Preprocessing and Inprocessing

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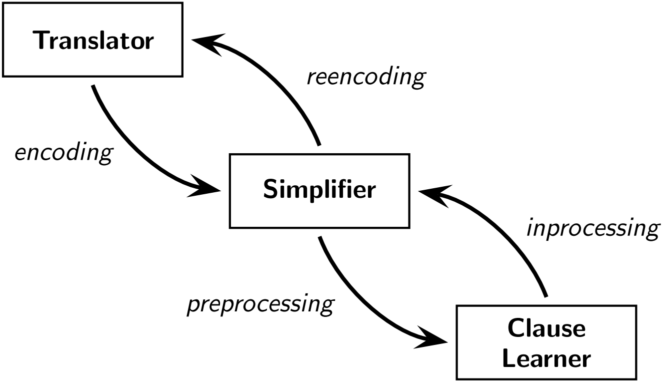
Interaction between different solving approaches



Translator

reencoding

encoding

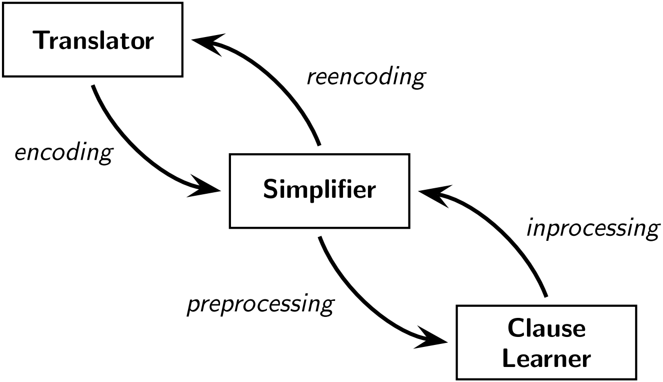
Simplifier

inprocessing

preprocessing

Clause Learner

Interaction between different solving approaches



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It all comes down to adding and removing redundant clauses

Redundant clauses

A clause is redundant with respect to a formula if adding it to the formula preserves satisfiability.

- For unsatisfiable formulas, all clauses can be added, including the empty clause $\bot$.

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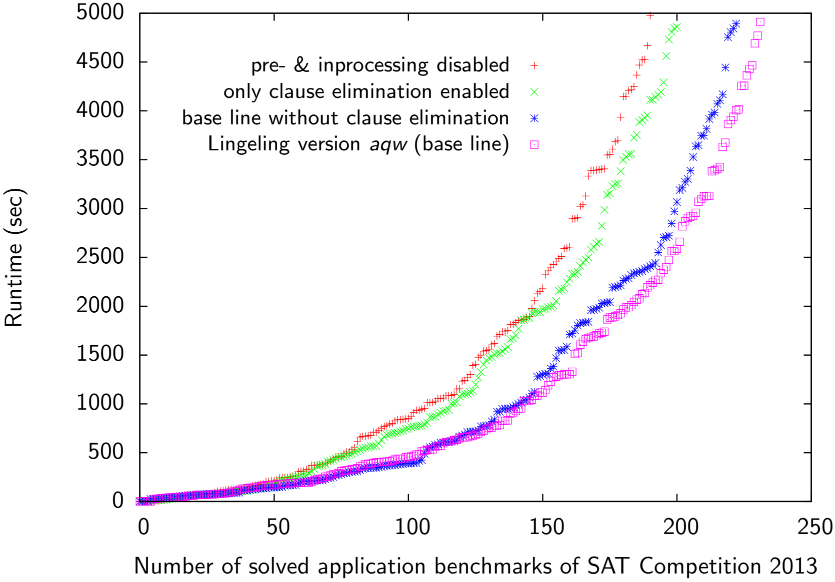
Challenge regarding redundant clauses:

- How to check redundancy in polynomial time?
- Ideally find redundant clauses in linear time

Preprocessing and Inprocessing in Practice



Outline

Subsumption

Variable Elimination

Bounded Variable Addition

Blocked Clause Elimination

Hyper Binary Resolution

Unhiding Redundancy

Subsumption

Tautologies and Subsumption

Definition (Tautology)

A clause C is a tautology if its contains two complementary literals $x, \bar{x}$.

Example

The clause $(a \vee b \vee \bar{b})$ *is a tautology.*

Definition (Subsumption)

Clause C subsumes clause D if and only if $C \subset D$.

Example

The clause $(a \vee b)$ *subsumes clause* $(a \vee b \vee \bar{c})$*.*

Self-Subsuming Resolution

Self-Subsuming Resolution

$$\frac{C \vee x \qquad D \vee \bar{x}}{D} \quad C \subseteq D \quad \frac{(a \vee b \vee x) \qquad (a \vee b \vee c \vee \bar{x})}{(a \vee b \vee c)}$$

resolvent D subsumes second antecedent $D \vee \bar{x}$

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Example

Assume a CNF contains both antecedents

$\ldots (a \vee b \vee x)(a \vee b \vee c \vee \bar{x}) \ldots$

If D is added, then $D \vee \bar{x}$ can be removed

which in essence removes $\bar{x}$ from $D \vee \bar{x}$

$\ldots (a \vee b \vee x)(a \vee b \vee c \quad) \ldots$

Initially in the SATeLite preprocessor, [EenBiere'07]
now common in most solvers (i.e., as pre- and inprocessing)

Self-Subsuming Example

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resolvent D subsumes second antecedent $D \vee \bar{x}$

Example: Remove literals using self-subsumption

$$\begin{array}{lcl}
(a \vee b \vee c) & \wedge & (\bar{a} \vee b \vee c) \wedge \\
(\bar{a} \vee b \vee \bar{c}) & \wedge & (a \vee \bar{b} \vee c) \wedge \\
(\bar{a} \vee \bar{b} \vee d) & \wedge & (\bar{a} \vee \bar{b} \vee \bar{d}) \wedge \\
(a \vee \bar{c} \vee d) & \wedge & (a \vee \bar{c} \vee \bar{d})
\end{array}$$

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Self-Subsuming Example

Self-Subsuming Resolution

$$\frac{C \lor x \qquad D \lor \bar{x}}{D} \quad C \subseteq D \quad \frac{(a \lor b \lor x) \qquad (a \lor b \lor c \lor \bar{x})}{(a \lor b \lor c)}$$

