Propositional Proof Theory

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Proofs of Unsatisfiability

Interference-Based Proofs

Propagation Redundancy

Satisfaction-Driven Clause Learning

Conclusions

Proofs of Unsatisfiability

Certifying Satisfiability and Unsatisfiability

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  - We can easily check that the assignment is satisfying:
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- Certifying unsatisfiability is not so easy:
  - If a formula has n variables, there are 2^n possible assignments.

  ➥ Checking whether every assignment falsifies the formula is costly.

  - More compact certificates of unsatisfiability are desirable.

    ➥ Proofs

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    ... but can be of exponential size with respect to a formula.

- Example: Resolution proofs
  - A resolution proof is a sequence $C_1, \ldots, C_m$ of clauses.
  - Every clause is either contained in the formula or derived from two earlier clauses via the resolution rule:

$$\frac{C \vee x \qquad \bar{x} \vee D}{C \vee D}$$

  - C_m is the empty clause (containing no literals), denoted by $\bot$.
  - There exists a resolution proof for every unsatisfiable formula.

Resolution Proofs

- Example: $F = (\bar{x} \vee \bar{y} \vee z) \wedge (\bar{z}) \wedge (x \vee \bar{y}) \wedge (\bar{u} \vee y) \wedge (u)$
- Resolution proof:
  $(\bar{x} \vee \bar{y} \vee z), (\bar{z}), (\bar{x} \vee \bar{y}), (x \vee \bar{y}), (\bar{y}), (\bar{u} \vee y), (\bar{u}), (u), \bot$

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- Drawbacks of resolution:
  - For many seemingly simple formulas, there are only resolution proofs of exponential size.
  - State-of-the-art solving techniques are not succinctly expressible.

Interference-Based Proofs

Traditional Proofs vs. Interference-Based Proofs

- In traditional proof systems, everything that is inferred, is logically implied by the premises.

$$\frac{C \vee x \qquad \bar{x} \vee D}{C \vee D}\ (\text{res}) \qquad \frac{A \qquad A \to B}{B}\ (\text{mp})$$

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↪ Inference rules reason about the presence of facts.

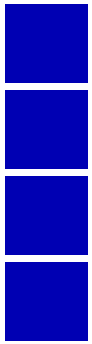
- If certain premises are present, infer the conclusion.

- Different approach: Allow not only implied conclusions.
  - Require only that the addition of facts preserves satisfiability.
  - Reason also about the absence of facts.

  ↪ This leads to interference-based proof systems.

Interference-Based Proofs

Formula



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Interference-Based Proofs

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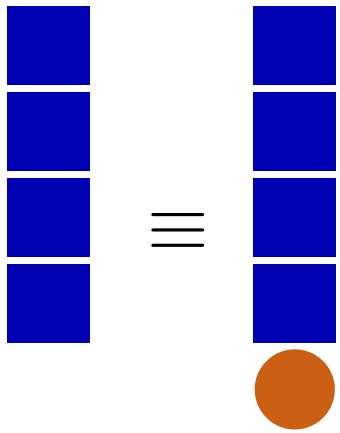

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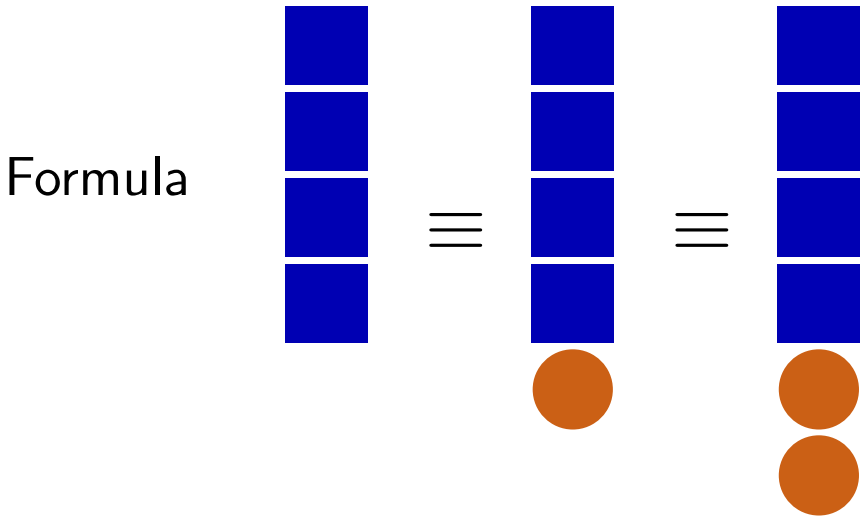

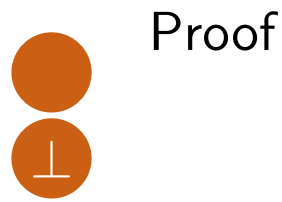

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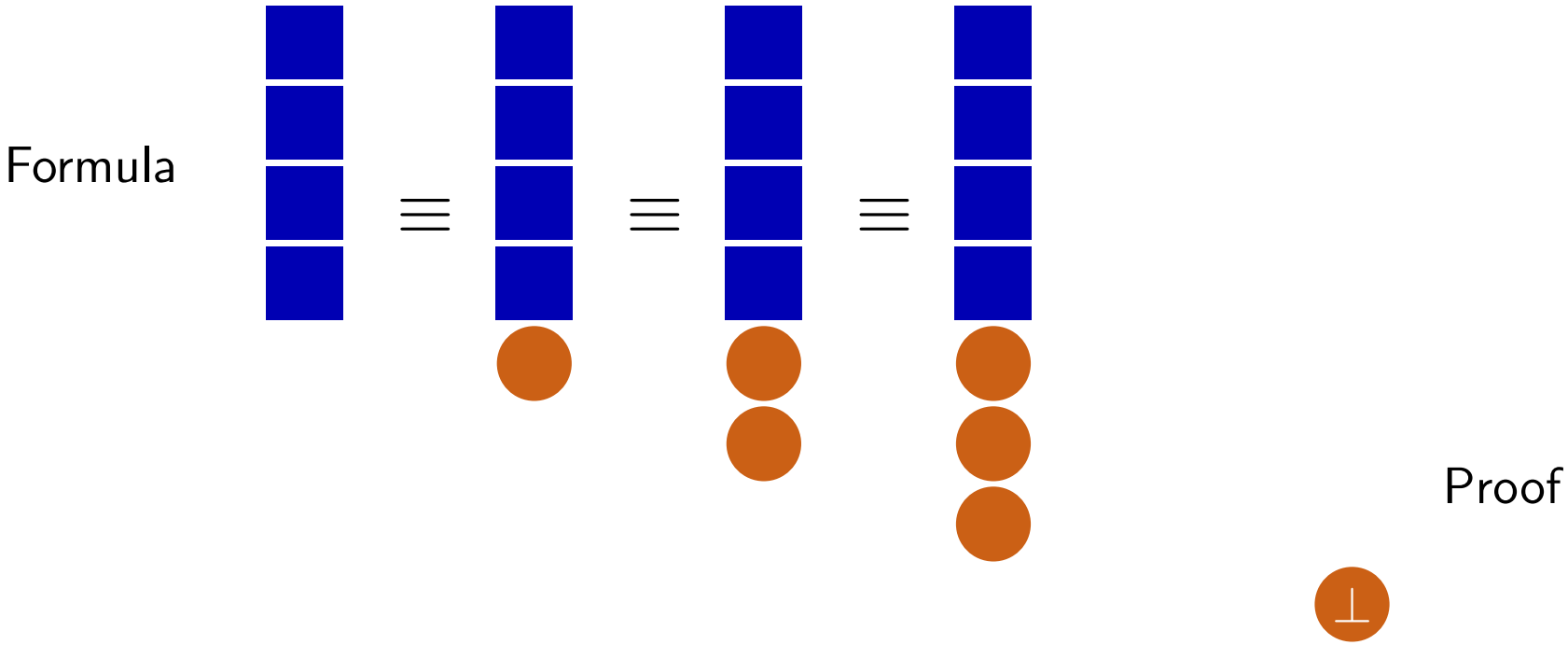

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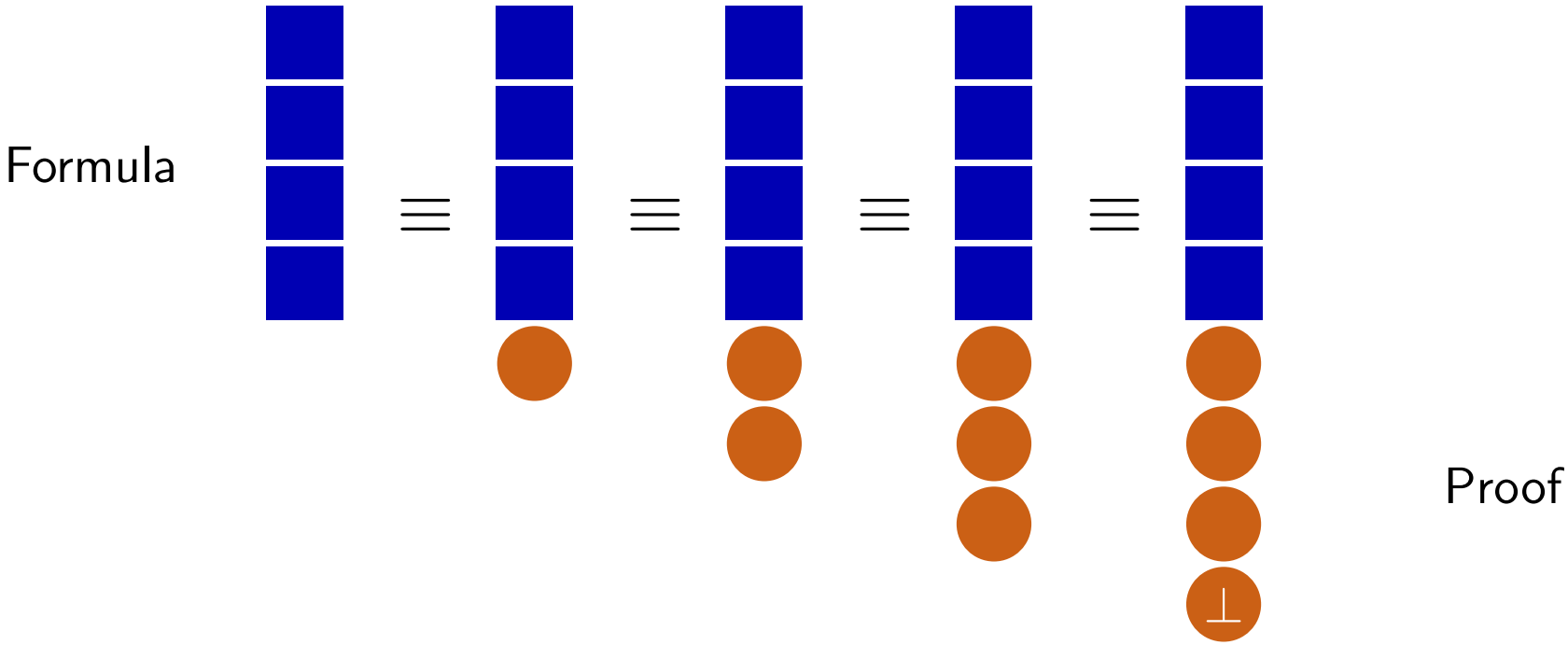

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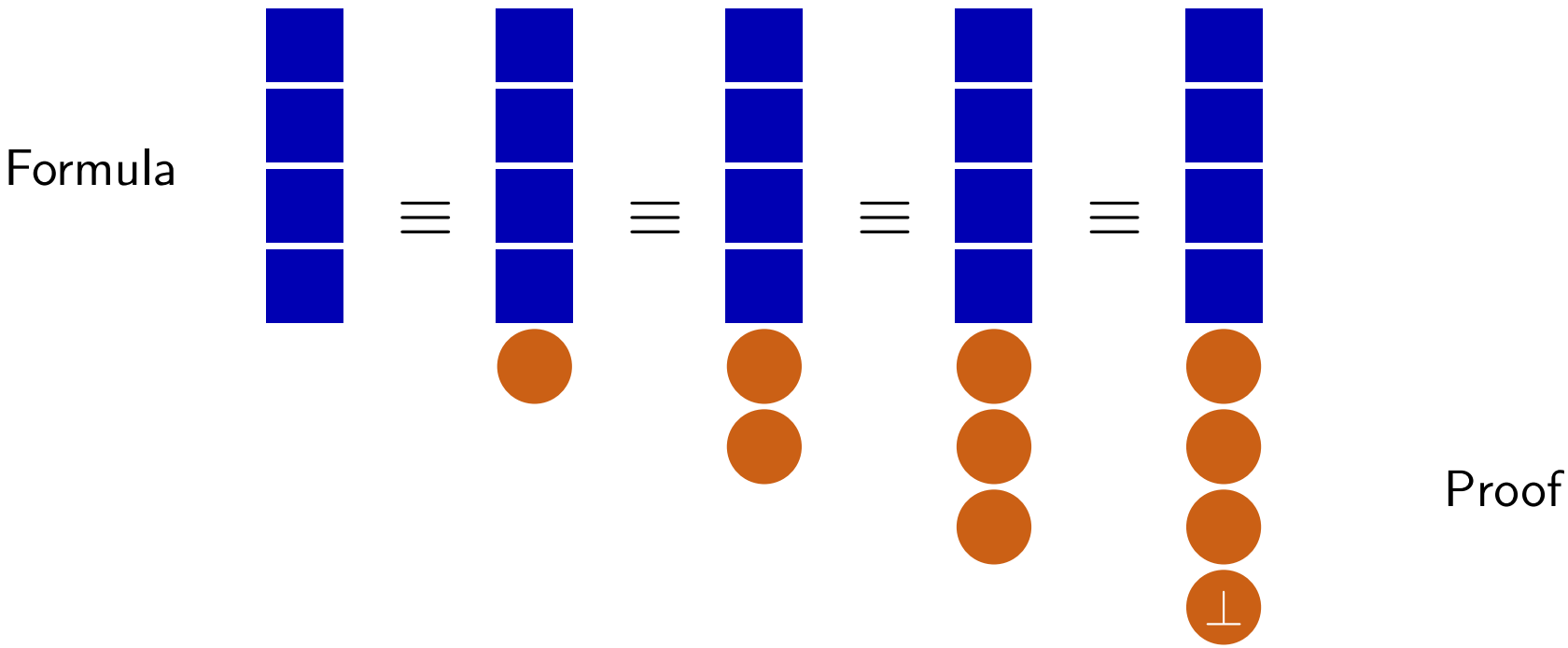

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- Clauses whose addition preserves satisfiability are called redundant.

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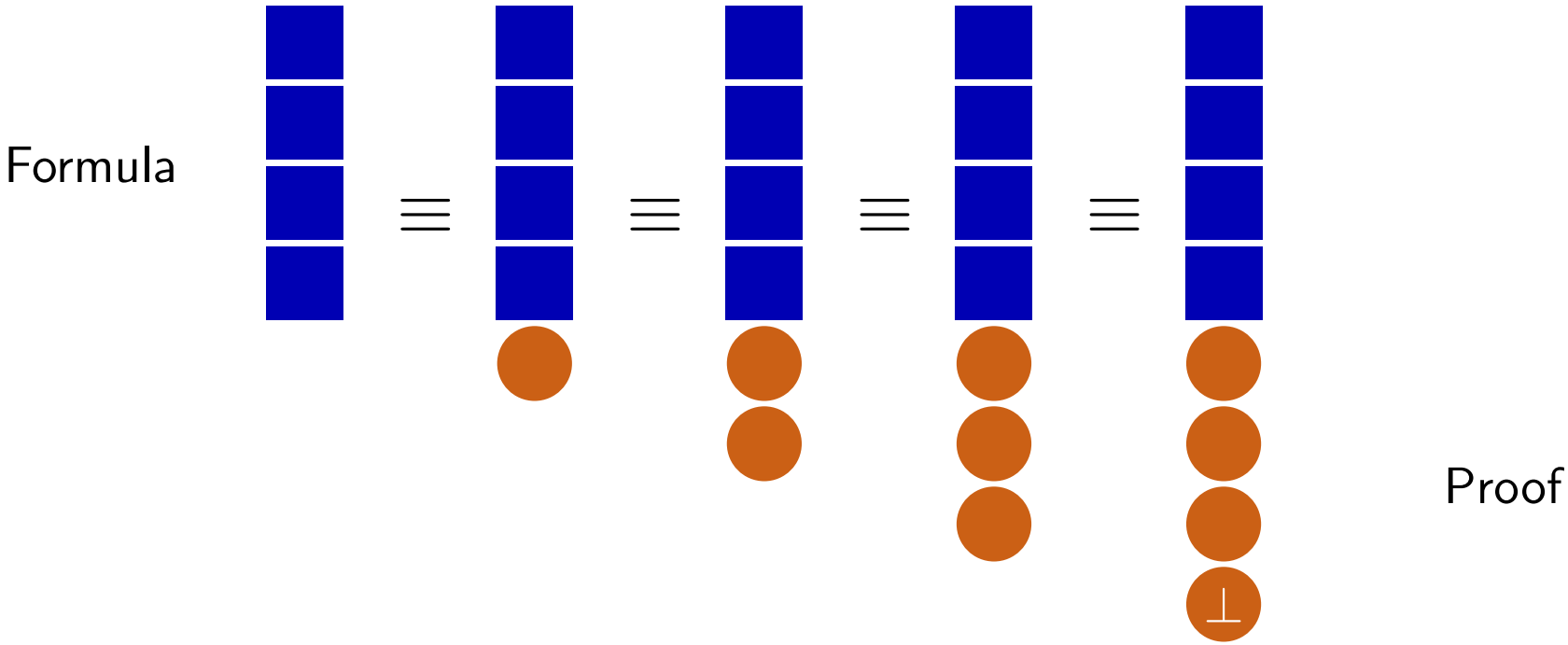

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↪ Idea: Allow only the addition of clauses that fulfill an efficiently checkable redundancy criterion.

DRAT: An Interference-Based Proof System

- Popular example of an interference-based proof system: DRAT
- DRAT allows the addition of so-called resolution asymmetric tautologies (RATs) to a formula (whatever that means).
  - It can be efficiently checked if a clause is a RAT.
  - RATs are not necessarily implied by the formula.
  - But RATs are redundant: their addition preserves satisfiability.
  - A RAT check involves reasoning about the absence of facts.
    - A clause is a RAT w.r.t. a formula if the formula contains no clause such that ...

Blocked Clauses [Kullmann'99]

Definition (Blocking literal)

A literal x in a clause C of a CNF F blocks C w.r.t. F if for every clause $D \in F_{\bar{x}}$, the resolvent $(C \setminus \{x\}) \cup (D \setminus \{\bar{x}\})$ obtained from resolving C and D on l is a tautology.

