Brownian motion, Ricci curvature, functional inequalities and geometric flows

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Outline

1. Stochastic analysis of static and evolving manifolds
2. Characterizing Ricci curvature by functional inequalities
3. Heat equations under geometric flows and entropy formulas

I. Stochastic analysis with respect to time dependent metrics

Let $g(t)$ be a C^1 family of Riemannian metrics on a manifold M, $t \in I$ where $I = [0, T^*[$ or $\mathbb{R}_+$.

- A continuous adapted process X is called Brownian motion with respect to $g(t)$ if

  $\forall f \in C^\infty(M)$,

  $$d(f(X_t)) - (\Delta_{g(t)} f)(X_t)\, dt = 0 \quad \text{(mod loc mart)}$$

- We call X shortly a $g(t)$-Brownian motion on M.

- We use the notation

  $$X_t = X_t^{(x,s)}, \quad t \geq s, \quad \text{if } X_s = x.$$

Geometries evolving in time: Deformation of Riemannian metrics $g(t)$ under certain evolution equations

Eminent example Ricci flow (R. Hamilton, 1982)

- Start with a given metric g_0 on M and let it evolve under

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)}, \quad g(0) = g_0$$

- Idea behind Ricci flow: Ricci flow works as heat equation on the space of Riemannian metrics.
- For instance, in terms of local coordinates x_i, if $\Delta x_i = 0$, then

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\Delta g_{ij} \ + \text{lower order terms.}$$

- The scalar curvature $\mathrm{Scal} := \mathrm{trace}\,\mathrm{Ric}$ satisfies the reaction-diffusion equation

$$\frac{\partial}{\partial t}\mathrm{Scal} = \Delta \mathrm{Scal} + 2|\mathrm{Ric}|^2.$$

Depending on the sign $\pm$ in

$$\frac{\partial}{\partial t} g(t) = \pm 2 \operatorname{Ric}_{g(t)}, \quad g(0) = g_0$$

we talk about backward/forward Ricci flow.

Corollary (Evolution equation for densities)

Let

$$\mathbb{P}\{X_t^{(x,s)} \in dy\} = p(x,s;y,t)\,\mathrm{vol}_t(dy), \quad s < t\,,$$

where $\mathrm{vol}_t(dy)$ *is the Riemannian volume on* $(M, g(t))$.

Then

$$P_{s,t}f(x) = \int_M p(x,s;y,t)\,f(y)\,\mathrm{vol}_t(dy).$$

Let $p_t = p(x,s;\cdot,t)$.

For the forward Ricci flow, we have:

$$\frac{d}{dt}p_t = \Delta_{g(t)}p_t + \mathrm{Scal}(\cdot,t)\,p_t.$$

For the backward Ricci flow, we have:

$$\frac{d}{dt}p_t = \Delta_{g(t)}p_t - \mathrm{Scal}(\cdot,t)\,p_t.$$

Heat equation under moving Riemannian metrics

- Study the heat equation under Ricci flow
- Consider positive solutions u to the heat equation:

$$\begin{cases} \frac{\partial}{\partial t} u - \Delta_{g(t)} u = 0 \\ \frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} \end{cases}$$

or to the conjugate heat equation

$$\begin{cases} \frac{\partial}{\partial t} u + \Delta_{g(t)} u - \mathrm{Scal}(t,\cdot) u = 0 \\ \frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} \end{cases}$$

Motivation comes from Perelman's work

Brownian motion on $(M, g(t))$

- Let $\mathbb{M} := M \times I$ be space time and consider the tangent bundle TM over $\mathbb{M}$:

$$TM \xrightarrow{\pi} \mathbb{M}, \quad \pi \text{ projection.}$$

- There is a natural *space-time connection* on TM, considered as bundle over space-time $\mathbb{M}$, defined by

$$\nabla_X Y = \nabla_X^{g_t} Y \quad \text{and} \quad \nabla_{\partial_t} Y = \partial_t Y + \frac{1}{2}(\partial_t g_t)(Y, \cdot)^{\sharp g_t}, \quad g_t = g(t).$$

- This connection is compatible with the metric, i.e.

$$\frac{d}{dt}|Y|^2_{g_t} = 2\langle Y, \nabla_{\partial_t} Y\rangle_{g_t}$$

- This connection allows to define parallel transport along curves, but curves in space-time $\mathbb{M}$, typically of the form

$$\gamma_t = (x_t, \rho_t), \quad t \in [0, T]$$

where ρ_t is a monotone C^1 transformation of $[0, T]$, e.g.

