Lie Algebras Generated by Extremal Elements

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Joint Work

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- J.C.H.W. In ’t Panhuis
- D.A. Roozemond
1 Problem and Preliminaries
Problem Statement

Find a nice presentation of all sufficiently big classical simple Lie algebras using geometric properties.

In particular, find a result of the following type:

**Goal**

Let $\mathcal{L}$ be a sufficiently big classical simple Lie algebra over a field $\mathbb{F}$ of characteristic not two. We find a “structure” $\Gamma_{\mathcal{L}}$ such that:

$\Rightarrow$ $\mathcal{L}$ is generated by elements with $\Gamma_{\mathcal{L}}$-structure;

$\Leftarrow$ if a Lie algebra is generated by elements with $\Gamma_{\mathcal{L}}$-structure, then it is almost always isomorphic to $\mathcal{L}$. 
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Extremal Elements and Root Elements

- Classical simple Lie algebras $\mathcal{L}$ have a Cartan subalgebra and root system.

- (Long) root elements $x$ satisfy:

$$\forall y \in \mathcal{L}: [x, [x, y]] \in \mathbb{F}x. \quad (1)$$
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- A multiple of an extremal element is again extremal.
Example: Two Extremal Elements

- $x$ and $y$ are extremal; consider $\langle x, y \rangle =: \mathcal{L}$.
- If $[x, y] = \alpha x + \beta y$, then
  
  $$
  [x, [x, y]] = [x, \alpha x + \beta y] = \alpha \beta x + \beta^2 y,
  $$

  so $\beta = 0$. Similarly $\alpha = 0$.
- $\mathcal{L}$ is two-dimensional $\iff [x, y] = 0$.
- Otherwise, $[x, [x, y]] \in \mathbb{F}x$ and $[y, [x, y]] = -[y, [y, x]] \in \mathbb{F}y$,
  so $\mathcal{L}$ is three-dimensional.
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The Extremal Form

For extremal \( x \in \mathcal{L} \), all \( a \in \mathcal{L} \) give a \( \lambda \) with \( [x, [x, a]] = \lambda x \); let \( f_x \in \mathcal{L}^* \) be such that

\[
[x, [x, a]] = f_x(a)x.
\]

- \( f_x(x) = 0 \).
- \( f_x(y) = f_y(x) \).
- If \( \mathcal{L} \) is generated by extremal elements, then it is spanned by extremal elements. Then the form can be extended to a (symmetric) bilinear form \( f(x, y) = f_x(y) \).
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or $\mathcal{L}$ is three-dimensional;

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    f([x, y], [x, y]) &= f(x, y)^2.
\end{align*}
\]

Either $f(x, y) = 0$; then $\mathcal{L}$ is the so-called Heisenberg algebra,

or we can make $f(x, y) = 2$ by rescaling $x$; then

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\begin{align*}
    x &\rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
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determines an isomorphism with $\mathfrak{sl}_2$.

In a sense, the generic case is $\mathfrak{sl}_2$. 

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E.J. Postma

Lie Algebras Generated by Extremal Elements
The geometrical structure we will find is a set of generating extremal elements, where we specify which pairs of elements commute and which don’t (necessarily).
Generators for the Classical Algebras
Definitions

- Take an $n$-dimensional vector space $V$.
- The special Lie algebra $\mathfrak{sl}_n$ consists of the trace 0 matrices. Multiplication is the commutator.
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Extremal Elements in $\mathfrak{sl}_n$

- Let $v \in V$ and $h \in V^*$ with $h(v) = 0$. Define

$$v \otimes h : V \rightarrow V, \quad u \mapsto h(u)v.$$  

- $v \otimes h \in \mathfrak{sl}_n$. We call $I + v \otimes h$ a transvection and $v \otimes h$ an infinitesimal transvection (if $h(v) = 0$).

- The infinitesimal transvections span $\mathfrak{sl}_n$.