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Example

Consider the formula $(a \vee b) \wedge (a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee c)$.
First clause is not blocked.
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Proposition

Adding or removing a blocked clause preserves satisfiability.

Blocked Clause Elimination (BCE)

Definition (BCE)

While there is a blocked clause C in a CNF F, remove C from F.

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Consider $(a \vee b) \wedge (a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee c)$.
After removing either $(a \vee \bar{b} \vee \bar{c})$ *or* $(\bar{a} \vee c)$, *the clause* $(a \vee b)$ *becomes blocked (no clause with either* $\bar{b}$ *or* $\bar{a}$*).*

An extreme case in which BCE removes all clauses!

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Proposition

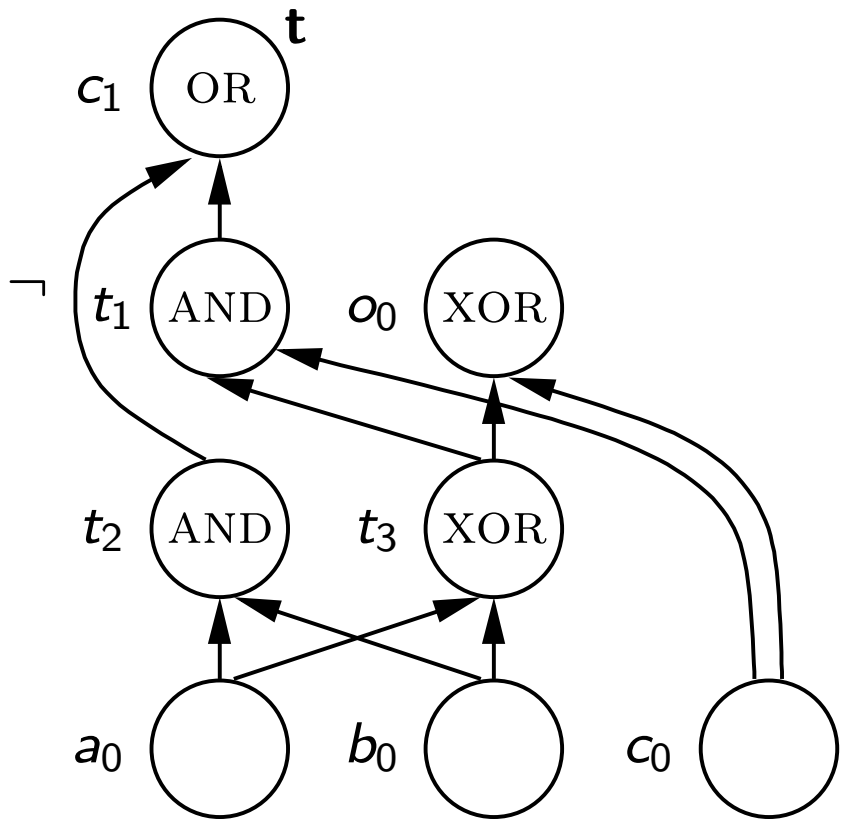
BCE is confluent, i.e., has a unique fixpoint

- Blocked clauses stay blocked w.r.t. removal

BCE very effective on circuits [JärvisaloBiereHeule'10]

BCE converts the Tseitin encoding to Plaisted Greenbaum
BCE simulates Pure literal elimination, Cone of influence, etc.

Example of circuit simplification by BCE on Tseitin encoding



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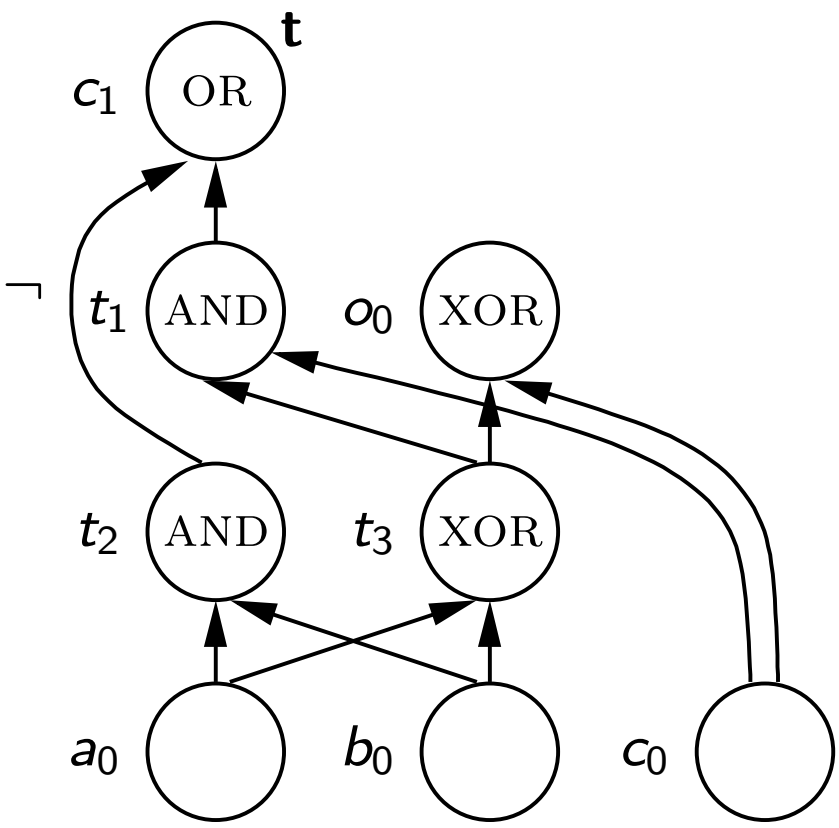

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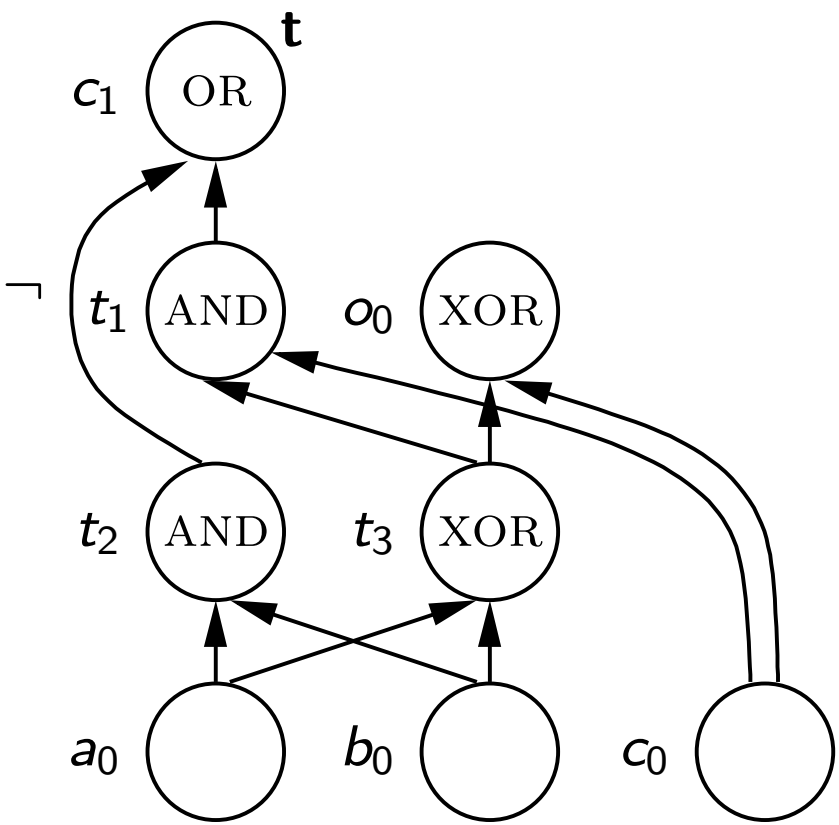

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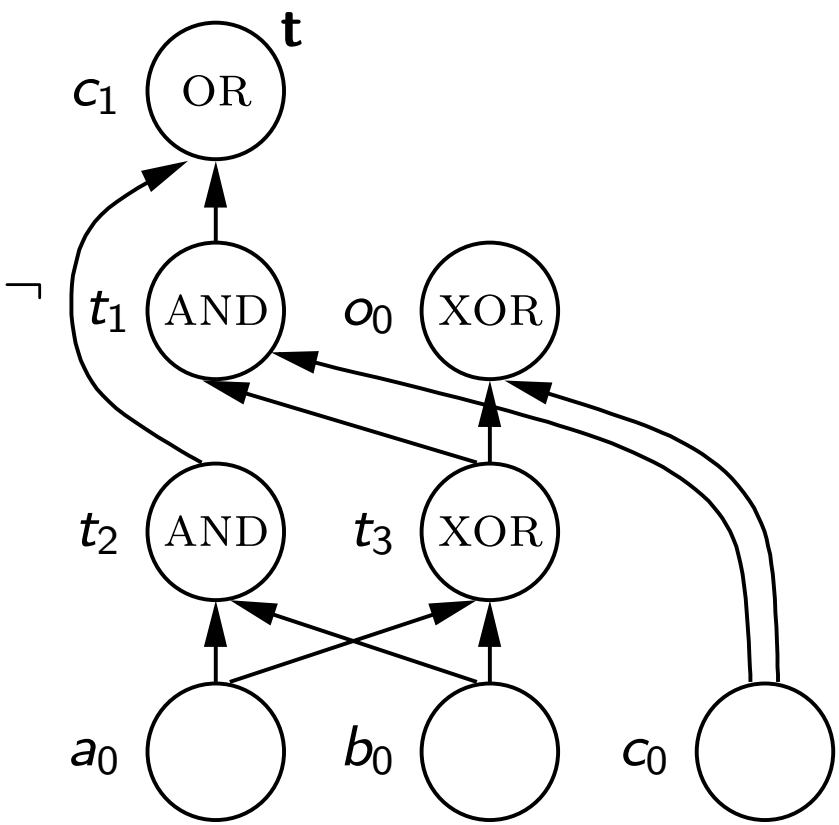

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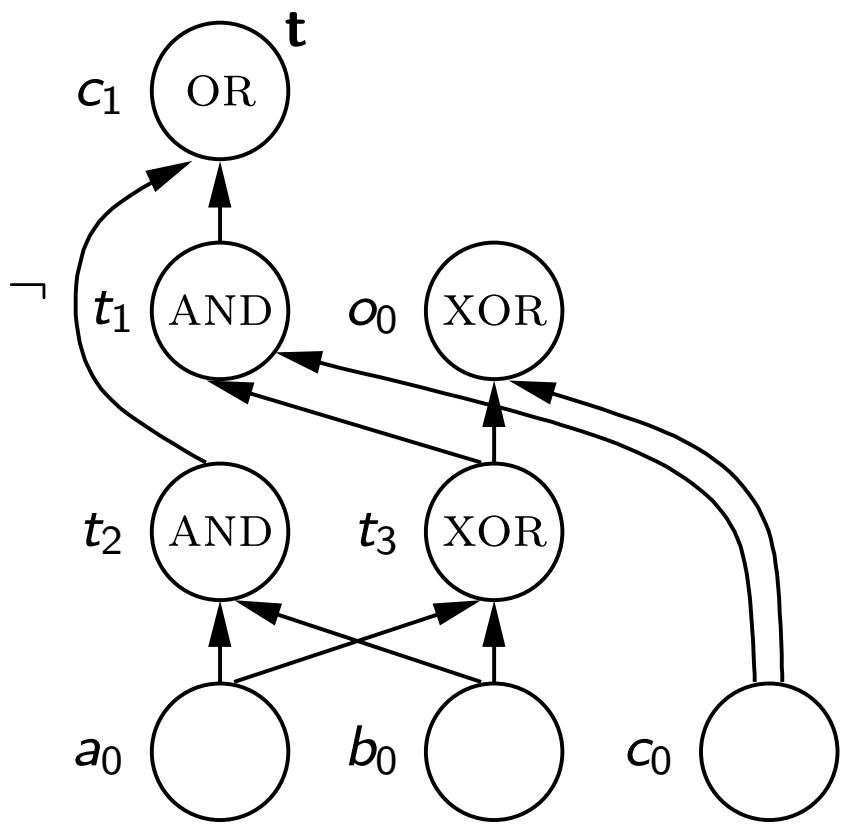

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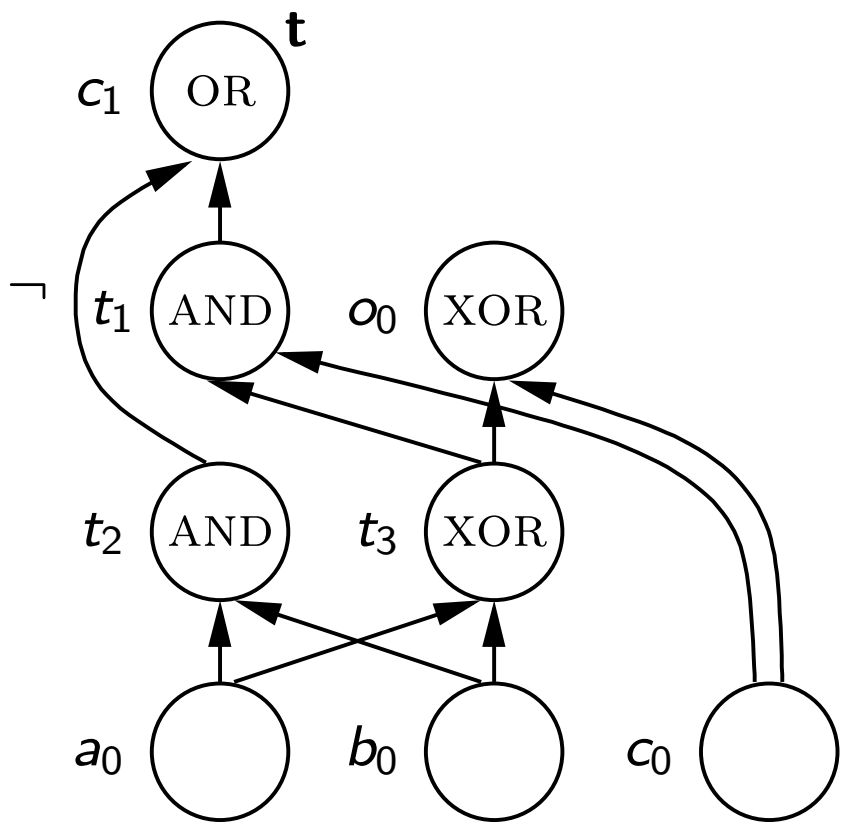

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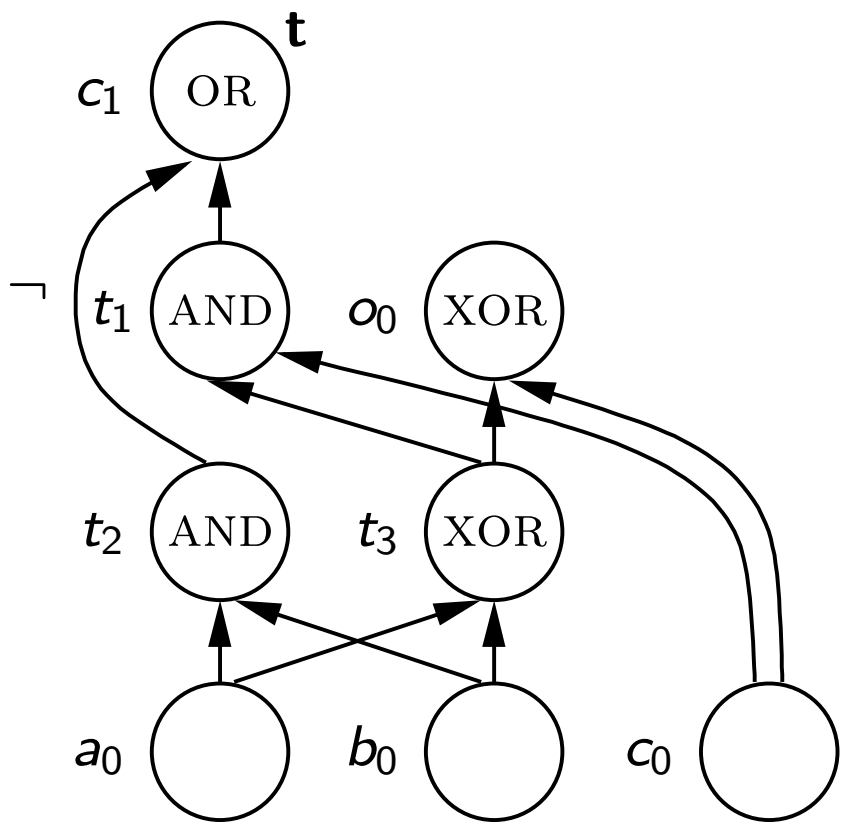

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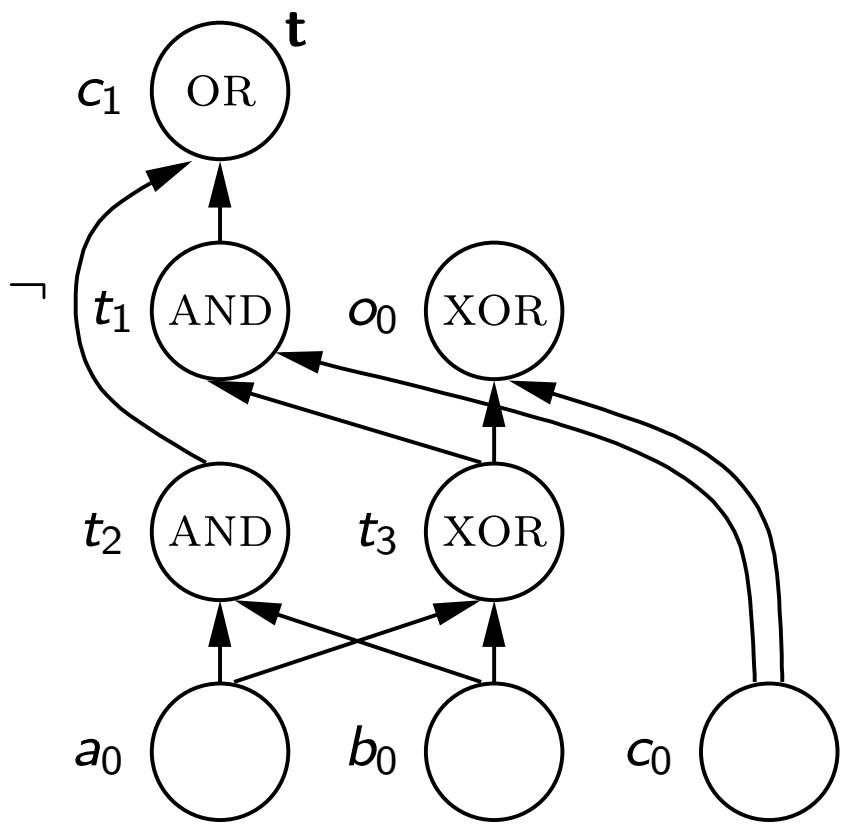

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~~$(o_0 \vee \bar{t}_3 \vee c_0)$~~

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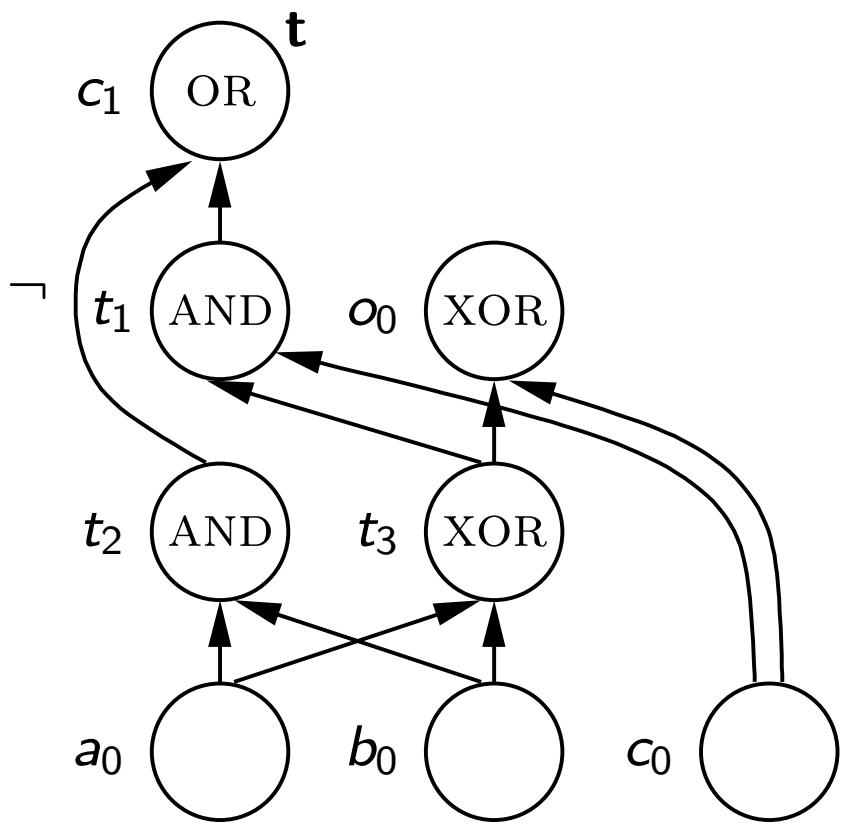

BCE very effective on circuits [JärvisaloBiereHeule'10]

BCE converts the Tseitin encoding to Plaisted Greenbaum
BCE simulates Pure literal elimination, Cone of influence, etc.

