Brauer algebras of type C

Shoumin Liu
joint work with Arjeh M. Cohen and Shona H. Yu

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Twisted Coxeter group of type $C_n$ from type $A_{2n-1}$

Brauer algebra of type $A$ and some results

The homomorphism $\phi$ from $\text{Br}(C_n)$ to $\text{Br}(A_{2n-1})$

The surjectivity of $\phi$

The injectivity of $\phi$
Definition for $W(A_m)$

The Coxeter group $W(A_m)$ has generators $R_1, \cdots, R_m$ and defining relations:

\[
\begin{align*}
R_i^2 &= 1 \\
R_i R_{i+1} R_i &= R_{i+1} R_i R_{i+1} \\
R_i R_j &= R_j R_i \text{ for } |i - j| \neq 1
\end{align*}
\]
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R_i R_j = R_j R_i \text{ for } |i-j| \neq 1
\]

$W(A_m) \cong \text{Sym}_{m+1}$, $R_i \leftrightarrow (i, i + 1)$
The Coxeter group \( W(C_n) \) has generators \( r_0, r_1, \cdots, r_{n-1} \) and defining relations:

\[
\begin{align*}
    r_i^2 &= 1 \\
    r_i r_{i+1} r_i &= r_{i+1} r_i r_{i+1} & i > 0, \\
    r_0 r_1 r_0 r_1 &= r_1 r_0 r_1 r_0 \\
    r_i r_j &= r_j r_i, \text{ for } |i - j| \neq 1
\end{align*}
\]

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root system for $W(A_m)$

$\Phi$: The root system of the Coxeter group $W(A_m)$
$\Phi = \{ \epsilon_i - \epsilon_j \mid 1 \leq i, j \leq m+1, \ i \neq j \} \subset \mathbb{R}^{m+1}$
$\{ \alpha_i \mid \alpha_i := \epsilon_i - \epsilon_{i+1} \}_{i=1}^{m}$: simple roots of $\Phi$.
$\Phi^+ = \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq m+1 \}$: positive roots.
$\Phi = \Phi^+ \cup (-\Phi^+)$
$\mathcal{A}$: all mutually orthogonal roots subsets of $\Phi^+$. 
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The set $\mathcal{A}$

Example

\[ B = \{ \alpha_1 + \alpha_2, \alpha_2 + \alpha_3 + \alpha_4 \} = \{ \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_5 \} \in \mathcal{A} \]
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**Action $\sigma$ on $\Phi$**

$$\sigma(\epsilon_i) = -\epsilon_{2n-i}$$
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The twisted group $W(C_n)$

$p : \mathbb{R}^{2n} \to \mathbb{R}^{2n}_\sigma$ be the orthogonal projection onto $\sigma$-invariant subspace,
$p(x) = (x + \sigma(x))/2$
The image $\Psi = p(\Phi)$ of $\Phi$ under $p$ is a root system of type $C_n$
with simple roots $\beta_0 = p(\alpha_n)$ and $\beta_i = p(\alpha_{n-i})$ for $i = 1, \ldots, n - 1$. 

\[
\begin{array}{ccccccc}
\beta_0 & | & & & & \beta_1 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
\end{array}
\]
Admissible root sets

\( \mathcal{B}' : \) the collection of all sets of mutually orthogonal roots in \( \Psi^+ \)

\( p : A_\sigma \rightarrow \mathcal{B}' \)

\( p(B) = \{ p(\alpha) \mid \alpha \in B \} \) for \( B \in A_\sigma \)

\( X \in \mathcal{B}' \text{ admissible} : \) lies in the image of \( p \)

\( \mathcal{B} : \) the set of all admissible elements of \( \mathcal{B}' \)
Embed $W(C_n)$ in to $W(A_{2n-1})$

The map $\phi$ from the generators of $W(C_n)$ to $W(A_{2n-1})$ below:

\[
\phi(r_0) = R_n \\
\phi(r_i) = R_{n-i}R_{n+i}
\]

induces an injective homomorphism.