- Infinitesimal transvections are extremal elements of $\mathfrak{sl}_n$:

$$[u \otimes g, [u \otimes g, v \otimes h]](w) = -2g(v)h(u)(u \otimes g)(w),$$

if $g(u) = h(v) = 0$, so $f(u \otimes g, v \otimes h) = -2g(v)h(u)$. 
Let $v \in V$ and $h \in V^*$ with $h(v) = 0$. Define

$$v \otimes h : V \to V, \ u \mapsto h(u)v.$$ 

Matrix for a suitable basis:

$$\begin{pmatrix}
0 & \ldots & 0 & 1 \\
0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
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Lie Algebras Generated by Extremal Elements
Geometric Configurations

\[ f(u \otimes g, v \otimes h) = -2h(u)g(v) \]

What geometrical setups do you get with a pair of infinitesimal transvections, \( v \otimes h \) and \( u \otimes g \)?
Geometric Configurations

\[ f(u \otimes g, v \otimes h) = -2h(u)g(v) \]

<table>
<thead>
<tr>
<th>( h(u) \neq 0 )</th>
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[u \otimes g, v \otimes h](w) = \\
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\]
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\[ [u \otimes g, v \otimes h](w) = g(v)(u \otimes h)(w) \]
Problem and Preliminaries
Generators for the Classical Algebras
Each Similarly Structured Algebra is Classical

Summary

Geometric Configurations

\[ f(u \otimes g, v \otimes h) = -2h(u)g(v) \]

\[ [u \otimes g, v \otimes h] = g(v)(u \otimes h). \]

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The last is also the case if two points or planes coincide.
A Concrete Realization

Let \( \{e_1, \ldots, e_n\} \) be a basis of \( V \) and \( \{f_1, \ldots, f_n\} \) its dual basis.

Let

\[
\begin{align*}
v_1 &= e_1 - e_2 \\
v_i &= e_{i-1} + e_i \\
h_1 &= f_1 + f_2, \\
h_i &= f_{i-1} - f_i \\
&\text{for } 1 < i \leq n.
\end{align*}
\]

Then \( \mathfrak{sl}_n = \langle v_i \otimes h_i \mid 1 \leq i \leq n \rangle \).
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  v_1 &= e_1 - e_2 & h_1 &= f_1 + f_2,
\end{align*}
\]

for \( 1 < i \leq n \).

Then \( \mathfrak{sl}_n = \langle v_i \otimes h_i \mid 1 \leq i \leq n \rangle \).

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Generators for the Classical Algebras
Each Similarly Structured Algebra is Classical
Summary

A Concrete Realization

- Let \( \{e_1, \ldots, e_n\} \) be a basis of \( V \) and \( \{f_1, \ldots, f_n\} \) its dual basis.
- Let
  
  \[
  v_1 = e_1 - e_2 \quad h_1 = f_1 + f_2, \\
  v_i = e_{i-1} + e_i \quad h_i = f_{i-1} - f_i \quad \text{for } 1 < i \leq n.
  \]

- Then \( \mathfrak{sl}_n = \langle v_i \otimes h_i \mid 1 \leq i \leq n \rangle \).
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Proof (1)

A group \( \langle I + \lambda v \otimes h \mid \lambda \in F \rangle \) is called a transvection group. It is contained in \( \text{SL}(V) \), because \( \det v_i \otimes h_i = 0 \). Transvection groups are isomorphic to \( F^+ \), because

\[
(I + \lambda v \otimes h)(I + \mu v \otimes h) = I + (\lambda + \mu)v \otimes h.
\]

Theorem (McLaughlin, 1967)

If a group \( H \) acting on \( V \) is generated by transvection subgroups, where \( V \) is spanned by an \( H \)-orbit of centres of these transvection subgroups and \( V^* \) is spanned by the axes, then either \( H = \text{SL}(V) \) or \( H = \text{Sp}(V, B) \).

Let \( G = \langle I + \lambda v_i \otimes h_i \mid \lambda \in F, 1 \leq i \leq n \rangle \). \( G \) satisfies the criteria, so either \( G = \text{SL}(V) \) or \( G = \text{Sp}(V, B) \).
A group \( \langle I + \lambda v \otimes h \mid \lambda \in \mathbb{F} \rangle \) is called a **transvection group**. It is contained in \( \text{SL}(V) \), because \( \det v_i \otimes h_i = 0 \). Transvection groups are isomorphic to \( \mathbb{F}^+ \), because

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Problem and Preliminaries
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Proof (1)

A group \(<I + \lambda \nu \otimes h \mid \lambda \in F>\) is called a transvection group. It is contained in \(SL(V)\), because \(\det \nu_i \otimes h_i = 0\). Transvection groups are isomorphic to \(F^+\), because

\[(I + \lambda \nu \otimes h)(I + \mu \nu \otimes h) = I + (\lambda + \mu) \nu \otimes h.\]

Theorem (McLaughlin, 1967)

If a group \(H\) acting on \(V\) is generated by transvection subgroups, where \(V\) is spanned by an \(H\)-orbit of centres of these transvection subgroups and \(V^*\) is spanned by the axes, then either \(H = SL(V)\) or \(H = Sp(V, B)\).