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(\bar{a} \vee & \bar{b} & \ ) & \wedge & (\ \bar{a} \vee \bar{b} \vee \bar{d}\ ) & \wedge \\
(a \vee & \bar{c} & \vee\ d\ ) & \wedge & (\ a \vee \bar{c} \vee \bar{d}\ ) &
\end{array}$$

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$$\frac{C \lor x \qquad D \lor \bar{x}}{D} \quad C \subseteq D \quad \frac{(a \lor b \lor x) \qquad (a \lor b \lor c \lor \bar{x})}{(a \lor b \lor c)}$$

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(\bar{a}\quad\quad) & \wedge & (\,a \quad\quad) \wedge \\
(\bar{a} \vee \bar{b}\quad) & \wedge & (\,\bar{a} \vee \bar{b} \vee \bar{d}\,) \wedge \\
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(& \bar{a} \vee \bar{b} & &) & \wedge & (\bar{a} \vee \bar{b} \vee \bar{d}) & \wedge \\
(& a \vee \bar{c} & &) & \wedge & (a \vee \bar{c} \vee \bar{d}) &
\end{array}$$

Implementing Subsumtion

Definition (Subsumption)

Clause C subsumes clause D if and only if $C \subset D$.

Example

The clause $(a \vee b)$ *subsumes clause* $(a \vee b \vee \bar{c})$*.*

Forward Subsumption

```
for each clause $C$ in formula $F$ do
  if $C$ is subsumed by a clause $D$ in $F \setminus C$ then
    remove $C$ from $F$
```

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The clause $(a \lor b)$ *subsumes clause* $(a \lor b \lor \bar{c})$.

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```
for each clause $C$ in formula $F$ do
  if $C$ is subsumed by a clause $D$ in $F \setminus C$ then
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```

Backward Subsumption

```
for each clause $C$ in formula $F$ do
  remove all clauses $D$ in $F$ that are subsumed by $C$
```

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Definition (Subsumption)

Clause C subsumes clause D if and only if $C \subset D$.

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Forward Subsumption

```
for each clause $C$ in formula $F$ do
    if $C$ is subsumed by a clause $D$ in $F \setminus C$ then
        remove $C$ from $F$
```

Backward Subsumption

```
for each clause $C$ in formula $F$ do
    pick a literal $x$ in $C$
    remove all clauses $D$ in $F_x$ that are subsumed by $C$
```

Variable Elimination

Variable Elimination [DavisPutnam'60]

Definition (Resolution)

Given two clauses $C = (x \vee a_1 \vee \cdots \vee a_i)$ and $D = (\bar{x} \vee b_1 \vee \cdots \vee b_j)$, the *resolvent* of C and D on variable x (denoted by $C \otimes_x D$) is $(a_1 \vee \cdots \vee a_i \vee b_1 \vee \cdots \vee b_j)$

Resolution on sets of clauses F_x and $F_{\bar{x}}$ (denoted by $F_x \otimes_x F_{\bar{x}}$) generates all non-tautological resolvents of $C \in F_x$ and $D \in F_{\bar{x}}$.

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Definition (Variable elimination (VE))

Given a CNF formula F, *variable elimination* (or DP resolution) removes a variable x by replacing F_x and $F_{\bar{x}}$ by $F_x \otimes_x F_{\bar{x}}$

Variable Elimination [DavisPutnam'60]

Definition (Resolution)

Given two clauses $C = (x \lor a_1 \lor \cdots \lor a_i)$ and $D = (\bar{x} \lor b_1 \lor \cdots \lor b_j)$, the *resolvent* of C and D on variable x (denoted by $C \otimes_x D$) is $(a_1 \lor \cdots \lor a_i \lor b_1 \lor \cdots \lor b_j)$

Resolution on sets of clauses F_x and $F_{\bar{x}}$ (denoted by $F_x \otimes_x F_{\bar{x}}$) generates all non-tautological resolvents of $C \in F_x$ and $D \in F_{\bar{x}}$.

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Proof procedure [DavisPutnam60]

VE is a complete proof procedure. Applying VE until fixpoint results in either the empty formula (satisfiable) or empty clause (unsatisfiable)

Example VE by clause distribution [DavisPutnam'60]

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Example of clause distribution

$$
\begin{array}{c|ccc}
 & \multicolumn{3}{c}{\overbrace{\hspace{22em}}^{F_x}} \\
 & (x \vee c) & (x \vee \bar{d}) & (x \vee \bar{a} \vee \bar{b}) \\
\hline
F_{\bar{x}} \left\{ \begin{array}{c} (\bar{x} \vee a) \\ (\bar{x} \vee b) \\ (\bar{x} \vee \bar{e} \vee f) \end{array} \right. &
\begin{array}{c} (a \vee c) \\ (b \vee c) \\ (c \vee \bar{e} \vee f) \end{array} &
\begin{array}{c} (a \vee d) \\ (b \vee d) \\ (d \vee \bar{e} \vee f) \end{array} &
\begin{array}{c} (a \vee \bar{a} \vee \bar{b}) \\ (b \vee \bar{a} \vee \bar{b}) \\ (\bar{a} \vee \bar{b} \vee \bar{e} \vee f) \end{array}
\end{array}
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Example of clause distribution

			F_x	
		$(x \lor c)$	$(x \lor \bar{d})$	$(x \lor \bar{a} \lor \bar{b})$
$F_{\bar{x}}$	$(\bar{x} \lor a)$	$(a \lor c)$	$(a \lor d)$	~~$(a \lor \bar{a} \lor \bar{b})$~~
	$(\bar{x} \lor b)$	$(b \lor c)$	$(b \lor d)$	~~$(b \lor \bar{a} \lor \bar{b})$~~
	$(\bar{x} \lor \bar{e} \lor f)$	$(c \lor \bar{e} \lor f)$	$(d \lor \bar{e} \lor f)$	$(\bar{a} \lor \bar{b} \lor \bar{e} \lor f)$