\[
W(C_n) \simeq \phi(W(C_n)) = W(A_{2n-1})_{\sigma}.
\]
\[
\sigma(R_i) = R_{2n-i}
\]
Definition of $\text{Br}(A_m)$

**Definition**

$Q$: a graph, Brauer monoid $\text{BrM}(Q)$

Generators: $R_i$ and $E_i$, for each node $i$ of $Q$ and $\delta, \delta^{-1}$

Relations: the following + Coxeter relations of $R_i$s with $\sim$, ...
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Definition

$$\delta\delta^{-1} = 1$$
$$R_iE_i = E_iR_i = E_i$$
$$E_i^2 = \delta E_i$$
$$E_iR_j = R_jE_i, \text{ for } i \sim j$$
$$E_iE_j = E_jE_i, \text{ for } i \sim j$$
$$R_jR_iE_j = E_iE_j, \text{ for } i \sim j$$
$$R_iE_jR_i = R_jE_iR_j, \text{ for } i \sim j$$

The Brauer algebra $Br(Q)$ is the free $\mathbb{Z}$-algebra for Brauer monoid $BrM(Q)$. 
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Diagram for $BrM(A_m)$

The monomials of $Br(A_m)$ was discovered by Richard Brauer in 1937 for the invariant theory of orthogonal group. Brauer gives a diagram description by diagram monoid with $2m + 2$ dots and $m + 1$ strands.

Example

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Example

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Brauer algebras of type $C$
Involuntary automorphism determined by its behavior on the generators

\[ \sigma(R_i) = R_{2n-i}, \quad \sigma(E_i) = E_{2n-i}. \]

A monomial in \( \text{BrM}(A_{2n-1}) \) is fixed by \( \sigma \) if and only if it represents a symmetric diagram.
The linear span \( \text{SBr}(A_{2n-1}) \) of all \( \sigma \)-fixed monomials is a subalgebra of \( \text{Br}(A_{2n-1}) \).
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Example

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Brauer algebras of type $C$
Question 1: Can we define the subalgebra with generators and relations not depending on $\text{Br}(A_{2n-1})$?

Question 2: Are $R_n, E_n, R_i R_{2n-i}$ and $E_i E_{2n-i}$ for $i = 1, \ldots, n-1$, generators of $S\text{Br}(A_{2n-1})$?
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**Definition**

For \( n \in \mathbb{N} \), the Brauer algebra of type \( C_n \), denoted by \( \text{Br}(C_n) \), is a unital associative \( \mathbb{Z}[\delta^{\pm 1}] \)-algebra generated by \( \{r_i, e_i\}_{i=0}^{n-1} \) subject to the following relations and relations for \( r_i \text{s in } W(C_n) \).

\[
\begin{align*}
\delta \delta^{-1} &= 1 \\
\delta x &= x \delta \text{ for each generator } x \\
r_i e_i &= e_i r_i = e_i \text{ for any } i \\
e_i^2 &= \delta^2 e_i \text{ for } i > 0 \\
e_0^2 &= \delta e_0 
\end{align*}
\]
Definition of $\text{Br}(C_n)$

\begin{align*}
    r_ir_j &= r_jr_i, \text{ for } i \not\sim j \\
    e_ir_j &= r_je_i, \text{ for } i \not\sim j \\
    e_ie_j &= e_je_i, \text{ for } i \not\sim j \\
    e_1r_0e_1 &= \delta e_1 \\
    r_je_i &= e_ie_j, \text{ for } i \not\sim j \text{ with } i, j > 0 \\
    r_1r_0e_1 &= r_0e_1
\end{align*}
Definition of $\text{Br}(C_n)$

Definition

\[
\begin{align*}
    r_i e_j r_i &= r_j e_i r_j, \text{ for } i \sim j \text{ with } i, j > 0 \\
e_1 e_0 e_1 &= \delta e_1 \\
e_1 r_0 r_1 &= e_1 r_0 \\
e_1 e_0 r_1 &= e_1 e_0 \\
r_1 e_0 r_1 e_0 &= e_0 e_1 e_0 \\
(r_1 r_0 r_1)e_0 &= e_0(r_1 r_0 r_1)
\end{align*}
\]

The submonoid of the multiplicative monoid of $\text{Br}(C_n)$ generated by $\delta$, $\{r_i\}_{i=0}^{n-1}$ and $\{e_i\}_{i=0}^{n-1}$ is denoted by $\text{BrM}(C_n)$. This is the monoid of monomials in $\text{Br}(C_n)$.