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Proof (2): $G \neq \text{Sp}(V)$

- All transvections in $\text{Sp}(V)$ are of shape $I + v \otimes h$ where $h(w) = B(v, w)$.
- If $I + u \otimes g, I + v \otimes h \in \text{Sp}(V)$ and $h(u) = B(v, u) = 0$, then also $g(v) = B(u, v) = 0$.
- So the situation where $h(u) = 0 \neq g(v)$ (Heisenberg) does not occur.
- $[v_2 \otimes h_2, v_1 \otimes h_1 + 4v_3 \otimes h_3 \pm 2[v_1 \otimes h_1, v_3 \otimes h_3]]$ are a pair of infinitesimal transvections in that situation.
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Proof (3)

\[ G = \langle I + \lambda v_i \otimes h_i \mid \lambda \in \mathbb{F}, 1 \leq i \leq n \rangle = \text{SL}(V). \]
\[ \langle v_i \otimes h_i \rangle = T_I(\text{SL}(V)) = \mathfrak{sl}_n. \]
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Other Classical Lie Algebras

\[ A_n \]

1 3 \( \ldots \) n n + 1

\[ 2 \]
Other Classical Lie Algebras

\[ A_n \]

\[
\begin{array}{cccc}
1 & & & n \\
2 & & & n + 1 \\
3 & & & \\
\end{array}
\]

\[ B_n \]

\[
\begin{array}{cccc}
1 & & & n - 2 \\
2 & & & n - 1 \\
3 & & & n \\
\end{array}
\]

\[ \text{Lie Algebras Generated by Extremal Elements} \]
Other Classical Lie Algebras

- $A_n$: $1 \rightarrow 3 \rightarrow \ldots \rightarrow n \rightarrow n+1$
- $B_n$: $1 \rightarrow 3 \rightarrow \ldots \rightarrow n-2 \rightarrow n-1 \rightarrow n \rightarrow n+1$
- $C_n$: $1 \rightarrow 2 \rightarrow \ldots \rightarrow 2n-1 \rightarrow 2n$
Other Classical Lie Algebras

- $A_n$: \[ \begin{array}{ccccccc}
1 & 3 & \ldots & n & n+1 \\
2 & & & & & &
\end{array} \]

- $B_n$: \[ \begin{array}{ccccccc}
1 & 3 & \ldots & n-2 & n-1 & n \\
2 & & & & & n+1 & &
\end{array} \]

- $C_n$: \[ \begin{array}{ccccccc}
1 & 2 & \ldots & 2n-1 & 2n \\
& & & & & &
\end{array} \]

- $D_n$: \[ \begin{array}{ccccccc}
1 & 3 & \ldots & n-3 & n-2 & n-1 \\
2 & & & & & n & &
\end{array} \]
Each Similarly Structured Algebra is Classical
Goal

- With two extremal elements: of the three-dimensional Lie algebras, the “generic” one is $\mathfrak{sl}_2$.
- Of the maximal-dimensional Lie algebras generated by $n$ extremal elements where the unconnected ones commute, the “generic” one is $\mathfrak{sl}_n$.
- Find a sort of presentation.
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- Find a sort of presentation.

\[ A : \begin{array}{c}
  1 \\
  2 \\
  3 \\
  \vdots \\
  n - 1 \\
  n 
\end{array} \]
Towards a Presentation

- For a presentation, we need some sort of free object and some defining relations to divide out.
- The free object will be the free Lie algebra $\mathcal{F}$ over $\mathbb{F}$ on $n$ generators $x_1, \ldots, x_n$.
- Every Lie algebra generated by $n$ elements is a quotient of $\mathcal{F}$.
Towards a Presentation

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  \[ \{1, \ldots, n\} \]

  \[ \mathcal{F} \quad \text{\rightarrow} \quad \mathcal{L} \]

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Towards a Presentation

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\[
\{1, \ldots, n\} \quad \xrightarrow{\mathcal{F}} \quad L
\]

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Lie Algebras Generated by Extremal Elements
We need to find an ideal that can be divided out, then after that $x_i$ are extremal elements and some of them commute.