Example of circuit simplification by BCE on Tseitin encoding

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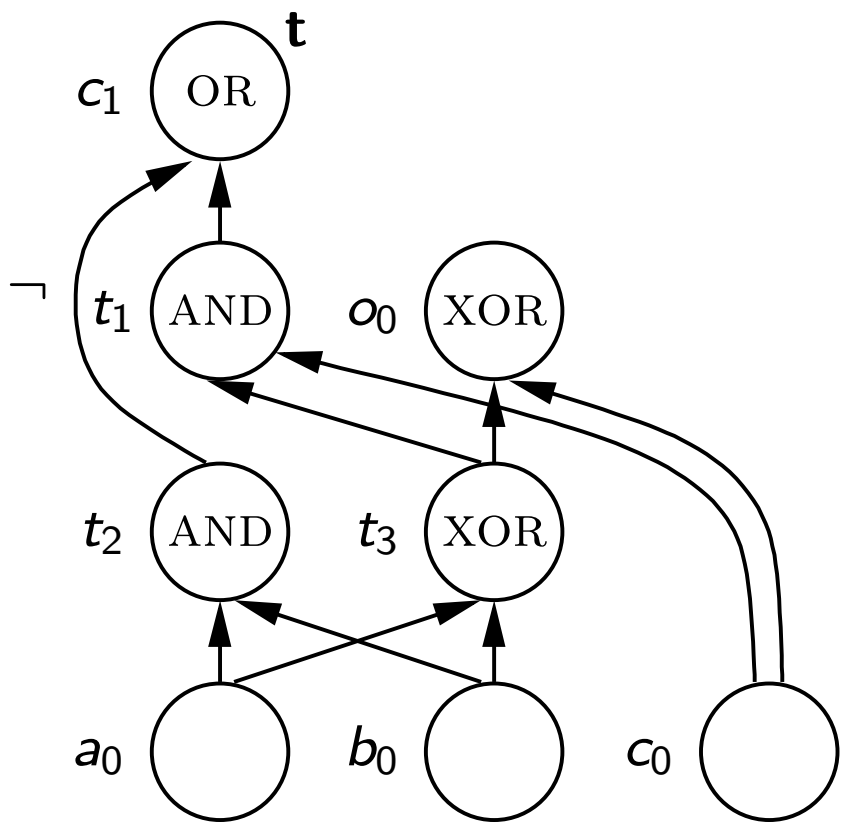

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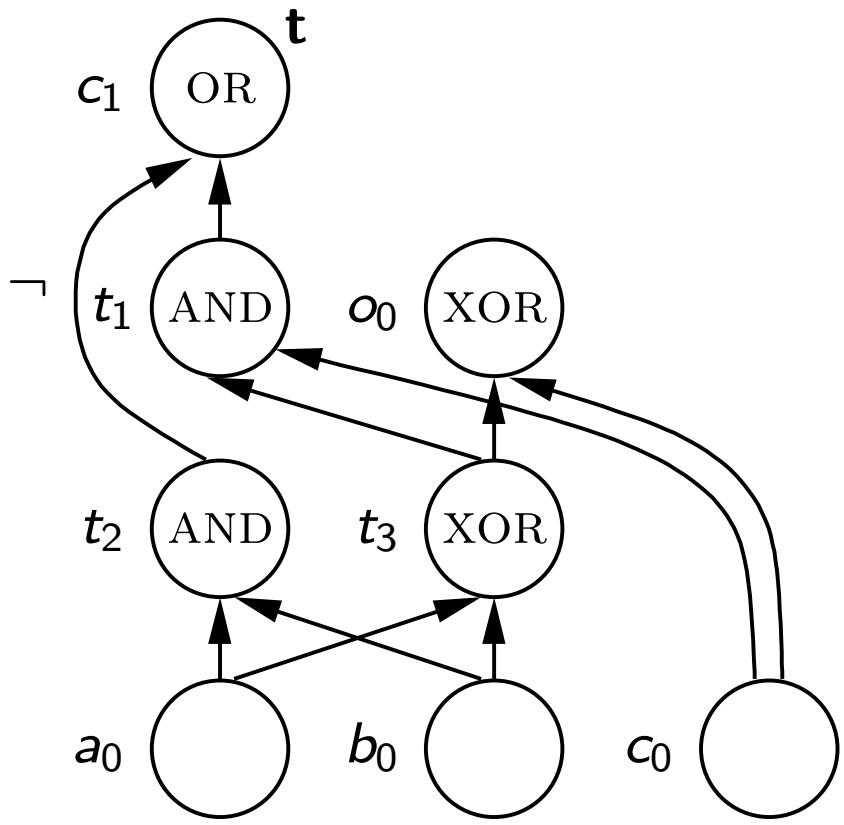

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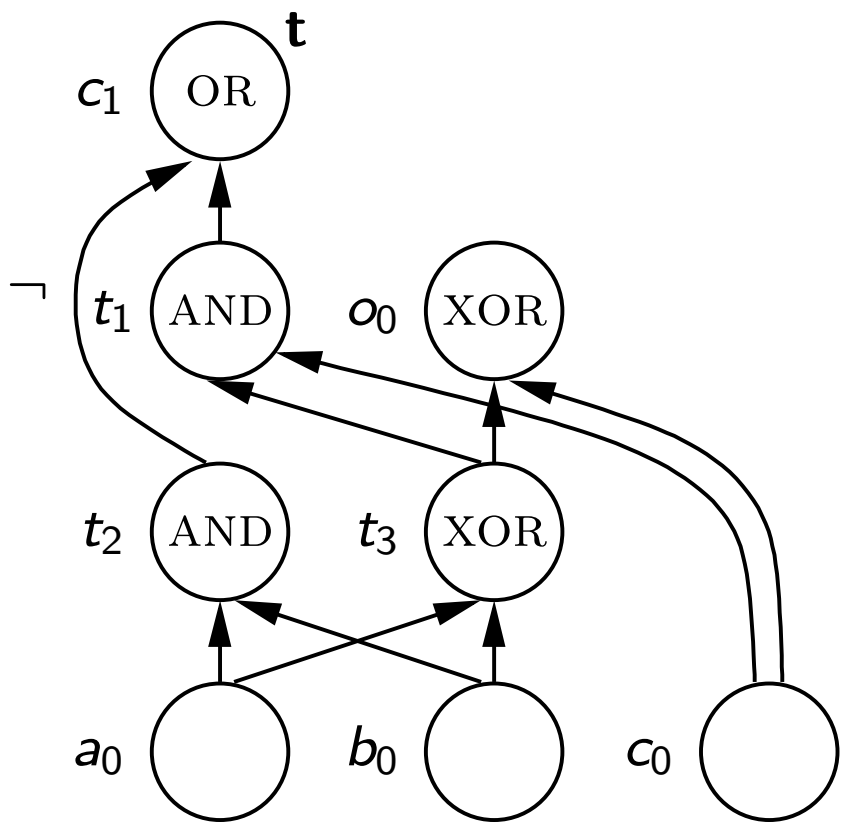
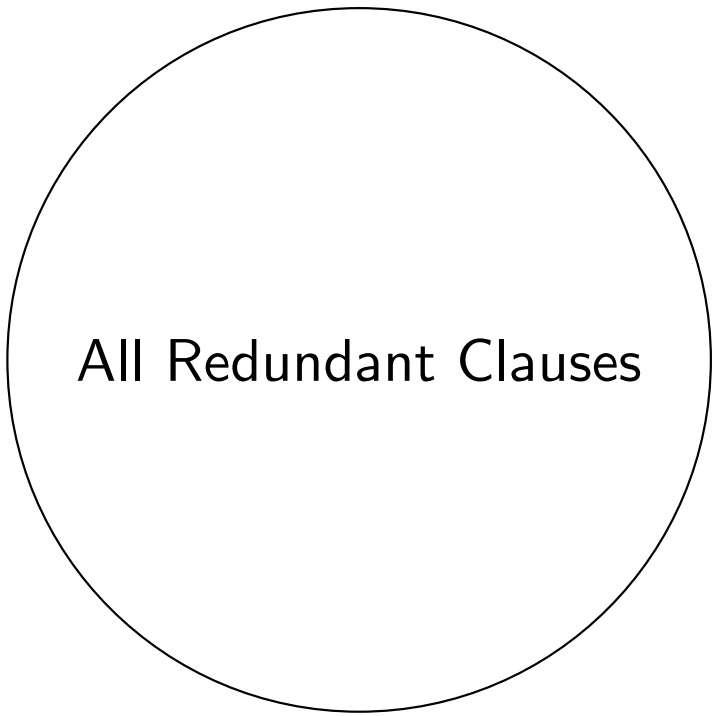


Propagation Redundancy

Redundant Clauses

- Strong proof systems allow addition of many redundant clauses.



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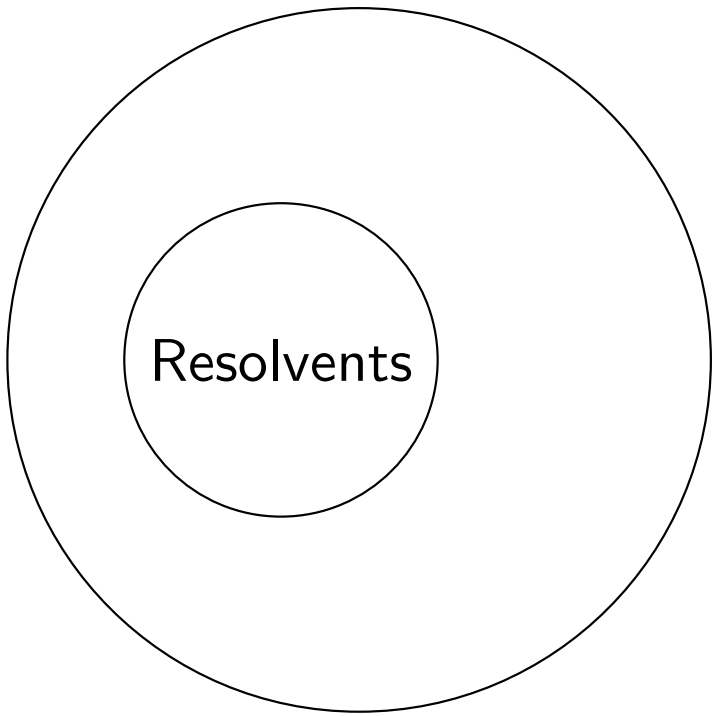

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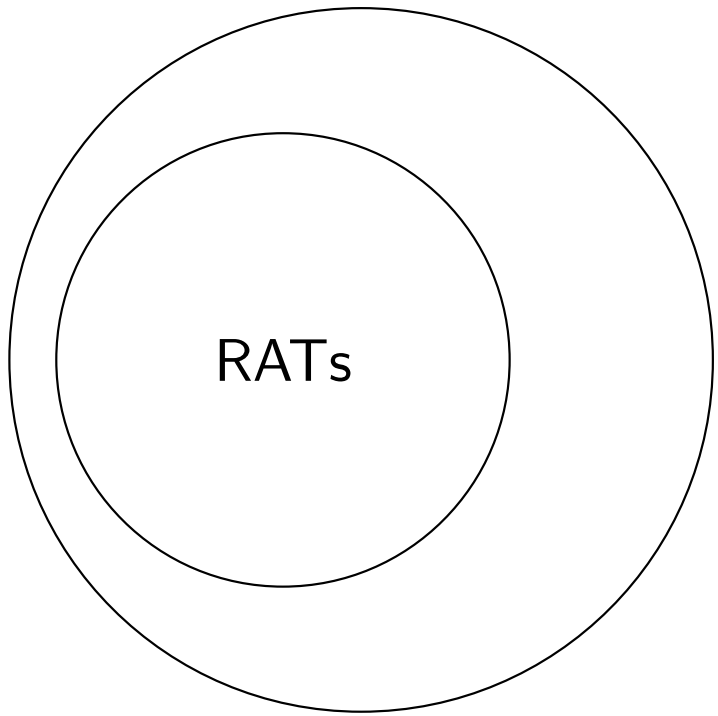

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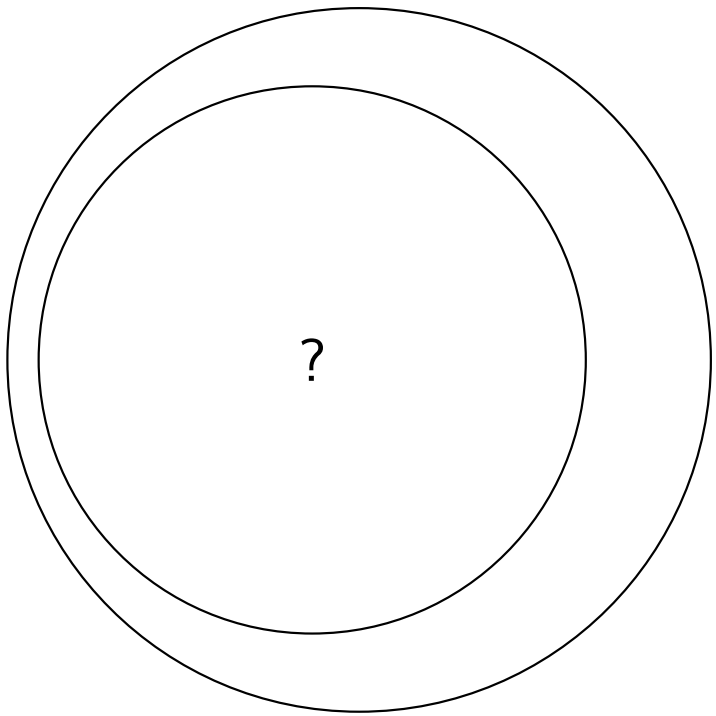
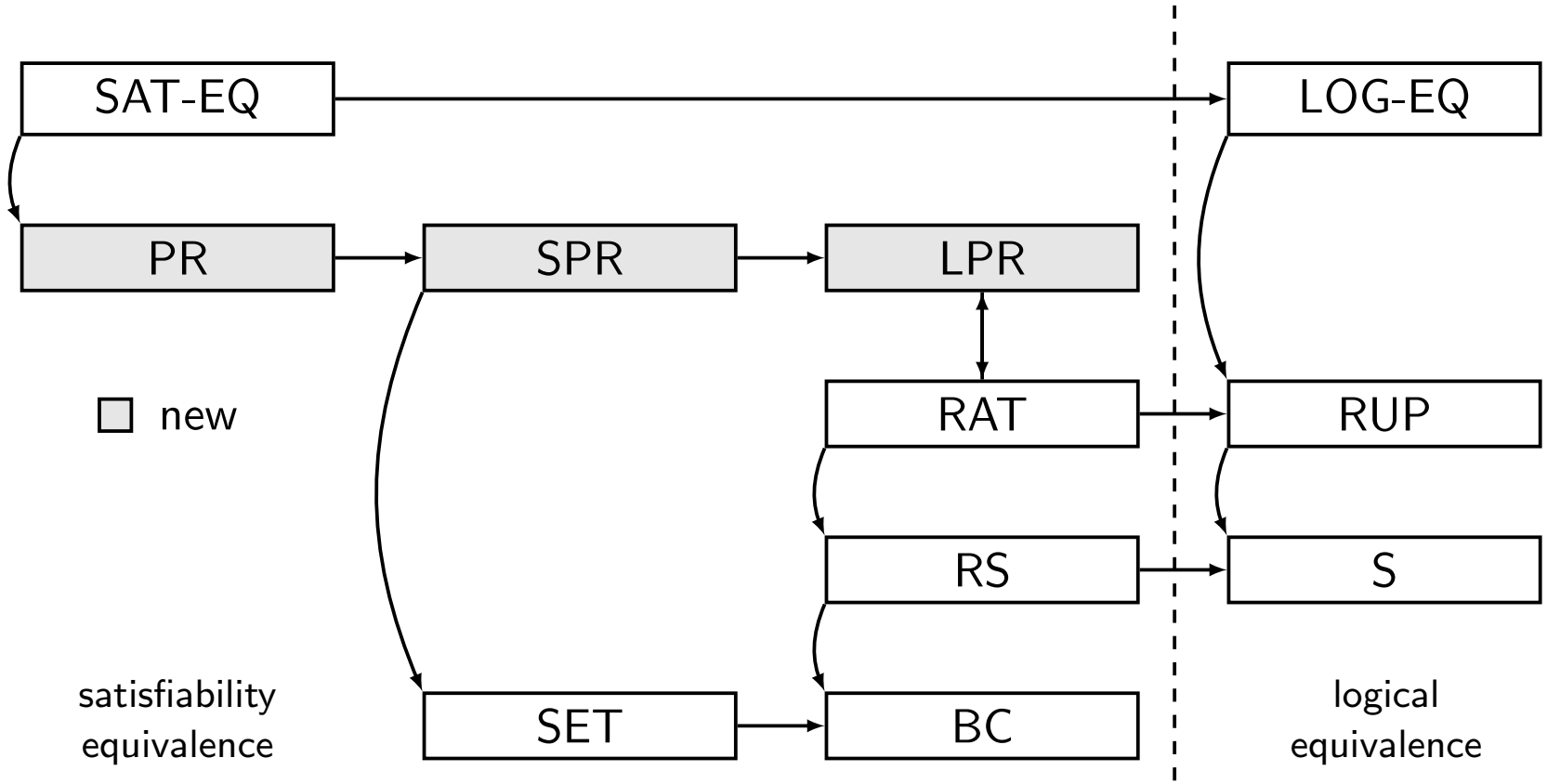


- Are stronger redundancy notions still efficiently checkable?