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Brauer algebras of type C
Main Theorem

**Theorem**

There exists an algebra isomorphism

\[ \phi : \text{Br}(C_n) \longrightarrow \text{SBr}(A_{2n-1}) \]

determined by \( \phi(r_0) = R_n, \phi(r_i) = R_{n-i}R_{n+i}, \phi(e_0) = E_n, \) and \( \phi(e_i) = E_{n-i}E_{n+i}, \) for \( 0 < i < n. \) In particular, the algebra \( \text{Br}(C_n) \) is free over \( \mathbb{Z}[\delta^{\pm 1}] \) of rank \( a_{2n}, \) where \( a_n \) is defined by \( a_0 = a_1 = 1 \) and the recursion

\[ a_n = a_{n-1} + 2(n-1)a_{n-2}. \]
Closed formula for $a_{2n}$

A closed formula for the rank of $\text{Br}(C_n)$ is

$$a_{2n} = \sum_{i=0}^{n} \left( \sum_{p+2q=i} \frac{n!}{p!q!(n-i)!} \right)^2 2^{n-i} (n-i)!.$$

Here we give a table for some $a_n$ for some small $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>25</td>
<td>81</td>
<td>331</td>
<td>1303</td>
<td>5937</td>
</tr>
</tbody>
</table>
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Diagram explanation for two formulas

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Brauer algebras of type $C$
Generalized $E$

For each $\alpha \in \Phi^+$ and $\alpha = R \alpha_i$, we can define

$$E_\alpha = RE_i R^{-1}.$$  

For $X \in \mathcal{A}$ consists of mutually orthogonal roots, we can define

$$E_X = \prod_{\alpha \in X} E_\alpha,$$

$$\hat{E}_X = \delta^{-|X|} E_X.$$
A monoid action of $\text{BrM}(A_m)$ on $\mathcal{A}$

Generators $R_i$’s just act on $\mathcal{A}$ as an Coxeter group element. The action of $E_i$’s are defined in the following.

$$E_iB = \begin{cases} B & \text{if } \alpha_i \in B, \\ B \cup \{\alpha_i\} & \text{if } \alpha_i \perp B, \\ R_\beta R_iB & \text{if } \beta \in B \setminus \alpha_i^\perp. \end{cases}$$

Alternatively, this action can be described by diagram multiplication.
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Example

$$e_1 \alpha_2 = \alpha_1$$
Height Function

**Definition**

The height of an admissible set $B$ is the minimal number of crossings needed in a completion of the top corresponding to $B$ to a Brauer diagram without increasing the number of horizontal strands at the top.

**Example**

The height of $\{\epsilon_5 - \epsilon_1, \epsilon_4 - \epsilon_2\}$ is 2.
**Rewritten form**

**Definition**

For every element $a \in \text{Br}_{M}(A_{m})$, we define the height of $a$, denoted by $\text{ht}(a)$, as the minimal number of generators $R_{i}$ needed to write $a$ as a word in the generators $R_{1}, \ldots, R_{m}, E_{1}, \ldots, E_{m}$ and $\delta, \delta^{-1}$.