$$\rho_t = t \quad \text{and} \quad \rho_t = T - t.$$

- Let $(M,g_t)_{t\in I}$ where $[0,T]\subset I\subset\mathbb{R}_+$. Stochastic development then gives space-time Brownian motions, like

$$(X_r, r) \quad \text{or} \quad (X_r, T-r)$$

- More precisely, consider the O(n)-principal bundle of orthonormal frames

$$\mathcal{F} \xrightarrow{\pi} \mathbb{M}$$

with fibres

$$\mathcal{F}_{(x,t)} = \{u\colon \mathbb{R}^n \to (T_xM, g_t) \mid u \text{ isometry}\}$$

and

$$\pi\colon \mathcal{F} \to \mathbb{M} = M\times I \quad \text{where } \pi(u) = (x,t) \text{ if } u\in\mathcal{F}_{(x,t)}.$$

- Let

$$T\mathcal{F} = V\oplus H := \ker d\pi \oplus h(\pi^* T\mathbb{M})$$

be the induced splitting of $T\mathcal{F}$.

- In terms of the *horizontal lift* of the G-connection,

$$h_u\colon\ T_{\pi(u)}\mathbb{M} \xrightarrow{\sim} H_u, \quad u \in \mathcal{F},$$

  we get to each

$$\alpha X + \beta\partial_t \in T_{(x,t)}\mathbb{M}$$

  and each frame $u \in \mathcal{F}_{(x,t)}$, a unique "horizontal lift"

$$\alpha X^* + \beta D_t \in H_u$$

  of $\alpha X + \beta\partial_t$ such that

$$\pi_*(\alpha X^* + \beta D_t) = \alpha X + \beta\partial_t.$$

- In terms of the standard-horizontal vector fields on $T\mathcal{F}$,

$$H_i \in \Gamma(T\mathcal{F}), \quad H_i(u) = (ue_i)^* \equiv h_u(ue_i), \quad i = 1,\ldots,n,$$

  we define Bochner's horizontal Laplacian on $\mathcal{F}$:

$$\Delta_{\mathrm{hor}} = \sum_{i=1}^{n} H_i^2.$$

- For $\rho_t : [0, T] \to [0, T]$ monotone (here: $\rho_t = t$ or $\rho_t = T - t$), define

$$D_t^\rho := \dot{\rho}(t)\, D_t = \pm D_t.$$

- Consider the following Stratonovich SDE on $\mathcal{F}$:

$$dU = \pm D_t(U)\, dt + \sum_{i=1}^{n} H_i(U) \circ dZ^i, \quad U_0 = u,$$

where Z is a continuous semimartingale taking values in $\mathbb{R}^n$.
- If U solves the SDE then

$$\pi(U_t) = (X_t, \rho_t)$$

for some process X on M, the *stochastic development* of Z.

- Modulo choice of initial conditions each of the three processes X, U, Z determines the two others.
  (1) We call (X_t, ρ_t) a (space-time) Brownian motion if Z is a Brownian motion on $\mathbb{R}^n$.
  (2) We call (X_t, ρ_t) a (space-time) martingale if Z is a local martingale on $\mathbb{R}^n$.
- Let

$$//_{r,s} := U_s \circ U_r^{-1} : (T_{x_r}M, g_{\rho_r}) \to (T_{x_s}M, g_{\rho_s}), \quad 0 \le r \le s \le T,$$

  be parallel transport along X_t (by construction consisting of isometries!). For the sake of brevity write $//_s := //_{0,s}$.
- In the special case $\rho_t = t$, resp. $\rho_t = T - t$, we call (X_t, t), resp. $(X_t, T - t)$ a Brownian motion on $\mathbb{M}$ based at $(x, 0)$, resp. based at (x, T), if $X_0 = x$. In the same way, we talk about martingales on $\mathbb{M}$ based at $(x, 0)$, resp. (x, T).

II. First application: gradient-entropy estimate

- Assume that all manifolds (M, g_t) are complete ($t \in I$). Let $u \colon \mathbb{M} \to \mathbb{R}$ be a positive solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta_{g_t} u.$$

- It is straightforward to check:

$$\left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right)(u \log u) = \frac{|\nabla u|^2}{u},$$

$$\left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right)\frac{|\nabla u|^2}{u} = u\left(2|\nabla\nabla \log u|^2 + \left(2\mathrm{Ric} + \frac{\partial g}{\partial t}\right)\left(\frac{\nabla u}{u}, \frac{\nabla u}{u}\right)\right)$$

- Now assume that

$$\frac{\partial g}{\partial t} \geq -2\mathrm{Ric},$$

i.e. (g_t) is a supersolution to the Ricci flow.