New Propositional Proof Systems

- We introduced new clause-redundancy notions:
  - Propagation-redundant (PR) clauses
  - Set-propagation-redundant (SPR) clauses
  - Literal-propagation-redundant (LPR) clauses
- LPR clauses coincide with RAT.
- SPR clauses strictly generalize RATs.
- PR clauses strictly generalize SPR clauses.
- The redundancy notions provide the basis for new proof systems.

New Landscape of Redundancy Notions



Stronger Proof Systems: What Are They Good For?

- The new proof systems can give short proofs of formulas that are considered hard.
- We have short SPR and PR proofs for the well-known pigeon hole formulas (linear in the size of the input).
  - Pigeon hole formulas have only exponential-size resolution proofs.
  - If the addition of new variables via definitions is allowed, there are polynomial-size proofs.
    - So-called extended resolution proofs.
- Our proofs do not require new variables.
  - ➥ Search space of possible clauses is finite.
  - ➥ Makes search for such clauses easier.

Redundancy as an Implication

A formula G is at least as satisfiable as a formula F if $F \models G$.

Given a formula F and assignment α, we denote with $F|_\alpha$ the reduced formula after removing from F all clauses satisfied by α and all literals falsified by α.

Theorem

Let F be a formula, C a clause, and α the smallest assignment that falsifies C. Then, C is redundant w.r.t. F iff there exists an assignment ω such that 1) ω satisfies C; and 2) $F|_\alpha \models F|_\omega$.

This is the strongest notion of redundancy. However, it cannot be checked in polynomial time (assuming P $\neq$ NP), unless bounded.

Checking Redundancy Using Unit Propagation

- Unit propagation (UP) satisfies unit clauses by assigning their literal to true (until fixpoint or a conflict).
- Let F be a formula, C a clause, and α the smallest assignment that falsifies C. C is implied by F via UP (denoted by $F \vdash_1 C$) if UP on $F|_\alpha$ results in a conflict.
- Implied by UP is used in SAT solvers to determine redundancy of learned clauses and therefore $\vdash_1$ is a natural restriction of $\models$.
- We bound $F|_\alpha \models F|_\omega$ by $F|_\alpha \vdash_1 F|_\omega$.
- Example: $F = (x \vee y \vee z) \wedge (\overline{x} \vee y \vee z) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee \overline{y} \vee z)$ and $G = (z)$. Observe that $F \models G$, but that $F \not\vdash_1 G$.

Hand-crafted PR Proofs of Pigeon Hole Formulas

We manually constructed PR proofs of the famous pigeon hole formulas and the two-pigeons-per-hole family.

- The proofs consist only of binary and unit clauses.
- Only original variables appear in the proof.
- All proofs are linear in the size of the formula.

↪ Our proofs are smaller than Cook's extended resolution proofs.

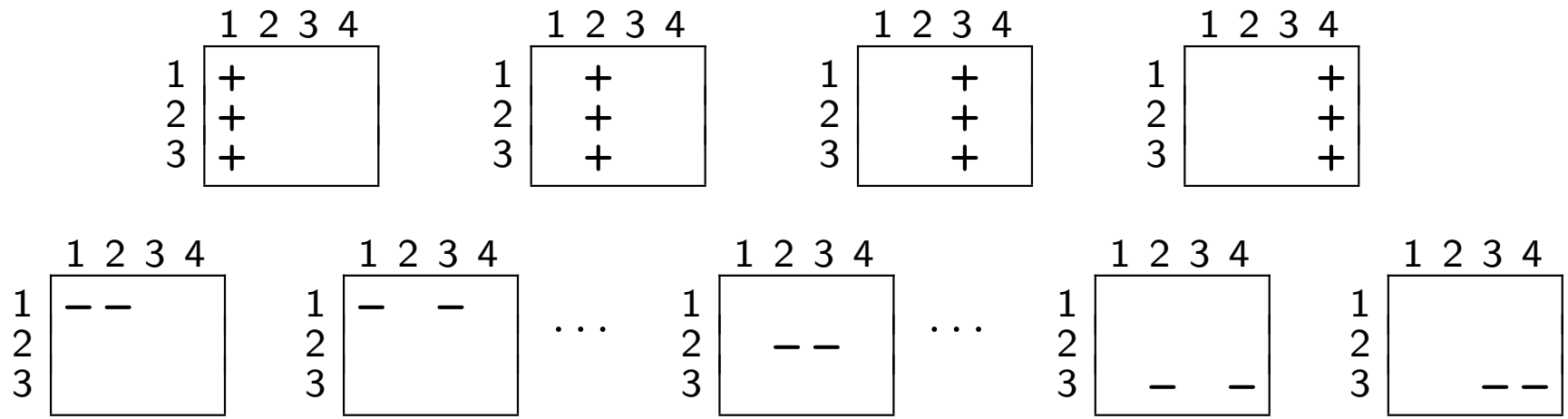
- All resolution proofs of these formulas are exponential in size.

Pigeon Hole Formulas

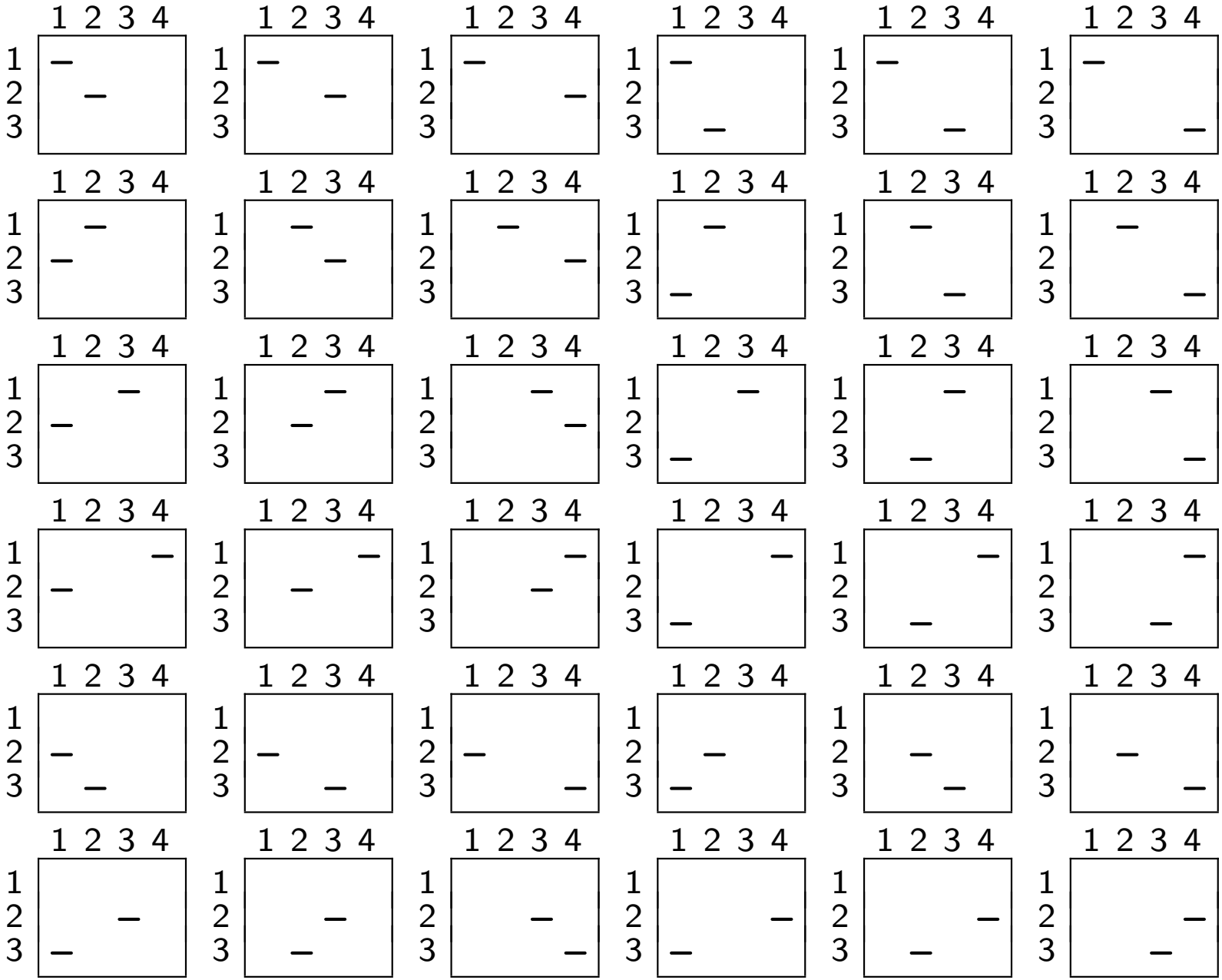
Can $n+1$ pigeons be placed in n holes (at most one pigeon per hole)?

$$PHP_n := \bigwedge_{1 \le p \le n+1} (x_{1,p} \vee \cdots \vee x_{n,p}) \wedge \bigwedge_{1 \le h \le n} \bigwedge_{1 \le p < q \le n+1} (\overline{x}_{h,p} \vee \overline{x}_{h,q})$$

Or in array notation for PHP_3 (inspired by Haken):

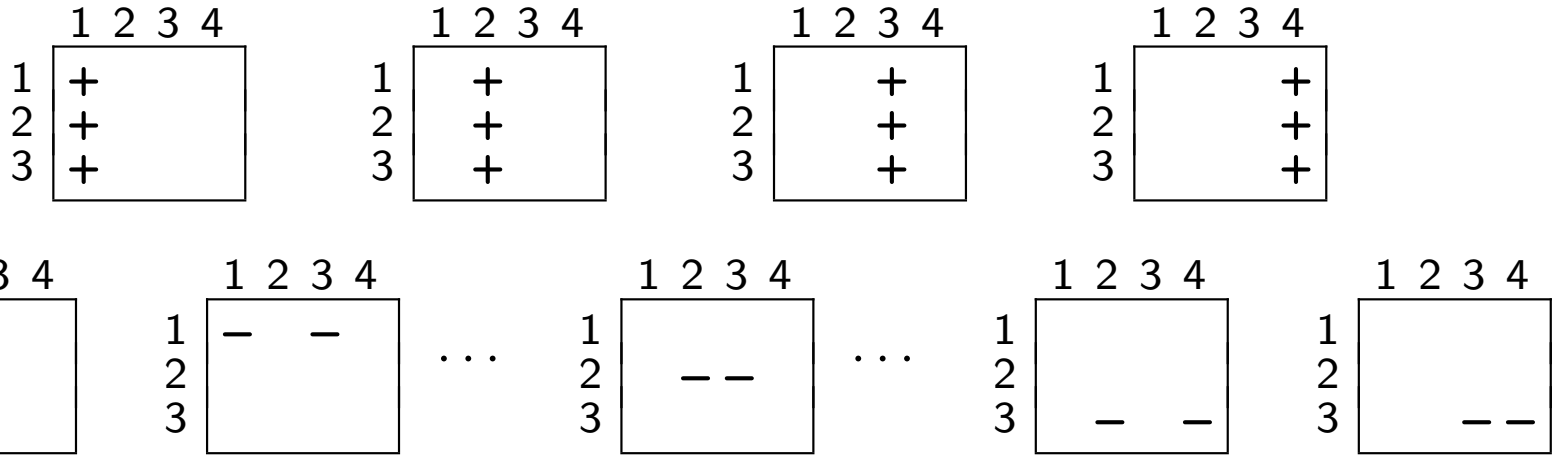


All Binary PR Clauses for PHP_3

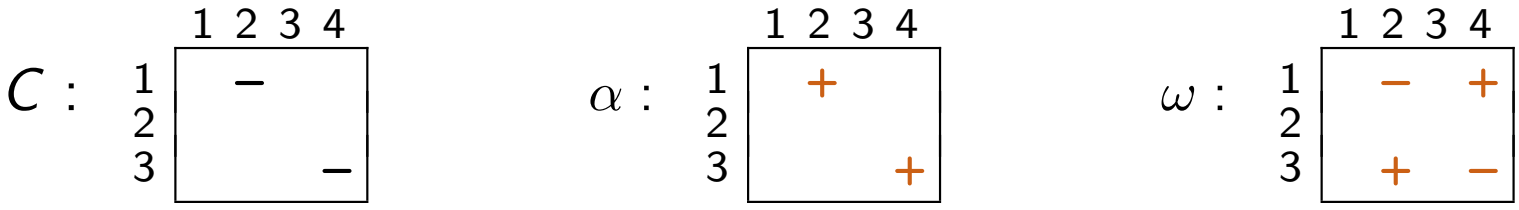


PR Clauses for Pigeon Hole Formulas

Array notation for PHP_3 (inspired by Haken):



Key observation: each $(\bar{x}_{h,p} \vee \bar{x}_{k,q})$ with $h \neq k$, $p \neq q$ is a PR clause.



One can learn a unit clause after learning n such binary clauses.

One can reduce PHP_n to PHP_{n-1} by learning n such unit clauses.

Efficient PR Proof Checker

We implemented an efficient PR proof checker on top of the `DRAT-trim` checker (used to validate SAT competition results).

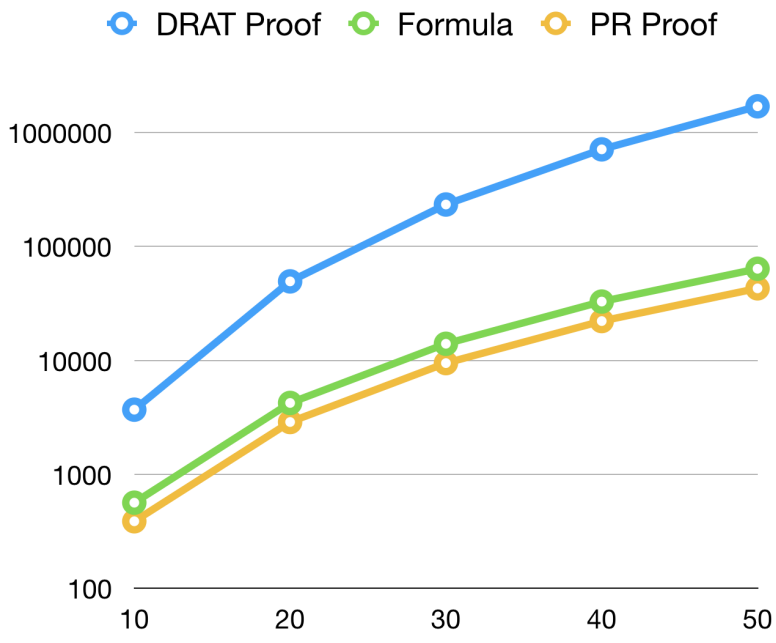
- Complexity is $\mathcal{O}(m^3)$ with m being the number of proof steps.
- However the worst-case is similar to DRAT proof checking...

➥ ..., and DRAT proof checking is in practice almost linear in the size of the formula and proof, by aggressively deleting clauses to limit the size of F.