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$F_{\bar{x}}$	$(\bar{x} \vee a)$	$(a \vee c)$	$(a \vee d)$	~~$(a \vee \bar{a} \vee \bar{b})$~~
	$(\bar{x} \vee b)$	$(b \vee c)$	$(b \vee d)$	~~$(b \vee \bar{a} \vee \bar{b})$~~
	$(\bar{x} \vee \bar{e} \vee f)$	$(c \vee \bar{e} \vee f)$	$(d \vee \bar{e} \vee f)$	$(\bar{a} \vee \bar{b} \vee \bar{e} \vee f)$

In the example: $|F_x \otimes F_{\bar{x}}| > |F_x| + |F_{\bar{x}}|$

Exponential growth of clauses in general

VE by substitution [EenBiere07]

General idea

Detect gates (or definitions) $x = \mathrm{GATE}(a_1, \ldots, a_n)$ in the formula and use them to reduce the number of added clauses

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Possible gates

gate	G_x	$G_{\bar{x}}$
$\mathrm{AND}(a_1, \ldots, a_n)$	$(x \vee \bar{a}_1 \vee \cdots \vee \bar{a}_n)$	$(\bar{x} \vee a_1), \ldots, (\bar{x} \vee a_n)$
$\mathrm{OR}(a_1, \ldots, a_n)$	$(x \vee \bar{a}_1), \ldots, (x \vee \bar{a}_n)$	$(\bar{x} \vee a_1 \vee \cdots \vee a_n)$
$\mathrm{ITE}(c, t, f)$	$(x \vee \bar{c} \vee \bar{t}), (x \vee c \vee \bar{f})$	$(\bar{x} \vee \bar{c} \vee t), (\bar{x} \vee c \vee f)$

VE by substitution [EenBiere07]

General idea

Detect gates (or definitions) $x = \mathrm{GATE}(a_1, \ldots, a_n)$ in the formula and use them to reduce the number of added clauses

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$\mathrm{ITE}(c, t, f)$	$(x \vee \bar{c} \vee \bar{t}), (x \vee c \vee \bar{f})$	$(\bar{x} \vee \bar{c} \vee t), (\bar{x} \vee c \vee f)$

Variable elimination by substitution [EenBiere07]

Let $R_x = F_x \setminus G_x$; $R_{\bar{x}} = F_{\bar{x}} \setminus G_{\bar{x}}$.

Replace $F_x \wedge F_{\bar{x}}$ by $G_x \otimes_x R_{\bar{x}} \wedge G_{\bar{x}} \otimes_x R_x$.
Always less than $F_x \otimes_x F_{\bar{x}}$!

VE by substitution [EenBiere'07]

Example of gate extraction: $x = \mathrm{AND}(a, b)$

$$F_x = (x \vee c) \wedge (x \vee \bar{d}) \wedge (x \vee \bar{a} \vee \bar{b})$$
$$F_{\bar{x}} = (\bar{x} \vee a) \wedge (\bar{x} \vee b) \wedge (\bar{x} \vee \bar{e} \vee f)$$

VE by substitution [EenBiere'07]

Example of gate extraction: $x = \mathrm{AND}(a, b)$

$$\begin{aligned} F_x &= (x \lor c) \land (x \lor \bar{d}) \land (x \lor \bar{a} \lor \bar{b}) \\ F_{\bar{x}} &= (\bar{x} \lor a) \land (\bar{x} \lor b) \land (\bar{x} \lor \bar{e} \lor f) \end{aligned}$$

Example of substitution

		R_x		G_x
		$(x \lor c)$	$(x \lor \bar{d})$	$(x \lor \bar{a} \lor \bar{b})$
$G_{\bar{x}}$	$(\bar{x} \lor a)$	$(a \lor c)$	$(a \lor d)$	
	$(\bar{x} \lor b)$	$(b \lor c)$	$(b \lor d)$	
$R_{\bar{x}}$	$(\bar{x} \lor \bar{e} \lor f)$			$(\bar{a} \lor \bar{b} \lor \bar{e} \lor f)$

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$R_{\bar{x}}$	$(\bar{x} \vee \bar{e} \vee f)$			$(\bar{a} \vee \bar{b} \vee \bar{e} \vee f)$

using substitution: $|F_x \otimes F_{\bar{x}}| < |F_x| + |F_{\bar{x}}|$

Bounded Variable Addition

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Main Idea

Given a CNF formula F, can we construct a (semi)logically equivalent F' by introducing a new variable $x \notin VAR(F)$ such that $|F'| < |F|$?

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Reverse of Variable Elimination

For example, replace the clauses

$$\begin{array}{ccc} (a \vee c) & (a \vee d) & \\ (b \vee c) & (b \vee d) & \\ (c \vee \bar{e} \vee f) & (d \vee \bar{e} \vee f) & (\bar{a} \vee \bar{b} \vee \bar{e} \vee f) \end{array}$$

by

$$\begin{array}{ccc} (\bar{x} \vee a) & (\bar{x} \vee b) & (\bar{x} \vee \bar{e} \vee f) \\ (x \vee c) & (x \vee d) & (x \vee \bar{a} \vee \bar{b}) \end{array}$$

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Main Idea

Given a CNF formula F, can we construct a (semi)logically equivalent F' by introducing a new variable $x \notin VAR(F)$ such that $|F'| < |F|$?

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$$\begin{array}{ccc} (a \vee c) & (a \vee d) & \\ (b \vee c) & (b \vee d) & \\ (c \vee \bar{e} \vee f) & (d \vee \bar{e} \vee f) & (\bar{a} \vee \bar{b} \vee \bar{e} \vee f) \end{array}$$

by

$$\begin{array}{ccc} (\bar{x} \vee a) & (\bar{x} \vee b) & (\bar{x} \vee \bar{e} \vee f) \\ (x \vee c) & (x \vee d) & (x \vee \bar{a} \vee \bar{b}) \end{array}$$

Challenge: how to find suitable patterns for replacement?

Factoring Out Subclauses

Example

Replace

$$(a \vee b \vee c \vee d) \quad (a \vee b \vee c \vee e) \quad (a \vee b \vee c \vee f)$$

by

$$(x \vee d) \quad (x \vee e) \quad (x \vee f) \quad (\bar{x} \vee a \vee b \vee c)$$

adds 1 variable and 1 clause *reduces* number of literals by 2

Not compatible with VE, which would eliminate x immediately!

. . . *so this does not work* . . .