Commutation relations: easy, put $[x_i, x_j]$ in the ideal whenever $x_i$ and $x_j$ need to commute.

Extremal elements: depends on extremal form! Divide out $[x, [x, y]] - f(x, y)x$ – but what is $f(x, y)$?
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Extremal elements: depends on extremal form! Divide out \([x, [x, y]] - f(x, y)x\) – but what is \( f(x, y) \)?
Solution

- For a given graph $\Gamma$ (ex.: $A$) on $n$ vertices, let $\mathcal{F}_\Gamma$ be the quotient of $\mathcal{F}$ by the ideal generated by $[x_i, x_j]$ whenever $x_i$ and $x_j$ are unconnected.

- Take $f_i \in \mathcal{F}_\Gamma^*$, the dual of $\mathcal{F}_\Gamma$. This will determine the “extremal form”. Let $f = (f_1, \ldots, f_n)$ and

$$I_{\Gamma, f} = \langle [x_i, [x_i, y]] - f_i(y)x_i \mid y \in \mathcal{F}_\Gamma \rangle.$$ 

$$L_{\Gamma, f} = \mathcal{F}_\Gamma / I_{\Gamma, f}.$$ 

- The images of $x_i$ in $L_{\Gamma, f}$ are extremal with $f(x_i, y) = f_i(y)$.
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Obtain an Upper and Lower Bound

- Zel’manov and Kostrikin [1990]: “complete graph” (no commutation relations) and each $f_i = 0$, then finite-dimensional.

- Cohen, Steinbach, Ushirobira, Wales [2001]: any other $f$, then every basis for $f = 0$ turns into a spanning set (dimension may go down).

- Hence find a finite list $M_\Gamma$ of monomials in $x_i$ that form a basis of $L_{\Gamma,0}$. This gives an upper bound for the dimension of $L_{\Gamma,f}$.

- The preceding section, with the explicit realization of $L_{\Gamma,f}$, gave a lower bound for a certain $f$.

- In the case of the classical Lie algebras, these upper and lower bounds are the same.
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Example Monomial List

Example for $\mathcal{A}$: all monomials of the following shapes:

- $[x_k, [x_{k-1}, \ldots, [x_{m+1}, x_m]]]$ for $1 \leq m \leq k \leq n$,
- $[x_k, [x_{k-1}, \ldots, [x_3, x_1]]]$ for $3 \leq k \leq n$,
- $[x_k, [x_{k-1}, \ldots, [x_{m+1}, [x_{m-1}, [x_m, [x_{m-2}, [x_{m-1}, \ldots, [x_2, [x_3, x_1] \ldots ]]]]]]]$ for $3 \leq m \leq k \leq n$. 

![Diagram of a graph with nodes 1, 2, 3, 4, 5, 6 connected in a specific pattern.]

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Lie Algebras Generated by Extremal Elements
Example Monomial List

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Diagram:

```
1 -- 3 -- 4
|     |
|     |
2 -- 5 -- 6
```
The Extremal Form

- For some values of $f$, we will find $\dim \mathcal{L}_\Gamma, f = \dim \mathcal{L}_\Gamma, 0$; for others, the dimension will be smaller.

- It turns out that those $f$ for which the dimension is maximal, form an algebraic variety $X_\Gamma$ in $(\mathcal{F}_\Gamma^*)^n$.

- For the (sufficiently large) classical Lie algebras, we show that for a generic choice of $f \in X_\Gamma$ (meaning: outside a certain closed subset) $\mathcal{L}_\Gamma, f$ is isomorphic to that Lie algebra: given a suitable $f$, we modify the realization we found earlier to have a corresponding extremal form.
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- It turns out that those $f$ for which the dimension is maximal, form an algebraic variety $X_{\Gamma}$ in $(\mathcal{F}_{\Gamma}^*)^n$.
- For the (sufficiently large) classical Lie algebras, we show that for a generic choice of $f \in X_{\Gamma}$ (meaning: outside a certain closed subset) $\mathcal{L}_{\Gamma, f}$ is isomorphic to that Lie algebra: given a suitable $f$, we modify the realization we found earlier to have a corresponding extremal form.
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Summary

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