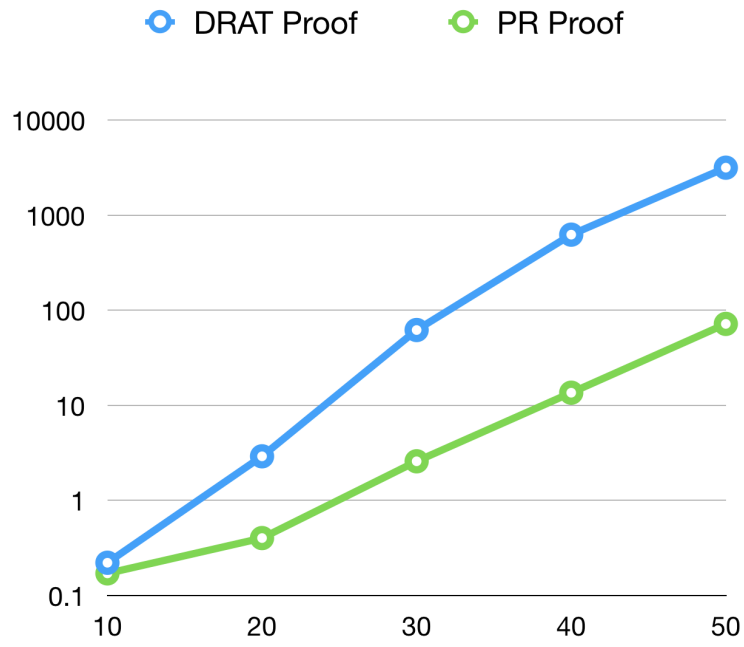
PRcheck (CNF formula F; PR proof $(C_1, \omega_1), \ldots, (C_m, \omega_m)$)

for $i \in \{i, \ldots, m\}$ **do**
  for $D \in F$ **do**
    if $D|\omega_i \neq \top$ **and** $(D|\alpha_i = \top$ **or** $D|\omega_i \subset D|\alpha_i)$ **then**
      if $F|\alpha_i \not\vdash_1 D|\omega_i$ **then return** *failure*
  $F := F \cup \{C_i\}$
return *success*

Comparison of Proof Size and Validation Times



size in the number of clauses



validation time in seconds

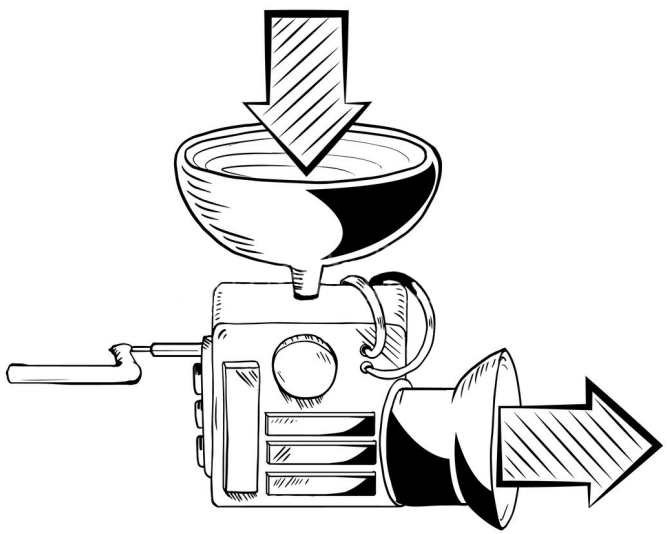
Satisfaction-Driven Clause Learning

Satisfiability Solving (Highly Simplified)

SAT problem:

Given a propositional formula, is it satisfiable?

$$(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$$



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Input Formula in CNF

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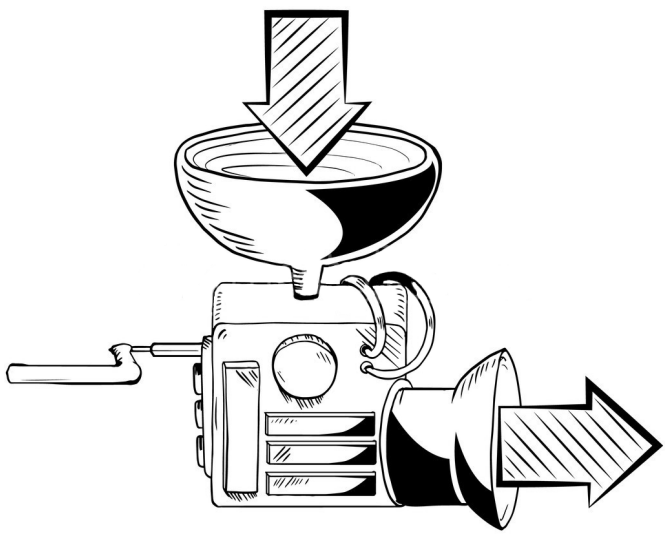

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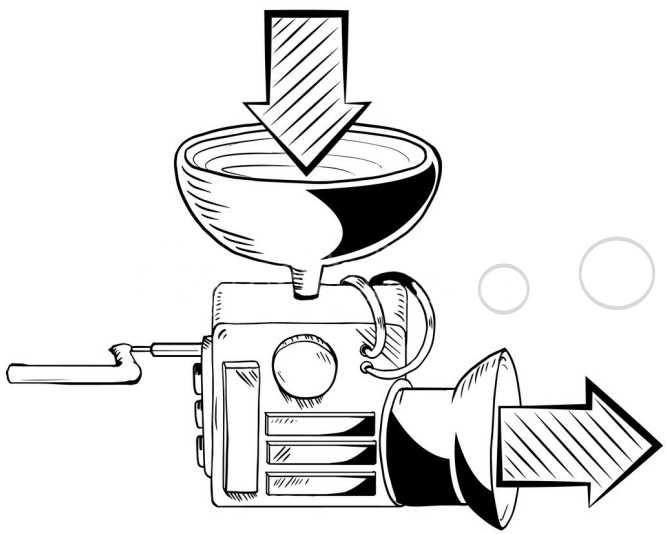

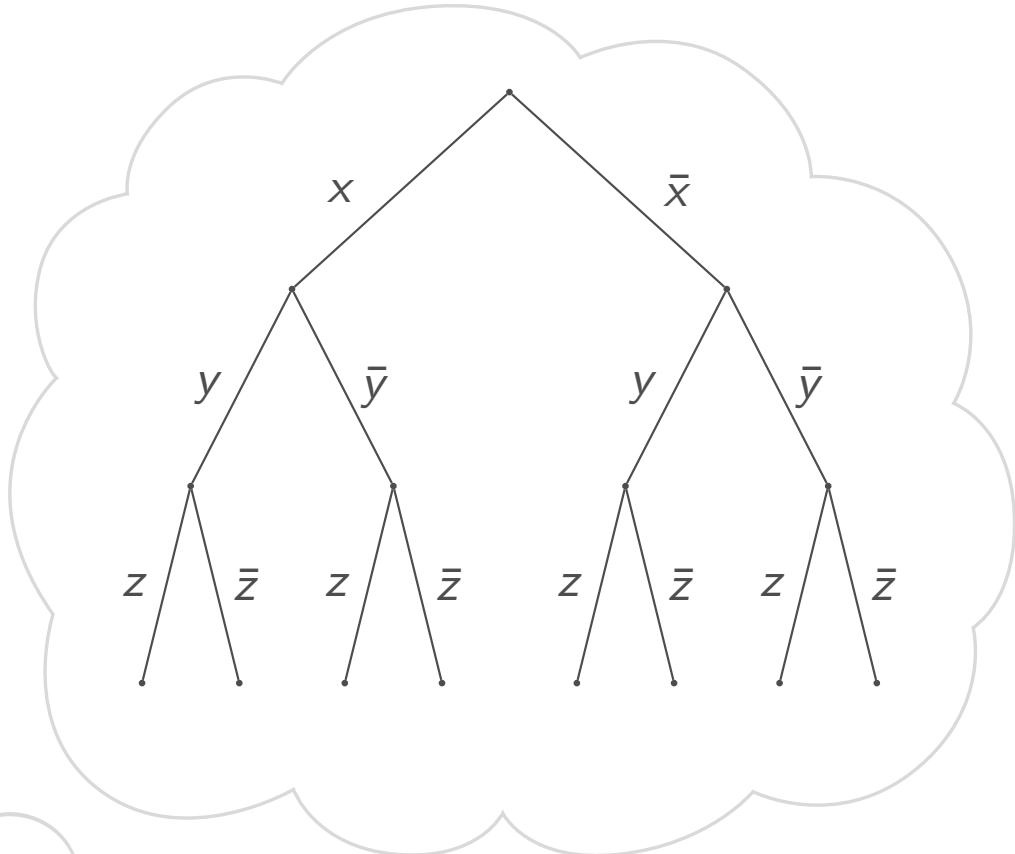

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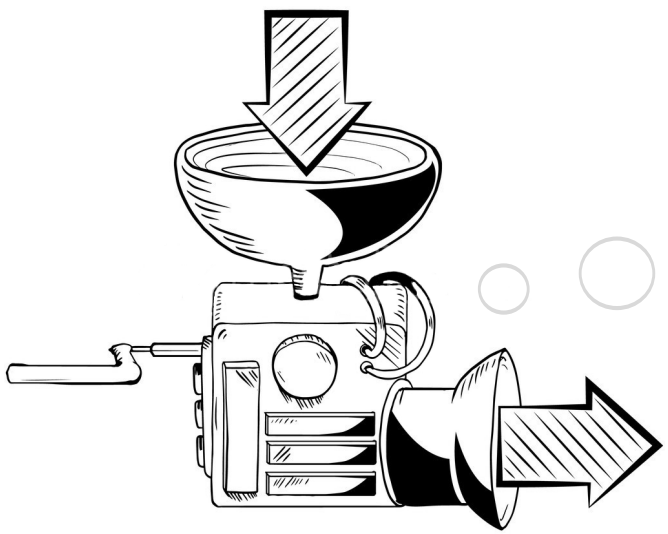

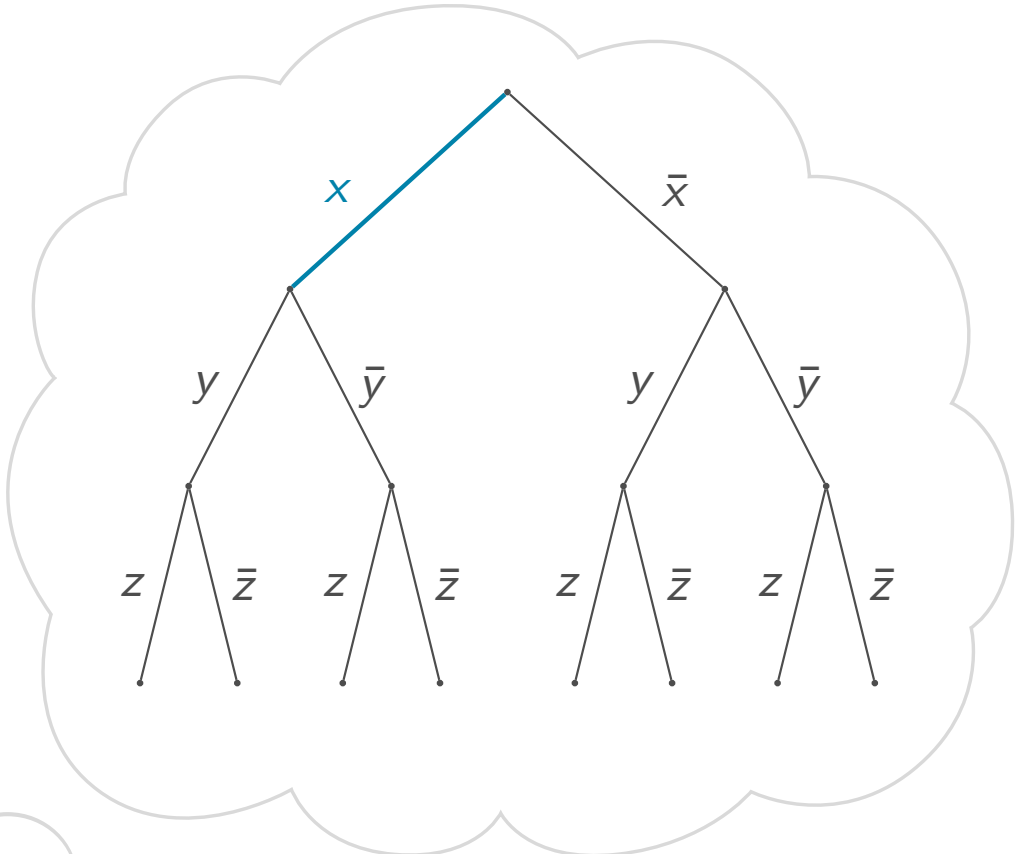

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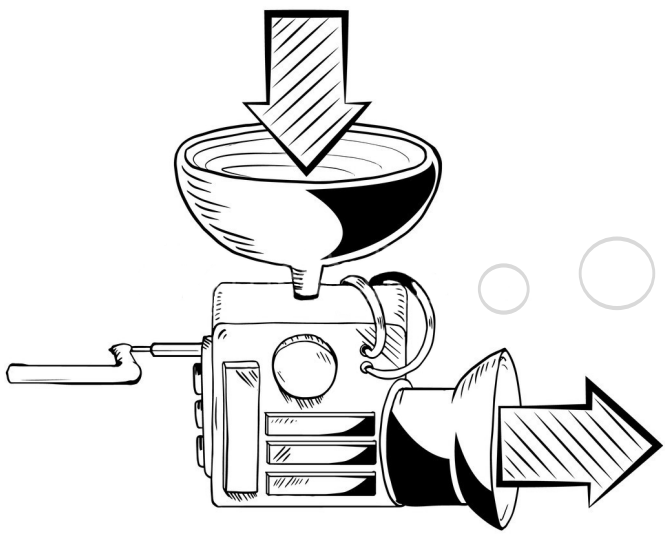

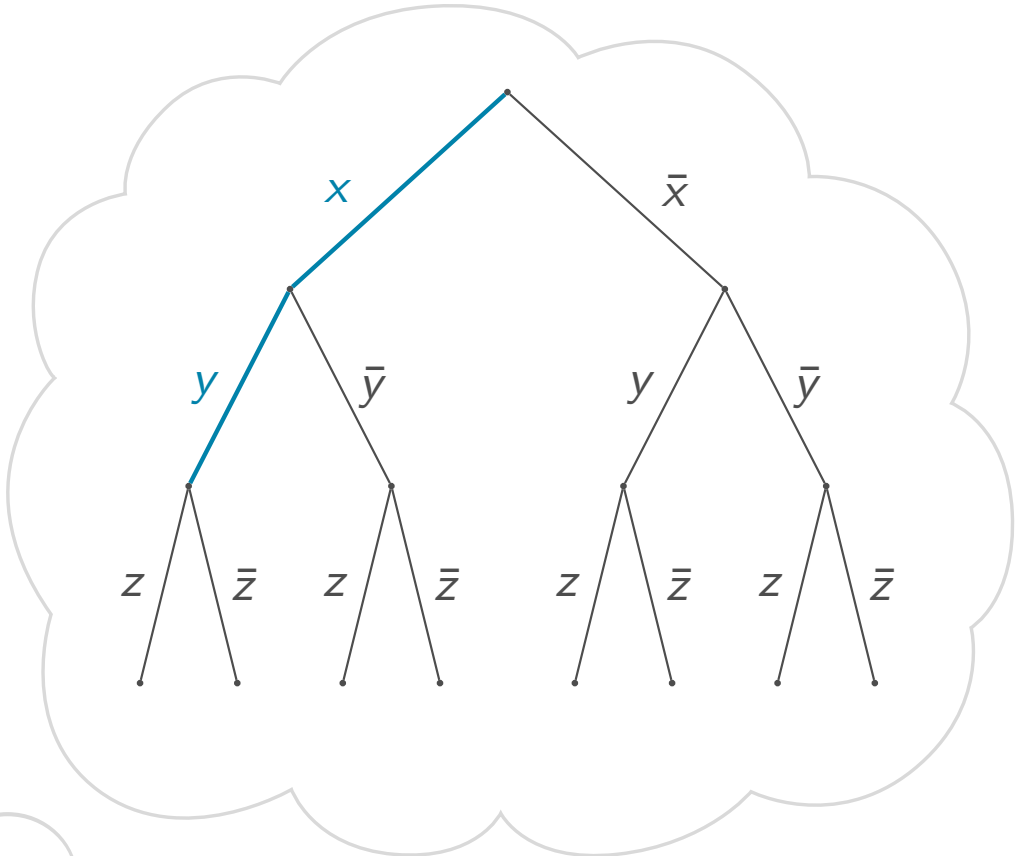

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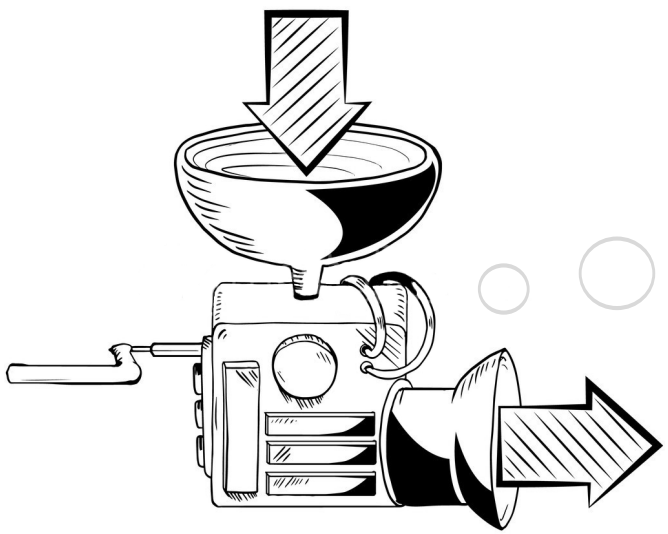

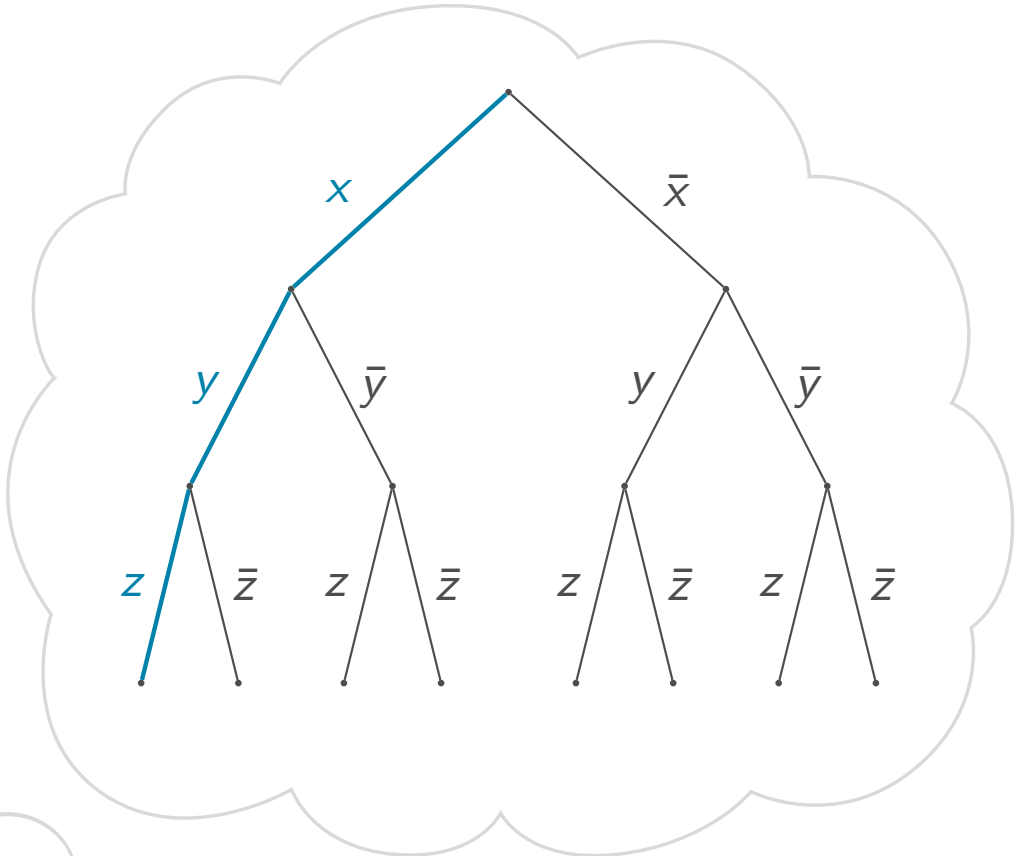

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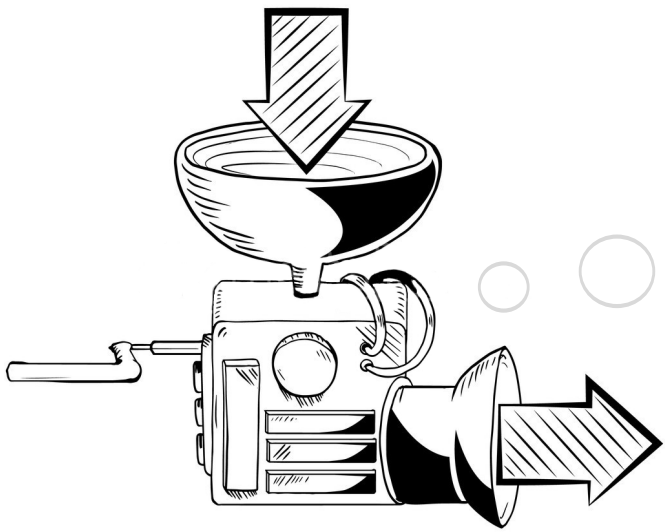

