^{(s)}_{\text{cor},j} = \left( K_{ii}^{(s)} - \Phi^{(s)} \Omega^{(s)-2} \Phi^{(s)T} \right) \left( M_{ii}^{(s)} K_{ii}^{(s)} -1 \right)^{-1} Y^{(s)} \]

### Vibration modes
- Make columns of \( X_{\text{cor}}^{(s)} \) M- and K- orthogonal: solve the reduced eigenvalue problem

\[ \begin{bmatrix} X_{\text{cor}}^{(s)T} & Y^{(s)} \end{bmatrix} Z^{(s)} = \begin{bmatrix} M_{\text{cor}}^{(s)} X_{\text{cor}}^{(s)} & A^{(s)} \end{bmatrix} \]

### Projected high order static correction modes
- If too many interface DOFs, compute correction modes only for interface approximation modes \( X_{\text{b}}^{(s)} \) (e.g., [Bourquin 92])

\[ Y^{(s)} \leftrightarrow X_{\text{b}}^{(s)} \]

Prof. Antoulas would call this “reduction in tangential directions for the inputs/outputs”?!?
New reduced matrices

\[
\begin{bmatrix}
S_{bb} & \Omega^{(1)^2} & \Lambda^{(1)} & \cdots & 0 \\
0 & \Omega^{(N_s)^2} & \Lambda^{(N_s)} \\
M_{bb}^* & M_{b\phi}^{(1)} & M_{bX}^{(1)} & \cdots & M_{b\phi}^{(N_s)} & M_{bX}^{(N_s)} \\
M_{qb}^{(1)} & I & I & \cdots & I \\
M_{Xb} & I & I & \cdots & I \\
M_{qb}^{(N_s)} & 0 & I & \cdots & I \\
M_{Xb}^{(N_s)} & 0 & I & \cdots & I \\
\end{bmatrix}
\]

Easy to implement in existing Craig-Bampton routines

Example

Truss frame with beams

Assume number of internal modes per substructure fixed

Relative eigenfrequency error

\[
\Delta \omega_r = \frac{\omega_{r} - \tilde{\omega}_r}{\omega_r}
\]

Relative force residual of modes

\[
\Delta f = \frac{\| (K_{\alpha} - \tilde{\omega}_r M_{\alpha}) \tilde{\phi}_r \|_2}{\| K_{\alpha} \tilde{\phi}_r \|_2}
\]

690 dof / substructure
### Example

**50 internal modes per subdomain**

10 interface modes for $X_{cor}^{(s)}$

<table>
<thead>
<tr>
<th>Mode number</th>
<th>$\Delta \omega$</th>
<th>$\Delta f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$10^0$</td>
<td>$10^0$</td>
</tr>
<tr>
<td>10</td>
<td>$10^{-5}$</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>20</td>
<td>$10^{-10}$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>30</td>
<td>$10^{-15}$</td>
<td>$10^{-8}$</td>
</tr>
</tbody>
</table>

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### Future possibilities

- What is the right mix of internal modes and corrections modes?
- Define strategies to optimally choose the expansion points for correction modes (IRKA?)
- Extend the to multi-physical problems usually non-symmetric (structure preserving ideas?)
- Substructuring for parametric (sub-)models (sampling strategies, parametric matching …)

### Conclusion

- Engineers would be well inspired to look at mathematical research in model reduction
- Mathematicians might get novel ideas for real problems from structural engineers
- More workshops like this are needed to stimulate both communities