Bounded Variable Addition

Example

Smallest pattern that is compatible: Replace

$$\begin{array}{cc} (a \vee d) & (a \vee e) \\ (b \vee d) & (b \vee e) \\ (c \vee d) & (c \vee e) \end{array}$$

by

$$\begin{array}{ccc} (\bar{x} \vee a) & (\bar{x} \vee b) & (\bar{x} \vee c) \\ (x \vee d) & (x \vee e) & \end{array}$$

adds 1 variable

removes 1 clause

Bounded Variable Addition

Possible Patterns

$$\begin{matrix} (X_1 \vee L_1) & \dots & (X_1 \vee L_k) \\ \vdots & & \vdots \\ (X_n \vee L_1) & \dots & (X_n \vee L_k) \end{matrix} \quad \equiv \quad \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{k} (X_i \vee L_j)$$

replaced by $\bigwedge_{i=1}^{n} (y \vee X_i) \wedge \bigwedge_{j=1}^{k} (\bar{y} \vee L_j)$

- Every k clauses share sets of literals L_j
- There are n sets of literals X_i that appear in clauses with L_j

Bounded Variable Addition

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Bounded Variable Addition

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Bounded Variable Addition on AtMostOneZero (1)

Example encoding of AtMostOneZero $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(x_1 \lor x_2) \land (x_9 \lor x_{10}) \land (x_8 \lor x_{10}) \land (x_7 \lor x_{10}) \land (x_6 \lor x_{10}) \land \\
(x_1 \lor x_3) \land (x_2 \lor x_3) \land (x_8 \lor x_9) \land (x_7 \lor x_9) \land (x_6 \lor x_9) \land \\
(x_1 \lor x_4) \land (x_2 \lor x_4) \land (x_3 \lor x_4) \land (x_7 \lor x_8) \land (x_6 \lor x_8) \land \\
(x_1 \lor x_5) \land (x_2 \lor x_5) \land (x_3 \lor x_5) \land (x_4 \lor x_5) \land (x_6 \lor x_7) \land \\
(x_1 \lor x_6) \land (x_2 \lor x_6) \land (x_3 \lor x_6) \land (x_4 \lor x_6) \land (x_5 \lor x_6) \land \\
(x_1 \lor x_7) \land (x_2 \lor x_7) \land (x_3 \lor x_7) \land (x_4 \lor x_7) \land (x_5 \lor x_7) \land \\
(x_1 \lor x_8) \land (x_2 \lor x_8) \land (x_3 \lor x_8) \land (x_4 \lor x_8) \land (x_5 \lor x_8) \land \\
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(x_1 \lor x_{10}) \land (x_2 \lor x_{10}) \land (x_3 \lor x_{10}) \land (x_4 \lor x_{10}) \land (x_5 \lor x_{10})
\end{array}
$$

Bounded Variable Addition on AtMostOneZero (1)

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&(x_1 \vee x_5) \wedge (x_2 \vee x_5) \wedge (x_3 \vee x_5) \wedge (x_4 \vee x_5) \wedge (x_6 \vee x_7) \wedge \\
&(x_1 \vee x_6) \wedge (x_2 \vee x_6) \wedge (x_3 \vee x_6) \wedge (x_4 \vee x_6) \wedge (x_5 \vee x_6) \wedge \\
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\end{aligned}
$$

Replace $(x_i \vee x_j)$ with $i \in \{1..5\}, j \in \{6..10\}$ by $(x_i \vee y)$, $(x_j \vee \bar{y})$

Bounded Variable Addition on AtMostOneZero (2)

Example encoding of AtMostOneZero $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(x_1 \vee x_2) \wedge (x_9 \vee x_{10}) \wedge (x_8 \vee x_{10}) \wedge (x_7 \vee x_{10}) \wedge (x_6 \vee x_{10}) \wedge \\
(x_1 \vee x_3) \wedge (x_2 \vee x_3) \wedge (x_8 \vee x_9) \wedge (x_7 \vee x_9) \wedge (x_6 \vee x_9) \wedge \\
(x_1 \vee x_4) \wedge (x_2 \vee x_4) \wedge (x_3 \vee x_4) \wedge (x_7 \vee x_8) \wedge (x_6 \vee x_8) \wedge \\
(x_1 \vee x_5) \wedge (x_2 \vee x_5) \wedge (x_3 \vee x_5) \wedge (x_4 \vee x_5) \wedge (x_6 \vee x_7) \wedge \\
(x_1 \vee y) \wedge (x_2 \vee y) \wedge (x_3 \vee y) \wedge (x_4 \vee y) \wedge (x_5 \vee y) \wedge \\
(x_6 \vee \bar{y}) \wedge (x_7 \vee \bar{y}) \wedge (x_8 \vee \bar{y}) \wedge (x_9 \vee \bar{y}) \wedge (x_{10} \vee \bar{y})
\end{array}
$$

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(x_1 \vee x_5) \wedge (x_2 \vee x_5) \wedge (x_3 \vee x_5) \wedge (x_4 \vee x_5) \wedge (x_6 \vee x_7) \wedge \\
(x_1 \vee y) \wedge (x_2 \vee y) \wedge (x_3 \vee y) \wedge (x_4 \vee y) \wedge (x_5 \vee y) \wedge \\
(x_6 \vee \bar{y}) \wedge (x_7 \vee \bar{y}) \wedge (x_8 \vee \bar{y}) \wedge (x_9 \vee \bar{y}) \wedge (x_{10} \vee \bar{y})
\end{array}
$$

Replace matched pattern

$$
\begin{array}{l}
(x_1 \vee z) \wedge (x_2 \vee z) \wedge (x_3 \vee z) \wedge \\
(x_4 \vee \bar{z}) \wedge (x_5 \vee \bar{z}) \wedge (y \vee \bar{z})
\end{array}
$$

Bounded Variable Addition on AtMostOneZero (3)