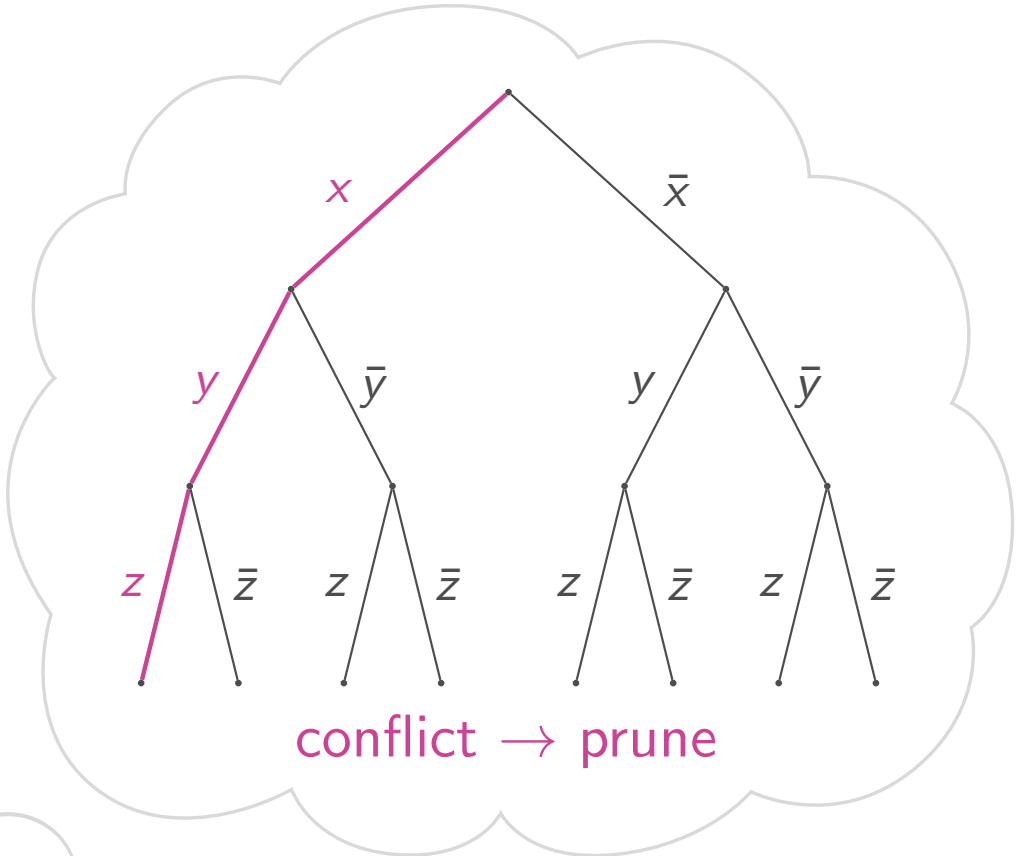

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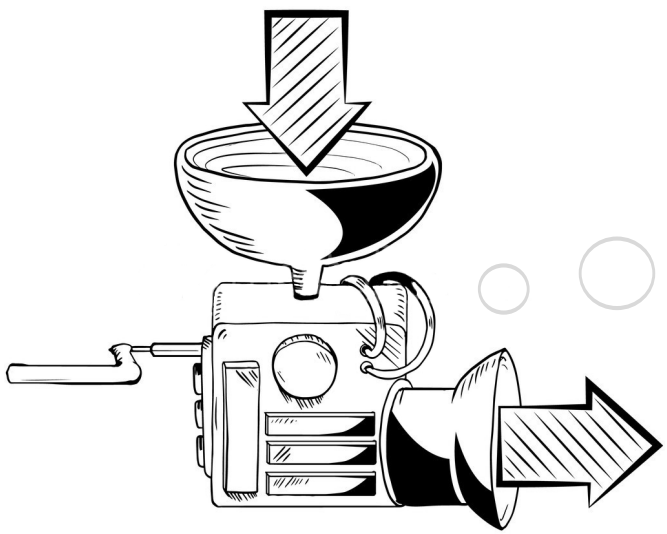

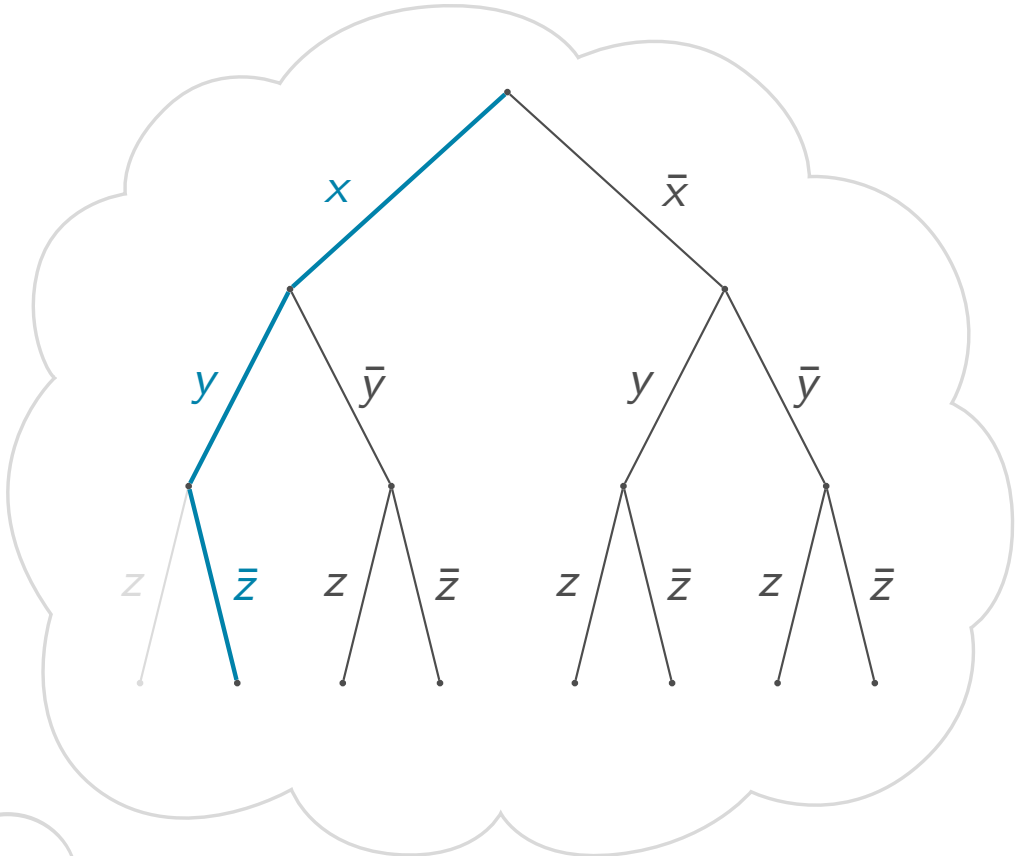

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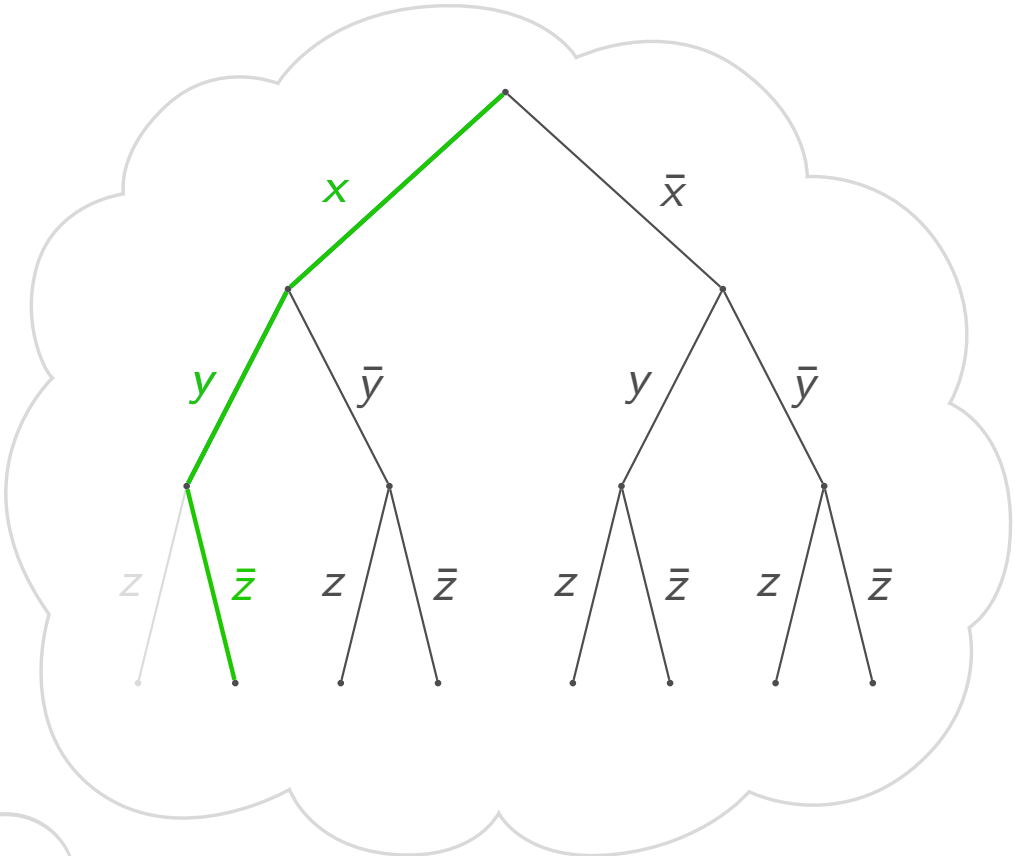

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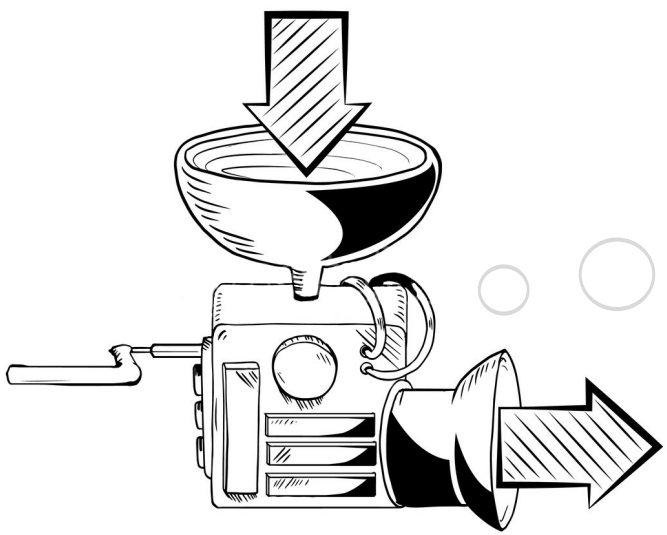
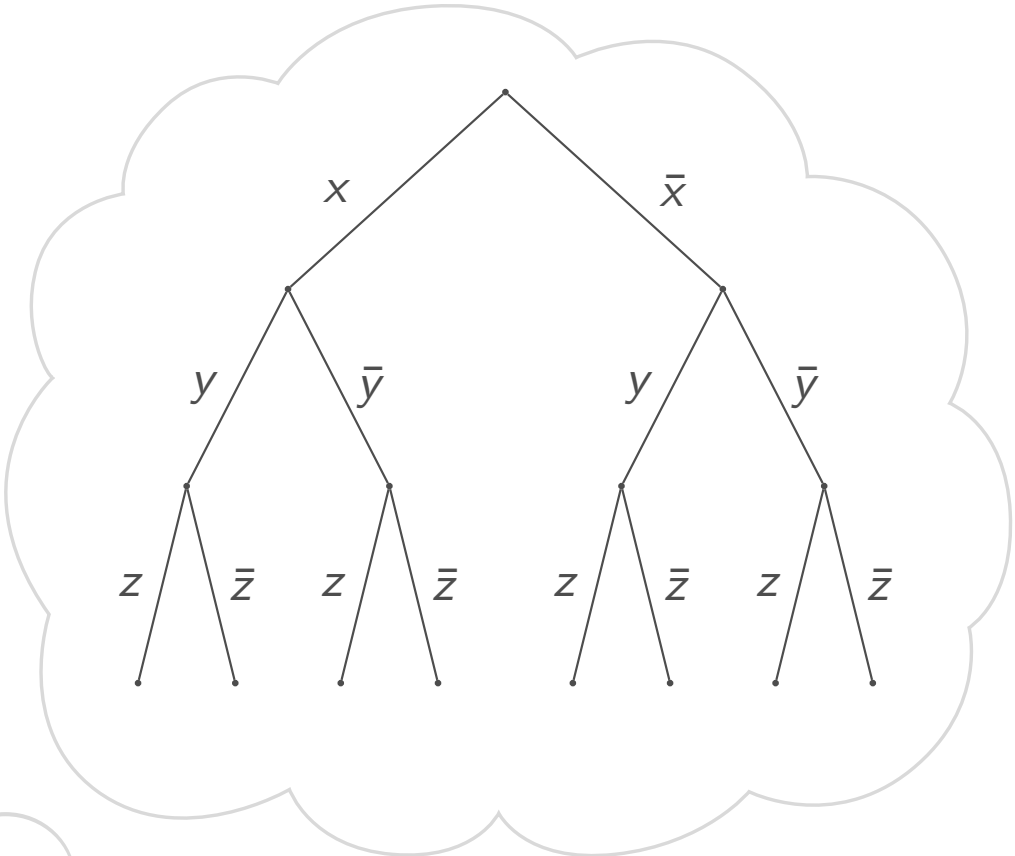


Satisfiable

Key Idea: Prune Less Satisfiable Branches

Can we prune earlier?
Even satisfiable branches?



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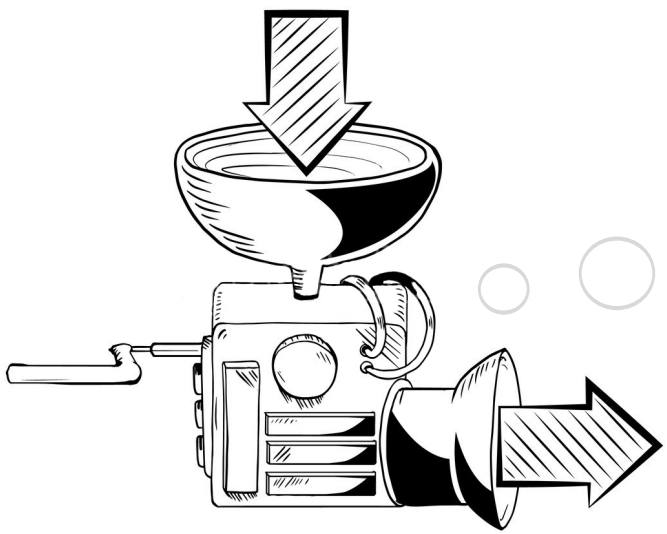

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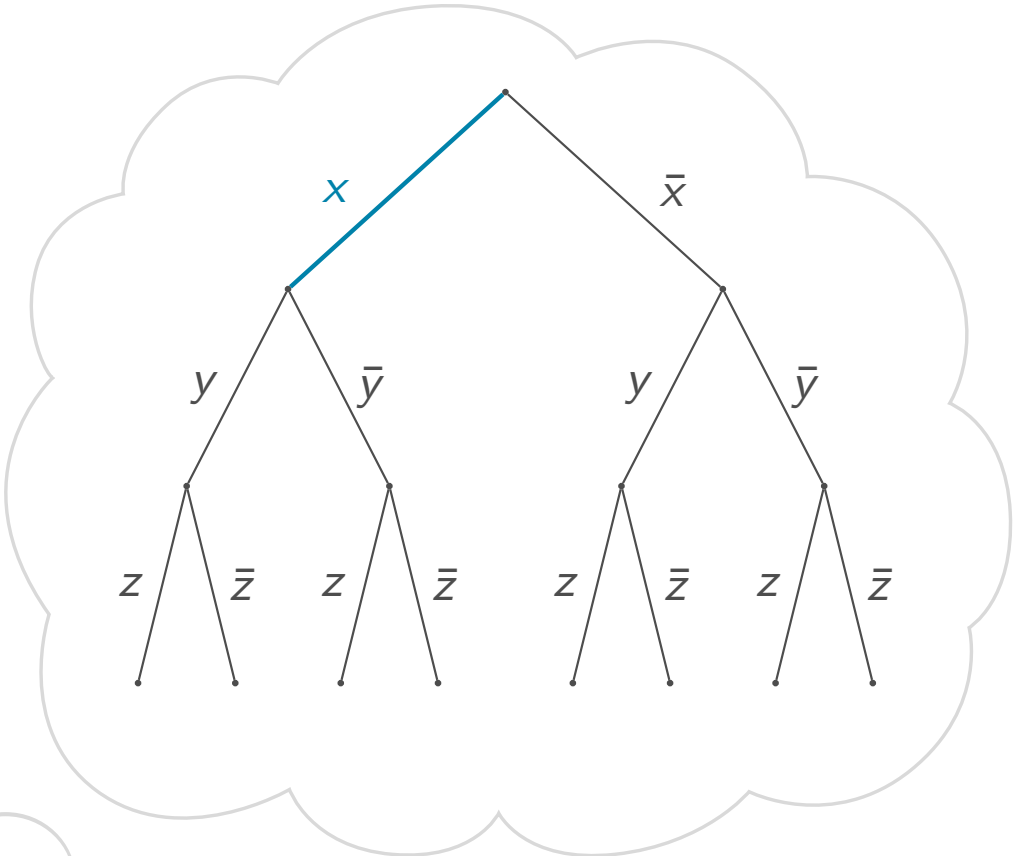
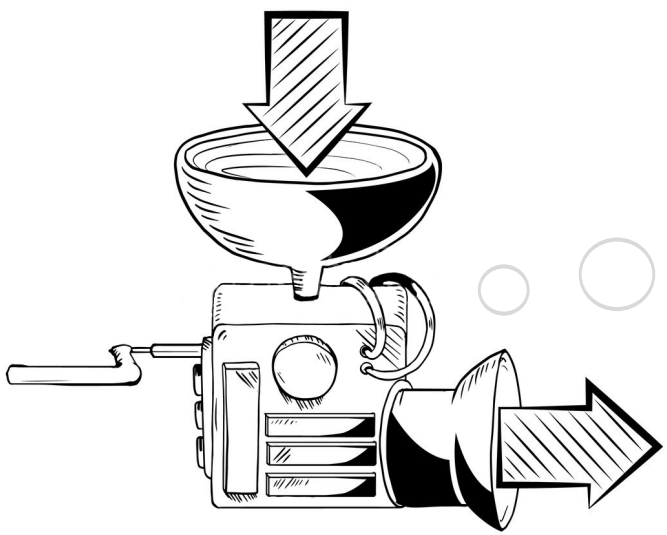
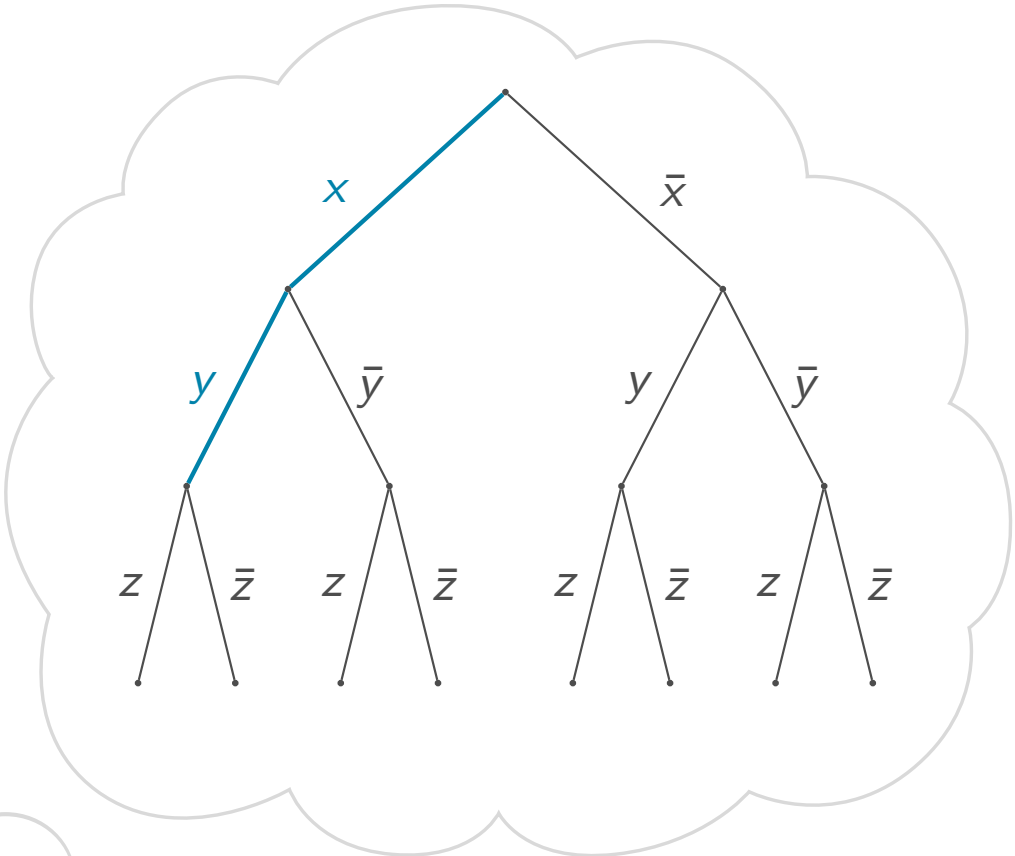