**Lemma**

*Suppose $B \in A$ has height 0. Then there are $r = m - 2|B|$ monomials of height 1 in the group of invertible elements in $\hat{E}_{B}\text{Br}_{M}(A_{m})\hat{E}_{B}$ forming a Coxeter system of type $A_{r}$, denoted by $K_{B}$.***
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Rewritten form

**Definition**

Suppose $m = 2n - 1$. For $i \in \{0, \ldots, n\}$ we let $Y_i \in \mathcal{A}$ be the following set of nodes of size $i$:

$$Y_i = \begin{cases} 
\{n, n \pm 2, n \pm 4, \ldots, n \pm (i - 1)/2\} & \text{if } i \equiv 1 \pmod{2} \\
\{n \pm 1, n \pm 3, \ldots, n \pm i/2\} & \text{if } i \equiv 0 \pmod{2}
\end{cases}$$

The corresponding set of positive roots $\{\alpha_y \mid y \in Y\}$ is denoted by $B_i$.
So, $B_i \in \mathcal{A}_\sigma$.
We will also write $E^{(i)} = \hat{E}_{B_i}$ and $K_i = K_{B_i}$. 
Rewritten form

Theorem

For each \( i \in \{0, 1, \ldots, \lfloor m/2 \rfloor \} \), select an admissible set \( B_i \) of size \( i \). Then each element \( a \) of \( \text{BrM}(A_m) \) can be written uniquely as

\[
\delta^k U E_{B_i} V W
\]

for certain \( k \in \mathbb{Z}, i \in \{0, \ldots, \lfloor m/2 \rfloor \}, U \in \text{BrM}(A_{2n-1}) \hat{E}_{B_i} \) with \( UB_i = a\emptyset \), \( W \in \hat{E}_{B_i} \text{BrM}(A_{2n-1}) \) with \( \emptyset a = B_i W \), and \( V \in K_{B_i} \) such that

\[
\text{ht}(a) = \text{ht}(U) + \text{ht}(V) + \text{ht}(W).
\]
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Example for rewriting

Example

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Brauer algebras of type $C$
By the above theorem and several small lemmas, we can get the homomorphism $\phi : \text{Br}(C_n) \to S\text{Br}(A_{2n-1})$ is surjective.
Basic admissible root sets

For $0 \leq p \leq i \leq n$ and $i - p$ even, the admissible root set $B_{i,p}$ corresponds to the following strands.

$\begin{align*}
1 & \quad n-i+1 & \quad n-i+2 \\
n-p+1 & \quad n & \quad n+1 & \quad n+p \\
n+i-1 & \quad n+i & \quad 2n
\end{align*}$
The homomorphism $\phi$ from $\operatorname{Br}(C_n)$ to $\operatorname{Br}(A_{2n-1})$

The surjectivity of $\phi$

The injectivity of $\phi$

$e_{i,p,p'} \in \operatorname{Br}M(C_n)$: whose image under $\phi$ having

its top corresponding to $B_{i,p}$

its the bottom corresponding to $B_{i,p'}$

Example

e_{5,3,1}
Some notation

\[ N_{i,p} = \{ a \in W(C_n) \mid ae_{i,p,i} = e_{i,p,i}a \} \]

\[ A_{i,p} = \{ a \in W(C_n) \mid ae_{i,p,i} = e_{i,p,i} \} \triangleleft N_{i,p} \]

\[ D_{i,p} \text{ the left coset of } N_{i,p} \text{ in } W(C_n) \]

\[ \exists \text{ a subgroup } \hat{K}_i \subset W(C_n) \text{ s.t. } N_{i,p} = A_{i,p} \rtimes \hat{K}_i \]
Rewritten forms in \( \text{BrM}(C_n) \)

**Theorem**

Each element \( a \) in the monoid \( \text{BrM}(C_n) \), there exist \( k \in \mathbb{Z} \) and \( i, p, p' \in \{0, \ldots, n \} \) are such that \( i - p \) and \( i - p' \) are even, such that

\[
a \in D_{i,p}e_{i,p,p'}K_iD_{i,p}'^{op}.
\]

In particular, \( \text{Br}(C_n) \) is free of rank \( a_{2n} \).

Hence we construct a cellular structure on \( \text{Br}(C_n) \) making it be a cellular algebra(Graham+Lehrer).
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Thanks!