Example encoding of AtMostOneZero $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(x_1 \vee x_2) \wedge (x_9 \vee x_{10}) \wedge (x_8 \vee x_{10}) \wedge (x_7 \vee x_{10}) \wedge (x_6 \vee x_{10}) \wedge \\
(x_1 \vee x_3) \wedge (x_2 \vee x_3) \wedge (x_8 \vee x_9) \wedge (x_7 \vee x_9) \wedge (x_6 \vee x_9) \wedge \\
(x_1 \vee z) \wedge (x_2 \vee z) \wedge (x_3 \vee z) \wedge (x_7 \vee x_8) \wedge (x_6 \vee x_8) \wedge \\
(x_4 \vee \bar{z}) \wedge (x_5 \vee \bar{z}) \wedge (y \vee \bar{z}) \wedge (x_4 \vee x_5) \wedge (x_6 \vee x_7) \wedge \\
(x_4 \vee y) \wedge (x_5 \vee y) \wedge (x_6 \vee \bar{y}) \wedge (x_7 \vee \bar{y}) \wedge (x_8 \vee \bar{y}) \\
(x_9 \vee \bar{y}) \wedge (x_{10} \vee \bar{y})
\end{array}
$$

Bounded Variable Addition on AtMostOneZero (3)

Example encoding of AtMostOneZero $(x_1, x_2, \ldots, x_n)$

$$
\begin{aligned}
&(x_1 \vee x_2) \wedge (x_9 \vee x_{10}) \wedge (x_8 \vee x_{10}) \wedge (x_7 \vee x_{10}) \wedge (x_6 \vee x_{10}) \wedge \\
&(x_1 \vee x_3) \wedge (x_2 \vee x_3) \wedge (x_8 \vee x_9) \wedge (x_7 \vee x_9) \wedge (x_6 \vee x_9) \wedge \\
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&(x_4 \vee y) \wedge (x_5 \vee y) \wedge (x_6 \vee \bar{y}) \wedge (x_7 \vee \bar{y}) \wedge (x_8 \vee \bar{y}) \\
&(x_9 \vee \bar{y}) \wedge (x_{10} \vee \bar{y})
\end{aligned}
$$

Replace matched pattern

$$
\begin{aligned}
&(x_6 \vee w) \wedge (x_7 \vee w) \wedge (x_8 \vee w) \wedge \\
&(x_9 \vee \bar{w}) \wedge (x_{10} \vee \bar{w}) \wedge (\bar{y} \vee \bar{w})
\end{aligned}
$$

Blocked Clause Elimination

Blocked Clauses [Kullmann'99]

Definition (Blocking literal)

A literal x in a clause C of a CNF F blocks C w.r.t. F if for every clause $D \in F_{\bar{x}}$, the resolvent $(C \setminus \{x\}) \cup (D \setminus \{\bar{x}\})$ obtained from resolving C and D on l is a tautology.

With respect to a fixed CNF and its clauses we have:

Definition (Blocked clause)

A clause is blocked if it contains a literal that blocks it.

Blocked Clauses [Kullmann'99]

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Example

Consider the formula $(a \vee b) \wedge (a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee c)$.
First clause is not blocked.
Second clause is blocked by both a *and* $\bar{c}$.
Third clause is blocked by c

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First clause is not blocked.
Second clause is blocked by both a *and* $\bar{c}$.
Third clause is blocked by c

Proposition

Removal of an arbitrary blocked clause preserves satisfiability.

Blocked Clause Elimination (BCE)

Definition (BCE)

While there is a blocked clause C in a CNF F, remove C from F.

Example

Consider $(a \vee b) \wedge (a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee c)$.
After removing either $(a \vee \bar{b} \vee \bar{c})$ *or* $(\bar{a} \vee c)$, *the clause* $(a \vee b)$ *becomes blocked (no clause with either* $\bar{b}$ *or* $\bar{a}$*).*

An extreme case in which BCE *removes all clauses!*

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An extreme case in which BCE *removes all clauses!*

Proposition

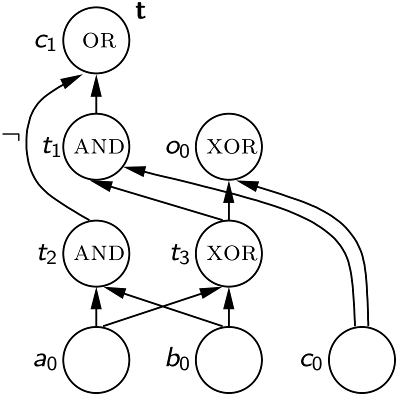
BCE is confluent, i.e., has a unique fixpoint

- Blocked clauses stay blocked w.r.t. removal

BCE very effective on circuits [JärvisaloBiereHeule'10]

BCE converts the Tseitin encoding to Plaisted Greenbaum
BCE simulates Pure literal elimination, Cone of influence, etc.

Example of circuit simplification by BCE on Tseitin encoding



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$(c_1 \vee \bar{t}_1)$
$(c_1 \vee t_2)$

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$(o_0 \vee t_3 \vee \bar{c}_0)$
$(o_0 \vee \bar{t}_3 \vee c_0)$

$(t_1 \vee \bar{t}_3 \vee \bar{c}_0)$
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$(\bar{t}_1 \vee c_0)$

$(t_2 \vee \bar{a}_0 \vee \bar{b}_0)$
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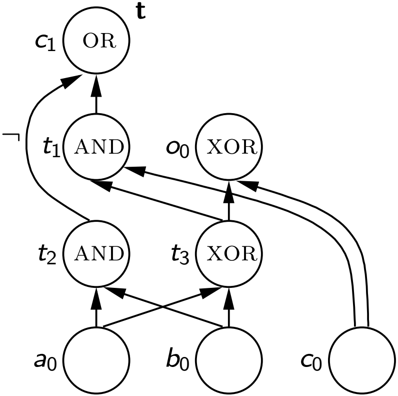

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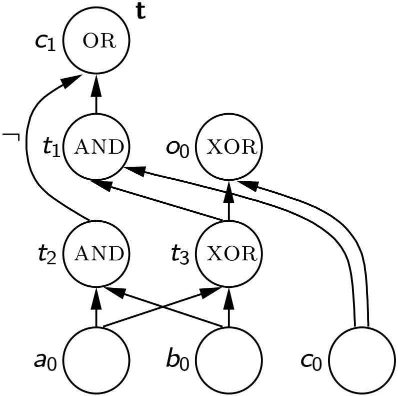