$(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$



Key Idea: Prune Less Satisfiable Branches

Can we prune earlier?
Even satisfiable branches?



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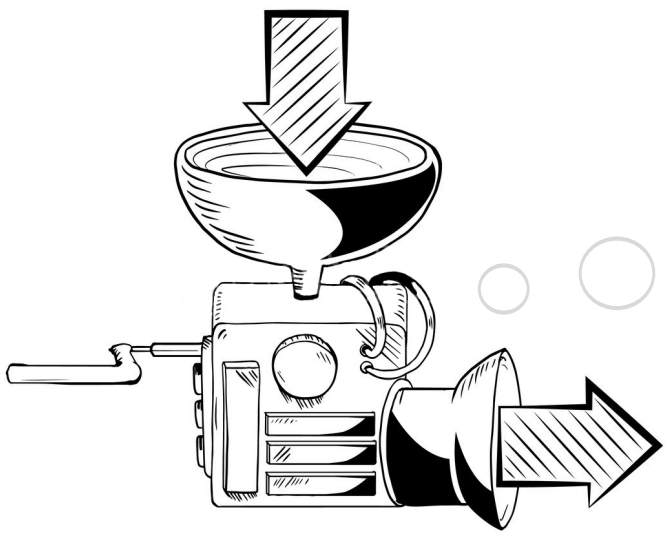

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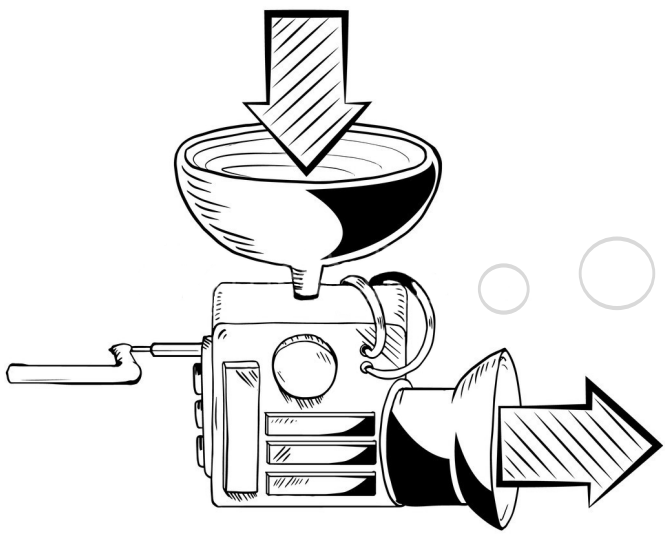

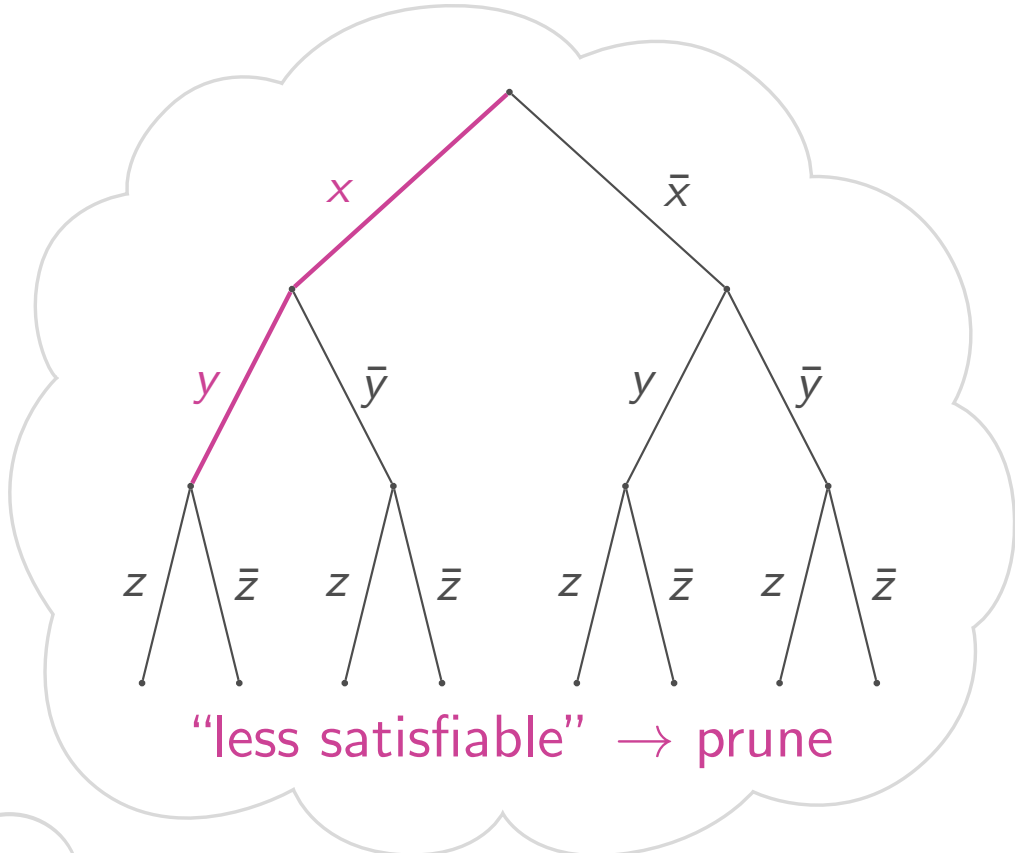

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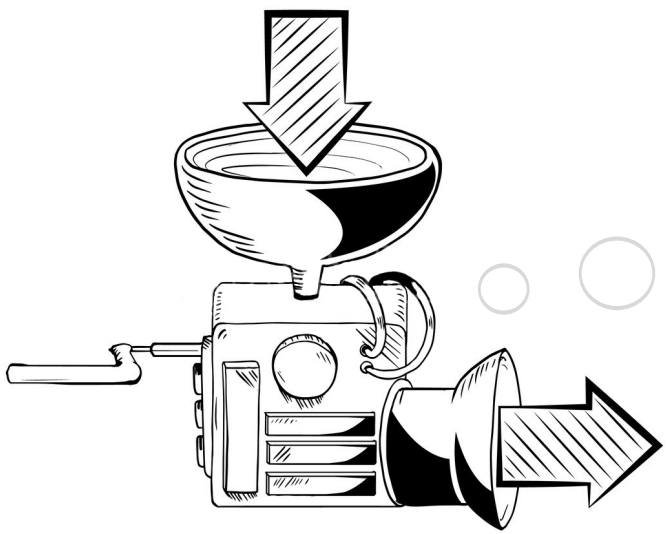
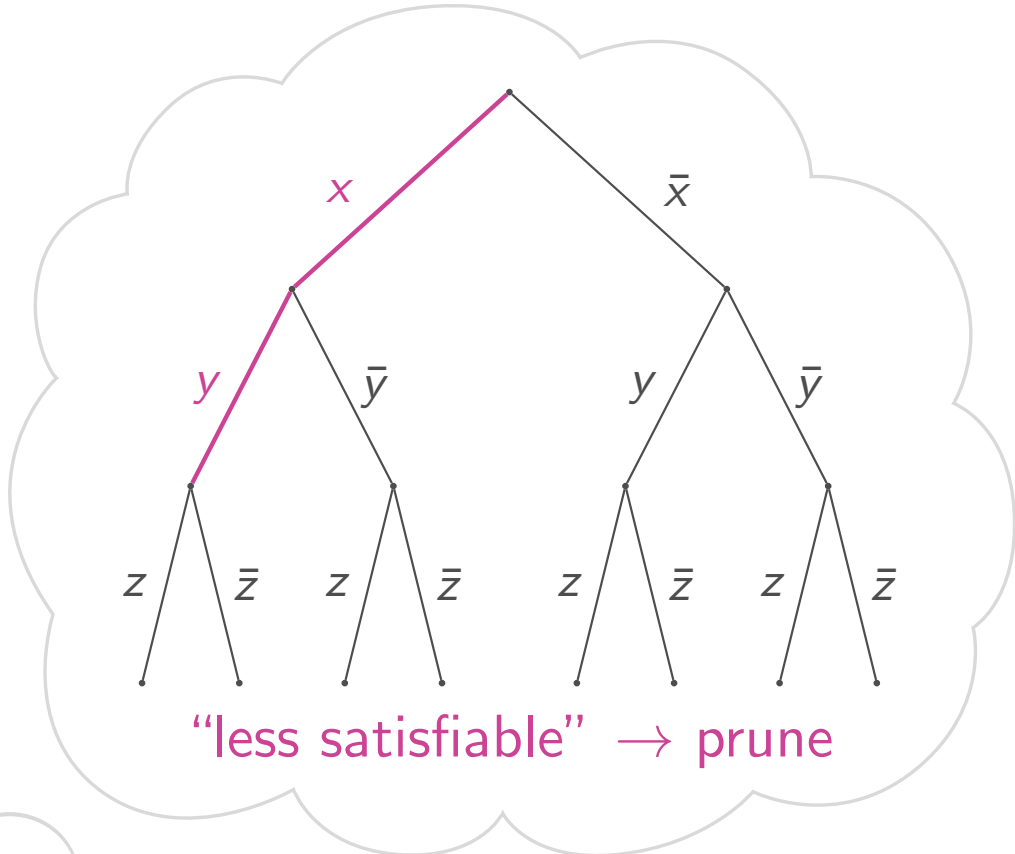
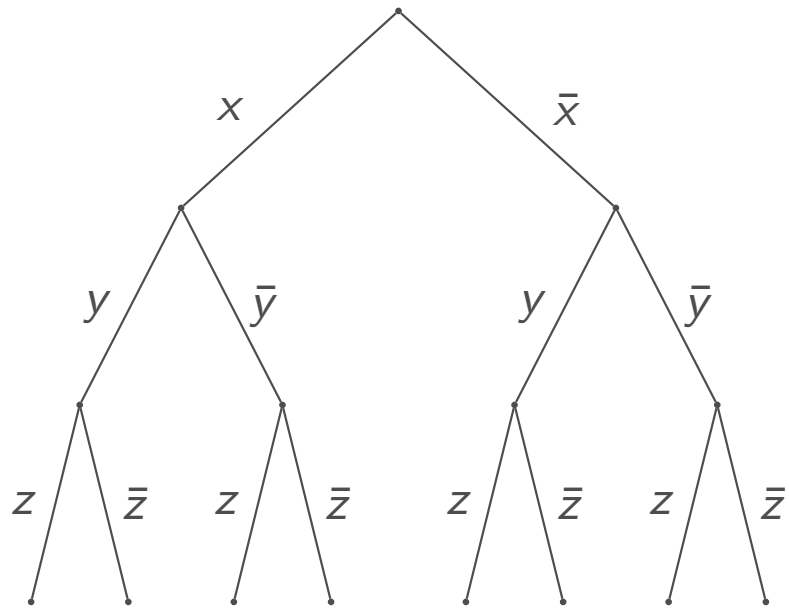




How to prune? Add redundant clauses!

Pruning via Clause Addition

- A clause prunes all branches that falsify the clause.



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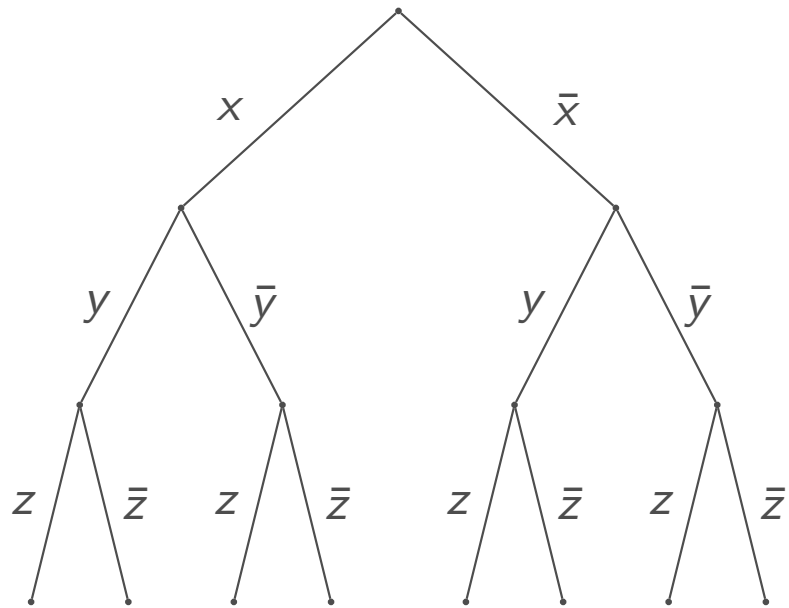

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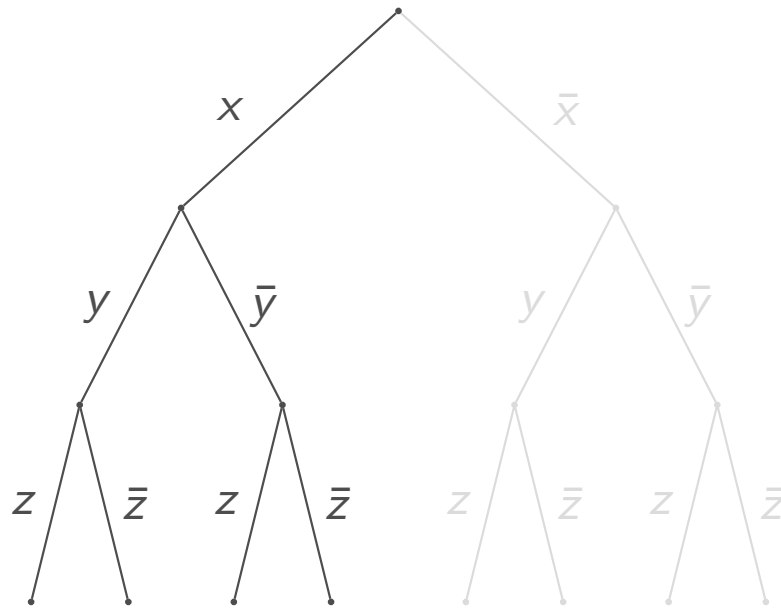

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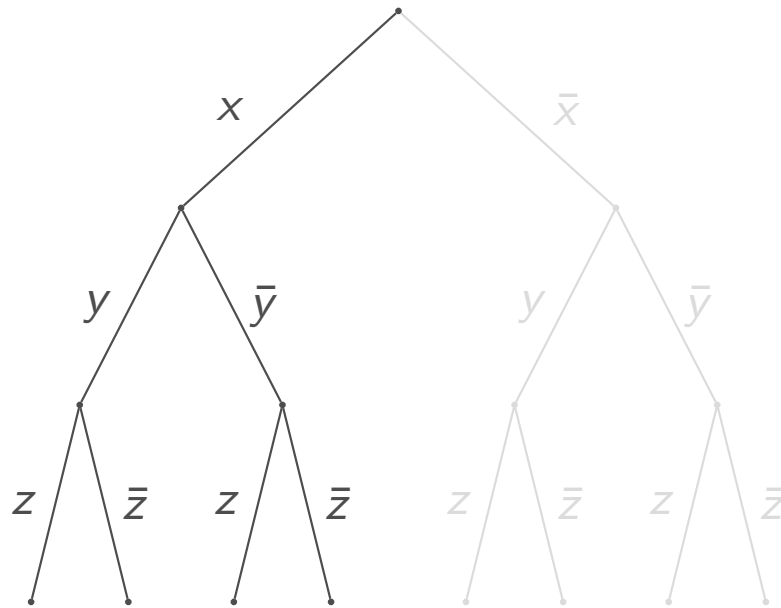