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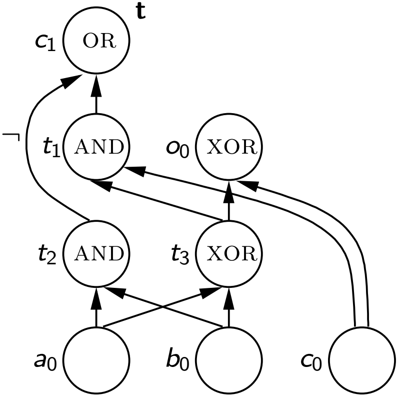

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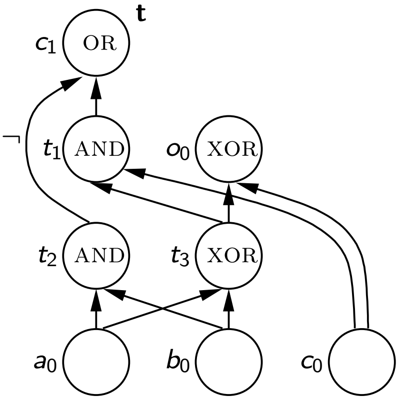

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$(\bar{t}_1 \lor t_3)$
$(\bar{t}_1 \lor c_0)$

$(t_2 \lor \bar{a}_0 \lor \bar{b}_0)$
~~$(\bar{t}_2 \lor a_0)$~~
~~$(\bar{t}_2 \lor b_0)$~~

$(\bar{t}_3 \lor a_0 \lor b_0)$
$(\bar{t}_3 \lor \bar{a}_0 \lor \bar{b}_0)$
$(t_3 \lor a_0 \lor \bar{b}_0)$
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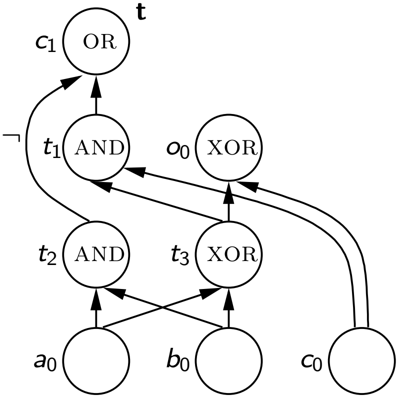

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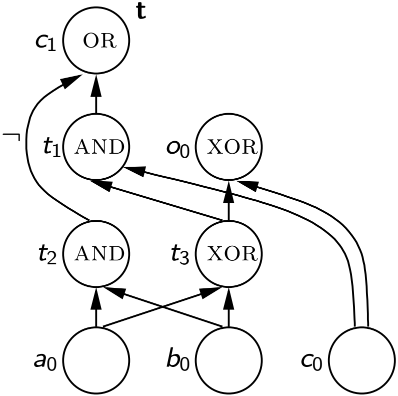

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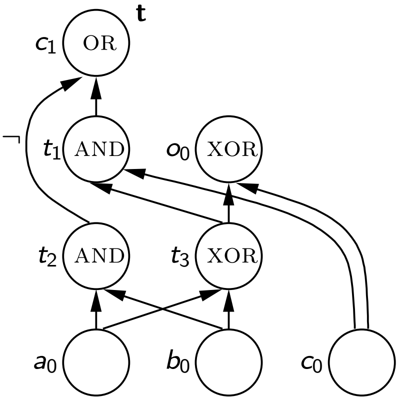

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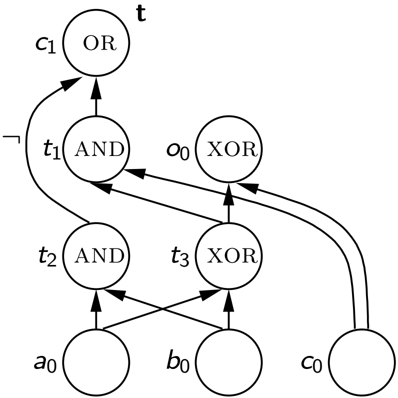

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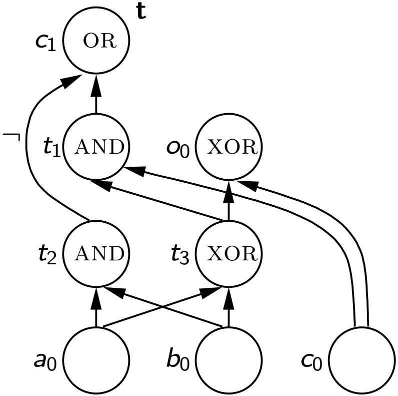

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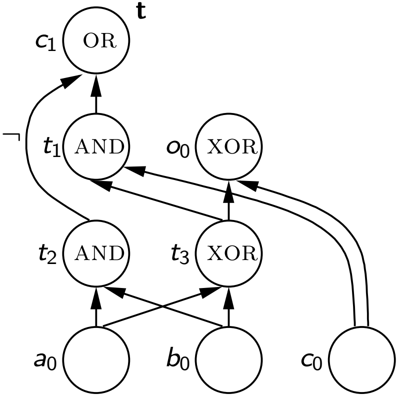

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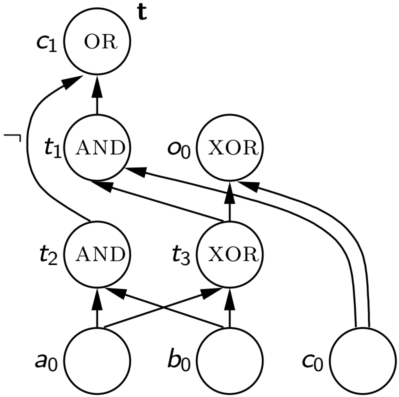
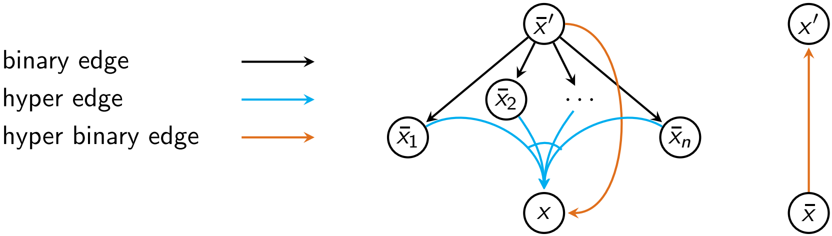


Hyper Binary Resolution

Hyper Binary Resolution [Bacchus-AAAI02]

Definition (Hyper Binary Resolution Rule)

$$\frac{(x \vee x_1 \vee x_2 \vee \cdots \vee x_n) \quad (\bar{x}_1 \vee x') \quad (\bar{x}_2 \vee x') \quad \dots \quad (\bar{x}_n \vee x')}{(l \vee l')}$$



Hyper Binary Resolution Rule:

- combines multiple resolution steps into one
- uses one n-ary clauses and multiple binary clauses
- special case *hyper unary resolution* where $x = x'$

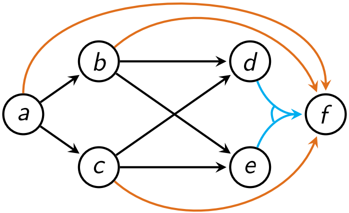
Hyper Binary Resolution (HBR)

Definition (Hyper Binary Resolution)

Apply the hyper binary resolution rule until fixpoint

Example

Consider
$(\bar{a} \vee b) \wedge (\bar{a} \vee c) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge (\bar{c} \vee d) \wedge (\bar{c} \vee e) \wedge (\bar{d} \vee \bar{e} \vee f)$.



hyper binary resolvents:
$(\bar{a} \vee f), (\bar{b} \vee f), (\bar{c} \vee f)$

HBR is confluent, i.e., has a unique fixpoint

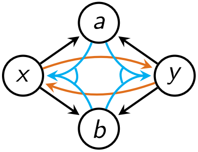
Structural Hashing of AND-gates via HBR

gate g	$g \Rightarrow f(g_1, \dots g_n)$ “positive”	$g \Leftarrow f(g_1, \dots g_n)$ “negative”
$g := \text{OR}(g_1, \dots, g_n)$	$(\bar{g} \vee g_1 \vee \dots \vee g_n)$	$(g \vee \bar{g_1}),\dots,(g \vee \bar{g_n})$
$g := \text{AND}(g_1, \dots, g_n)$	$(\bar{g} \vee g_1),\dots,(\bar{g} \vee g_n)$	$(g \vee \bar{g_1} \vee \dots \vee \bar{g_n})$
$g := \text{XOR}(g_1, g_2)$	$(\bar{g} \vee \bar{g_1} \vee \bar{g_2})$, $(\bar{g} \vee g_1 \vee g_2)$	$(g \vee \bar{g_1} \vee g_2)$, $(g \vee g_1 \vee \bar{g_2})$
$g := \text{ITE}(g_1, g_2, g_3)$	$(\bar{g} \vee \bar{g_1} \vee g_2)$, $(\bar{g} \vee g_1 \vee g_3)$	$(g \vee \bar{g_1} \vee \bar{g_2})$, $(g \vee g_1 \vee \bar{g_3})$

Definition (Structural Hashing of AND-gates)

Given a Boolean circuit with two equivalent gates, merge the gates.

Example



$$x = \text{AND(a,b)} : (\bar{x} \vee a) \wedge (\bar{x} \vee b) \wedge (x \vee \bar{a} \vee \bar{b})$$

$$y = \text{AND(a,b)} : (\bar{y} \vee a) \wedge (\bar{y} \vee b) \wedge (y \vee \bar{a} \vee \bar{b})$$

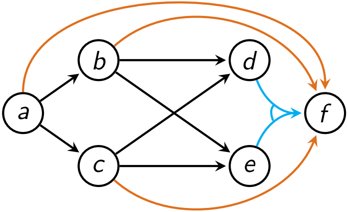
the two HBRs $(\bar{x} \vee y)$ and $(x \vee \bar{y})$ express that $x = y$

Non-transitive Hyper Binary Resolution (NHBR)

A problem with classic HBR is that it adds many transitive binary clauses

Example

Consider
$(\bar{a} \vee b) \wedge (\bar{a} \vee c) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge (\bar{c} \vee d) \wedge (\bar{c} \vee e) \wedge (\bar{d} \vee \bar{e} \vee f)$.



adding $(\bar{b} \vee f)$ or $(\bar{c} \vee f)$
makes $(\bar{a} \vee f)$ transitive

Solution [HeuleJärvisaloBiere 2013]

Add only non-transitive hyper binary resolvents
Can be implemented using an alternative unit propagation style

Space Complexity of NHBR: Quadratic

Question regarding complexity [Biere 2009]

- Are there formulas where the transitively reduced hyper binary resolution closure is quadratic in size w.r.t. to the size of the original?
- where size $=$ #clauses or size $=$ #literals or size $=$ #variables

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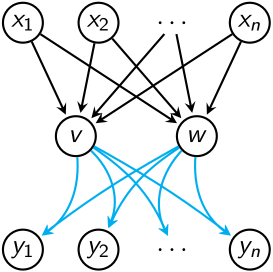
Yes!

Consider the formula $F_n = \bigwedge_{1 \leq i \leq n} \left((\bar{x}_i \vee v) \wedge (\bar{x}_i \vee w) \wedge (\bar{v} \vee \bar{w} \vee y_i) \right)$



#variables: $2n + 2$

#clauses: $3n$

#literals: $7n$

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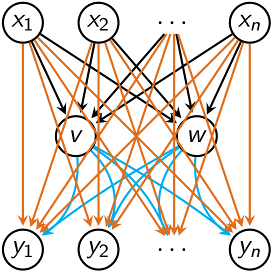

#variables: $2n + 2$

#clauses: $3n$

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n^2 hyper binary resolvents:

$(\bar{x}_i \vee y_j)$ for $1 \leq i, j \leq n$

Unhiding Redundancy

Redundancy

Redundant clauses:

- Removal of $C \in F$ preserves unsatisfiability of F
- Assign all $x \in C$ to false and check for a conflict in $F \setminus \{C\}$

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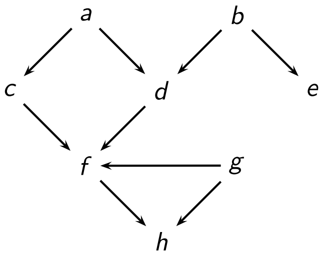
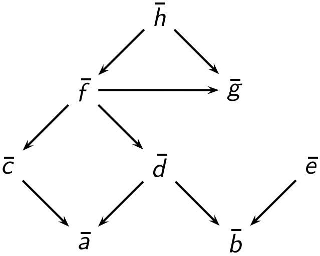
Redundancy elimination during pre- and in-processing