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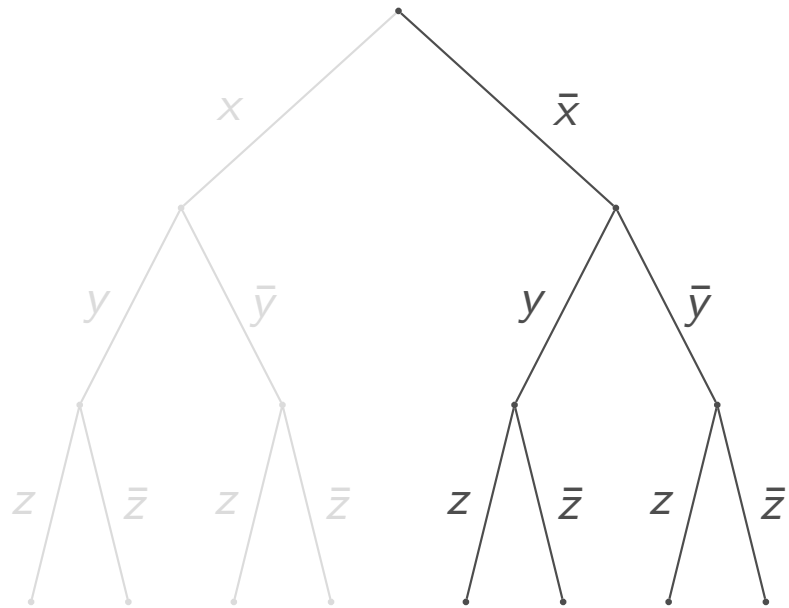

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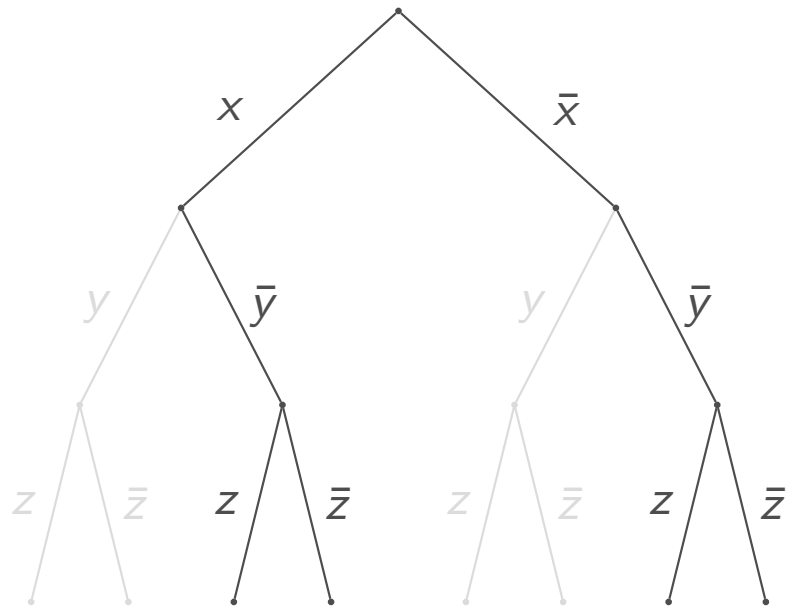

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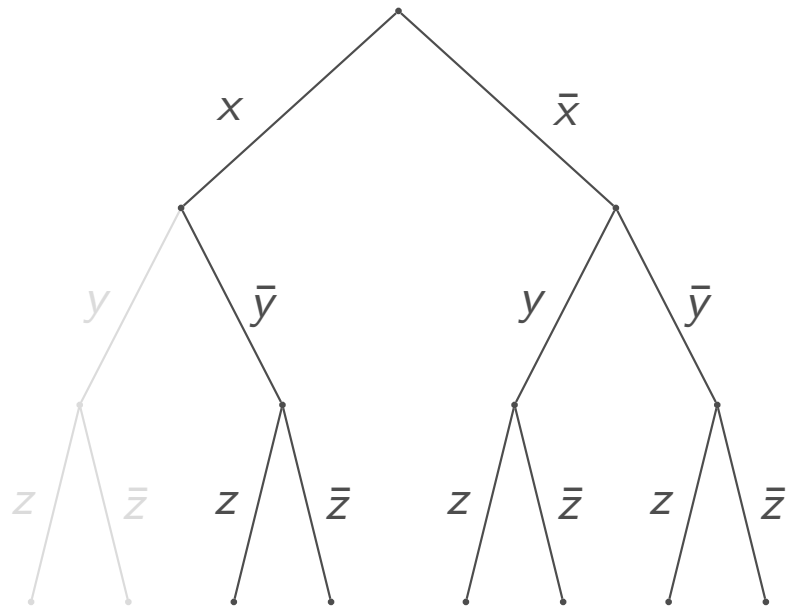

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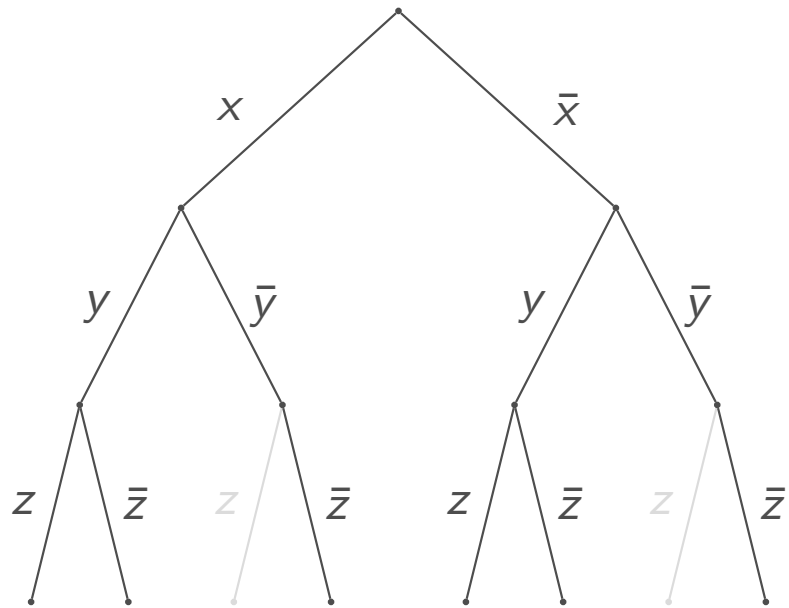

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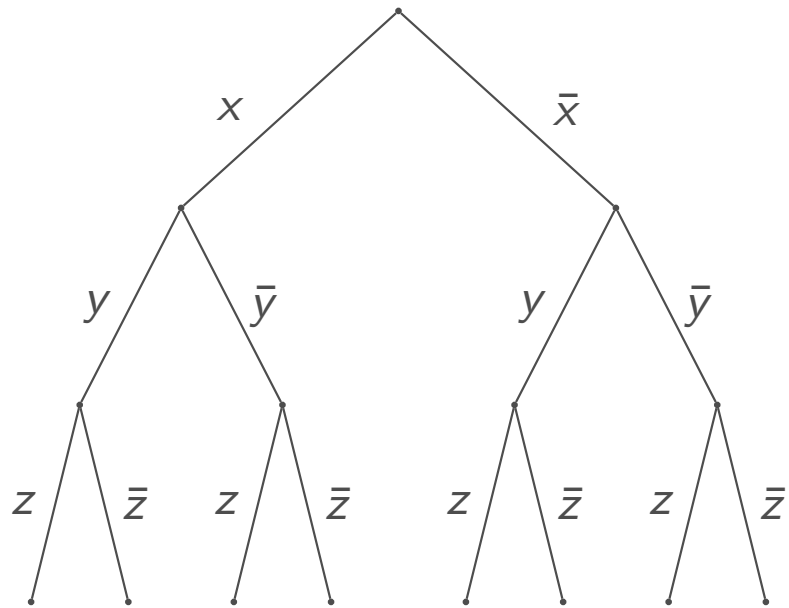
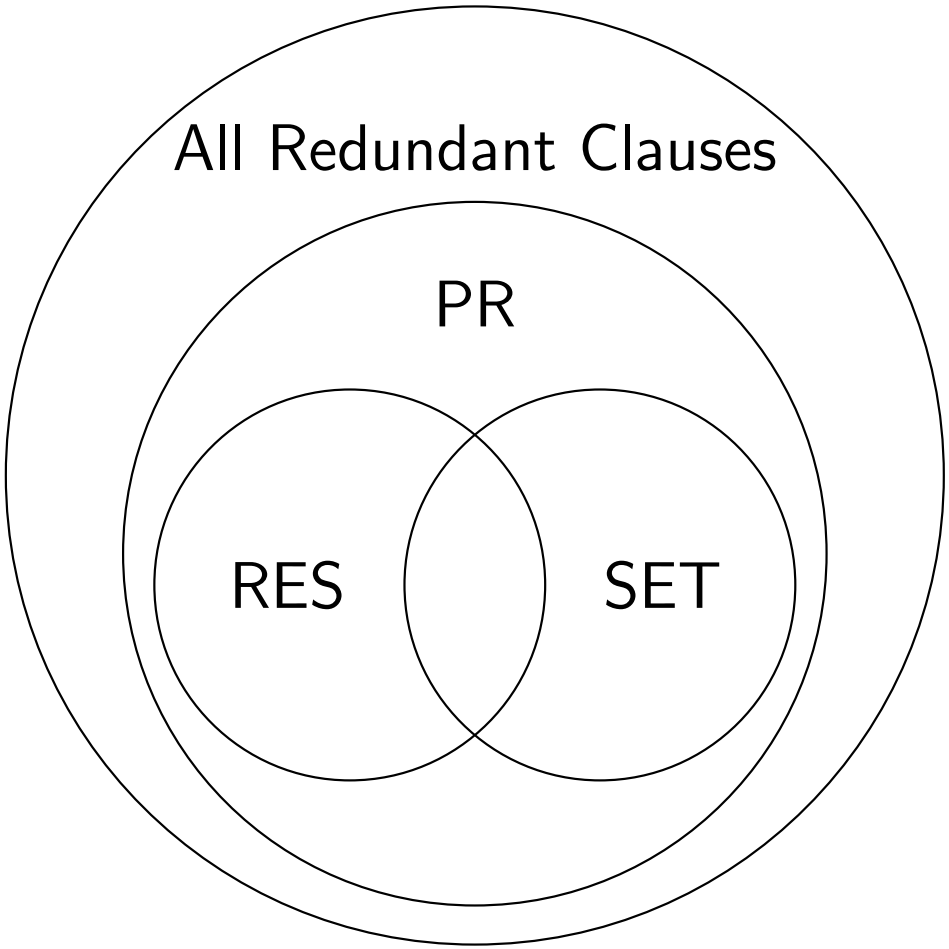


Redundant Clauses

A clause C is redundant w.r.t. a formula F if and only if F and $F \wedge C$ are either both satisfiable or both unsatisfiable.



PR = Propagation Redundant Clauses [CADE'17]

RES = Resolvents

SET = Set-Blocked Clauses [IJCAR'16]

Finding Redundant Clauses: The Positive Reduct

Determining whether a clause C is SET or PR w.r.t. a formula F is an NP-complete problem.

How to find SET and PR clauses? Encode it in SAT!

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Positive reducts are typically very easy to solve!

Key Idea: While solving a formula F, check whether the positive reduct of F and the current assignment α is satisfiable. In that case, prune the branch α.

The Positive Reduct: An Example

Given a formula F and a clause C. Let α denote the smallest assignment that falsifies C. The positive reduct of F and α, denoted by $p(F, \alpha)$, is the formula that contains C and all *assigned*(D, α) with $D \in F$ and D is satisfied by α.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.

Let $C_1 = (\bar{x})$, so $\alpha_1 = x$.
The positive reduct $p(F, \alpha_1) = (\bar{x}) \wedge (x) \wedge (x)$ is unsatisfiable.

Let $C_2 = (\bar{x} \vee \bar{y})$, so $\alpha_2 = x\,y$.
The positive reduct $p(F, \alpha_2) = (\bar{x} \vee \bar{y}) \wedge (x \vee y) \wedge (x \vee \bar{y})$ is satisfiable.

Autarkies

A non-empty assignment α is an autarky for formula F if every clause $C \in F$ that is touched by α is also satisfied by α.

A pure literal and a satisfying assignment are autarkies.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.
Assignment $\alpha_1 = \bar{z}$ is an autarky: $(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.
Assignment $\alpha_2 = x\,\bar{y}\,z$ is an autarky: $(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.

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Given an assignment α, $F|_\alpha$ denotes a formula F without the clauses satisfied by α and without the literals falsified by α.

Theorem ([Monien and Speckenmeyer 1985])

Let α be an autarky for formula F.
Then, F and $F|_\alpha$ are satisfiability equivalent.

Conditional Autarkies

An assignment $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ is a conditional autarky for formula F if α_{aut} is an autarky for $F|_{\alpha_{\text{con}}}$.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.
Let $\alpha_{\text{con}} = x$ and $\alpha_{\text{aut}} = \bar{y}$, then $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}} = x\,\bar{y}$ is a conditional autarky for F:

$$\alpha_{\text{aut}} = \bar{y} \text{ is an autarky for } F|_{\alpha_{\text{con}}} = (\bar{y} \vee \bar{z}).$$

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Let $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ be a conditional autarky for formula F.
Then F and $F \wedge (\alpha_{\text{con}} \rightarrow \alpha_{\text{aut}})$ are satisfiability-equivalent.

In the above example, we could therefore learn $(\bar{x} \vee \bar{y})$.

Learning PR clauses

Theorem
Given a formula F and an assignment α. Every satisfying assignment ω of $p(F, \alpha)$ is a conditional autarky of F.

Recall: Given a formula F and a clause C. Let α denote the smallest assignment that falsifies C. C is SET w.r.t. F if and only if $p(F, \alpha)$ is satisfiable.

Let assignment ω satisfy $p(F, \alpha)$. Removing all but one of the literals in C that are satisfied by ω results in a PR clause w.r.t. F.

Pseudo-Code of CDCL (formula F)

```
1    α := ∅
2    forever do
3       α := Simplify (F, α)
4       if F|α contains a falsified clause then
5          C := AnalyzeConflict ()
6          if C is the empty clause then return unsatisfiable
7          F := F ∪ {C}
8          α := BackJump (C, α)
13      else
14         l := Decide ()
15         if l is undefined then return satisfiable
16         α := α ∪ {l}
```

Pseudo-Code of SDCL (formula F)

```
1   α := ∅
2   forever do
3     α := Simplify (F, α)
4     if F|α contains a falsified clause then
5       C := AnalyzeConflict ()
6       if C is the empty clause then return unsatisfiable
7       F := F ∪ {C}
8       α := BackJump (C, α)
9     else if p(F, α) is satisfiable then
10      C := AnalyzeWitness ()
11      F := F ∪ {C}
12      α := BackJump (C, α)
13    else
14      l := Decide ()
15      if l is undefined then return satisfiable
16      α := α ∪ {l}
```

Benchmark Suite: Pigeon Hole Formulas

Can $n+1$ pigeons be placed in n holes (at-most-one pigeon per hole)?

$$PHP_n := \bigwedge_{1 \leq p \leq n+1} (x_{1,p} \vee \cdots \vee x_{n,p}) \wedge \bigwedge_{1 \leq h \leq n,\, 1 \leq p < q \leq n+1} (\overline{x}_{h,p} \vee \overline{x}_{h,q})$$

The binary clauses encode the constraint $\leq_1 (x_{h,1}; \ldots; x_{h,n+1})$.

There exists more compact encodings, such as the sequential counter and minimal encoding, for at-most-one constraints.

We include these encodings to evaluate the robustness of the solver.

Tool Comparison

We used three tools in our evaluation:

- EBDDRES: A tool based on binary decision diagrams that can convert a refutation into an extended resolution proof.
- GLUCOSER: A SAT solver with extended learning, i.e., a technique that introduces new variables and could potentially solve pigeon hole formulas in polynomial time.
- LINGELING (PR): Our SDCL solver.

Results on Small Pigeon Hole Formulas

formula	input #var	input #cls	EBDDRES time	EBDDRES #node	GLUCOSER time	GLUCOSER #lemma	LINGELING (PR) time	LINGELING (PR) #lemma
PHP_{10}-std	110	561	1.00	3M	22.71	329,470	0.07	329
PHP_{11}-std	132	738	3.47	9M	146.61	1,514,845	0.11	439
PHP_{12}-std	156	949	10.64	27M	307.29	2,660,358	0.16	571
PHP_{13}-std	182	1,197	30.81	76M	982.84	6,969,736	0.22	727
PHP_{10}-seq	220	311	OF	——	1.62	25,712	0.07	327
PHP_{11}-seq	264	375	OF	——	6.94	77,747	0.10	437
PHP_{12}-seq	312	445	OF	——	19.40	174,084	0.14	569
PHP_{13}-seq	364	521	OF	——	172.76	1,061,318	0.18	725
PHP_{10}-min	180	281	28.60	81M	0.64	15,777	0.06	329
PHP_{11}-min	220	342	143.92	399M	1.82	34,561	0.10	439
PHP_{12}-min	264	409	OF	——	9.87	121,321	0.13	571
PHP_{13}-min	312	482	OF	——	57.66	483,789	0.18	727

OF = 32-bit overflow

Results on Large Pigeon Hole Formulas

formula	input #var	input #cls	EBDDRES time	EBDDRES #node	GLUCOSER time	GLUCOSER #lemma	LINGELING (PR) time	LINGELING (PR) #lemma
PHP_{20}-std	420	4,221	OF	——	TO	——	1.61	2,659
PHP_{30}-std	930	13,981	OF	——	TO	——	13.45	8,989
PHP_{40}-std	1,640	32,841	OF	——	TO	——	67.41	21,319
PHP_{50}-std	2,550	63,801	OF	——	TO	——	241.14	41,649
PHP_{20}-seq	840	1,221	OF	——	TO	——	1.05	2,657
PHP_{30}-seq	1,860	2,731	OF	——	TO	——	6.55	8,987
PHP_{40}-seq	3,280	4,841	OF	——	TO	——	27.10	21,317
PHP_{50}-seq	5,100	7,551	OF	——	TO	——	86.30	41,647
PHP_{20}-min	760	1,161	OF	——	TO	——	1.03	2,659
PHP_{30}-min	1,740	2,641	OF	——	TO	——	6.30	8,989
PHP_{40}-min	3,120	4,721	OF	——	TO	——	26.65	21,319
PHP_{50}-min	4,900	7,401	OF	——	TO	——	85.00	41,649

OF = 32-bit overflow TO = timeout of 9000 seconds

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  - They allow for short proofs without new variables.

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- We introduced new redundancy notions for SAT.
- The redundancy notions strictly generalize RAT.
- Proof systems based on these redundancy notions are strong.
  - They allow for short proofs without new variables.
- SDCL generalizes the well-known CDCL paradigm by allowing to prune branches that are potentially satisfiable:
  - Such branches can be found using the positive reduct;
  - Pruning can be expressed in the PR proof system;
  - Runtime and proofs can be exponentially smaller.