- Distillation [JinSomenzi2005]
- ReVivAl [PietteHamadiSaïs2008]
- Unhiding [HeuleJärvisaloBiere2011]

Unhide: Binary implication graph (BIG)

unhide: use the binary clauses to detect redundant clauses and literals





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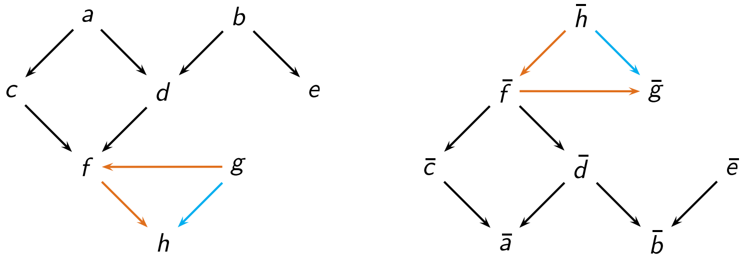
$$(\bar{c} \vee f) \wedge (\bar{d} \vee f) \wedge (\bar{g} \vee f) \wedge (\bar{f} \vee h) \wedge$$
$$(\bar{g} \vee h) \wedge \underbrace{(\bar{a} \vee \bar{e} \vee h) \wedge (\bar{b} \vee \bar{c} \vee h) \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)}_{\text{non binary clauses}}$$

Unhide: Transitive reduction (TRD)

transitive reduction: remove shortcuts in the binary implication graph



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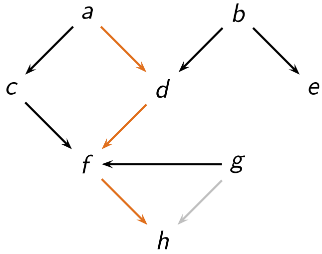
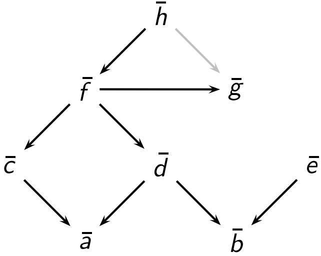
$$\underset{\text{TRD}}{\cancel{(\bar{g} \vee h)}} \wedge (\bar{a} \vee \bar{e} \vee h) \wedge (\bar{b} \vee \bar{c} \vee h) \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)$$

$g \rightarrow f \rightarrow h$

Unhide: Hidden tautology elimination (HTE) (1)

HTE removes clauses that are subsumed by an implication in BIG





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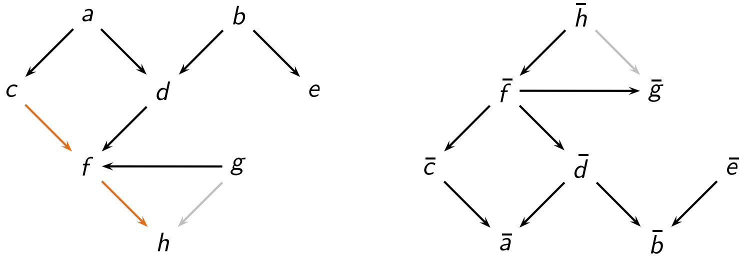
~~$(\bar{a} \vee \bar{e} \vee h)$~~ $\wedge (\bar{b} \vee \bar{c} \vee h) \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)$

HTE

$a \rightarrow d \rightarrow f \rightarrow h$

Unhide: Hidden tautology elimination (HTE) (2)

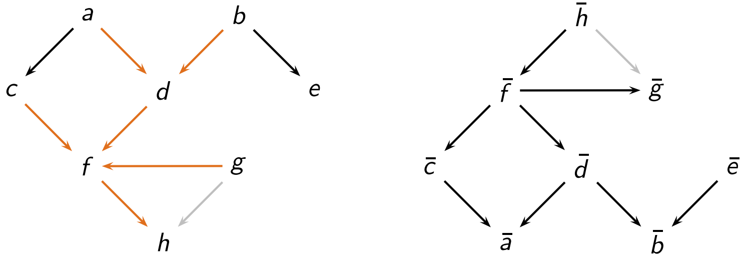
HTE removes clauses that are subsumed by an implication in BIG



$$(\bar{a} \lor c) \land (\bar{a} \lor d) \land (\bar{b} \lor d) \land (\bar{b} \lor e) \land$$
$$(\bar{c} \lor f) \land (\bar{d} \lor f) \land (\bar{g} \lor f) \land (\bar{f} \lor h) \land$$
$$\underset{\substack{\text{HTE} \\ c \to f \to h}}{\cancel{(\bar{b} \lor \bar{c} \lor h)}} \land (a \lor b \lor c \lor d \lor e \lor f \lor g \lor h)$$

Unhide: Hidden literal elimination (HLE)

HLE removes literal using the implication in BIG



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$$(\cancel{a \lor b \lor c \lor d \lor}\, e \lor \cancel{f \lor g \lor}\, h)$$

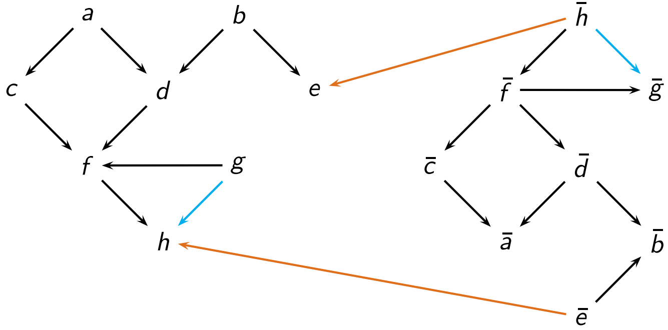
HLE

all but e imply h

also b implies e

Unhide: TRD + HTE + HLE

unhide: redundancy elimination removes and adds arcs from BIG(F)



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Conclusions

Many pre- or in-processing techniques in SAT solvers:

- (Self-)Subsumption
- Variable Elimination
- Blocked Clause Elimination
- Hyper Binary Resolution
- Bounded Variable Addition
- Equivalent Literal Substitution
- Failed Literal Elimination
- Autarky Reasoning
- ...

Preprocessing and Inprocessing

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IPA Course: Formal Methods June 11, 2018