- Then, if $(X_t, T-t)$ is a Brownian motion based at (x, T) with $T \in I$, it is straightforward to check that the process

$$N_t := (T-t)\,\frac{|\nabla u|^2}{u}(X_t, T-t) + \big(u \log u\big)(X_t, T-t), \quad 0 \le t \le T,$$

  is a local submartingale.
- Hence assuming that N_t is a true submartingale, we obtain $\mathbb{E}[N_0] \le \mathbb{E}[N_T]$ which gives

$$T\,\frac{|\nabla u|^2}{u}(x, T) + \big(u \log u\big)(x, T) \le \mathbb{E}\Big[\big(u \log u\big)(X_T, 0)\Big].$$

Theorem

Keeping assumptions as above. For each positive solution $u : [0,T] \times M \to \mathbb{R}_+$ *to the time-dependent heat equation, we have*

$$\left|\frac{\nabla u}{u}\right|^2 (x,T) \leq \frac{1}{T}\,\mathbb{E}\left[\frac{u(X_T,0)}{u(x,T)} \log \frac{u(X_T,0)}{u(x,T)}\right].$$

In particular,

(1) *Then, for any* $\delta > 0$,

$$\left|\frac{\nabla u}{u}\right|(x,T) \leq \frac{\delta}{2T} + \frac{1}{2\delta}\,\mathbb{E}\left[\frac{u(X_T,0)}{u(x,T)} \log \frac{u(X_T,0)}{u(x,T)}\right]$$

(2) *(Hamilton's gradient estimate in global form)*
If $m_T := \sup_{M\times[0,T]} u$, *then*

$$\frac{|\nabla u|}{u}(x,T) \leq \frac{1}{T^{1/2}} \sqrt{\log \frac{m_T}{u(x,T)}}.$$

III. Characterization of bounded Ricci curvature (static case)

Our setting

- (Process) X_t is an L-diffusion where

$$L = \Delta + Z \quad \text{with } Z \in \Gamma(TM)$$

- Assume that $\mathrm{Ric}^Z = \mathrm{Ric} - \nabla Z$

$$\mathrm{Ric}^Z(X, Y) = \mathrm{Ric}(X, Y) - \langle \nabla_X Z, Y \rangle,$$

  is bounded below, i.e., for some constant K,

$$\mathrm{Ric}^Z(X, X) \geq K|X|^2, \quad X \in TM.$$

- Probabilistic ingredients:
  - (Semigroup formula)

$$P_t f(x) = \mathbb{E}[f(X_t^x)], \quad t \geq 0.$$

  - (Derivative formula)

$$(\nabla P_t f)(x) = \mathbb{E}\left[Q_t //_t^{-1} \nabla f(X_t^x)\right], \quad t \geq 0.$$

Our focus

- For real constants $k_1 \leq k_2$, how to characterize

$$k_1 \leq \mathrm{Ric}^Z \leq k_2.$$

  in terms of functional inequalities for the semigroup P_t.
- Natural extensions:
  - Pointwise pinched curvature conditions

$$k_1(x) \leq \mathrm{Ric}^Z_x \leq k_2(x), \quad x \in M$$

  - Riemannian manifolds with a boundary
  - Manifolds evolving under a geometric flow

Well-known classical results: Let K be a real constant. The following conditions are equivalent:

- **(Bakry-Émery lower curvature bound)**

$$\mathrm{CD}(K,\infty) \qquad \mathrm{Ric}^Z(X,X) \geq K|X|^2, \quad X \in TM;$$

- **(gradient estimate)** for all $f \in C_c^\infty(M)$,

$$|\nabla P_t f|^2 \leq e^{-2Kt} P_t |\nabla f|^2;$$

- **(Poincaré inequality)** for all $p \in (1,2]$ and $f \in C_c^\infty(M)$,

$$\frac{p}{4(p-1)}\left(P_t f^2 - (P_t f^{2/p})^p\right) \leq \frac{1-e^{-2Kt}}{2K} P_t |\nabla f|^2;$$

- **(log-Sobolev inequality)** for all $f \in C_c^\infty(M)$,

$$P_t(f^2 \log f^2) - (P_t f^2)\log(P_t f^2) \leq \frac{2(1-e^{-2Kt})}{K} P_t |\nabla f|^2.$$

- ...

Equivalent statements to Bakry-Émery curvature condition

- **Gradient inequalities (Gaussian isoperimetric function), Poincaré inequalities, log-Sobolev inequalities**
  Bakry (1994,1997); Bakry-Émery (1984);
  Bakry-Ledoux (1996); Ledoux (2000); …

- **Transportation-cost inequalities; convexity properties of the entropy**
  von Renesse-Sturm (2005); Lott-Villani, Sturm, etc …

- **Wang's dimension-free Harnack inequalities**
  Wang (1997, 2004, 2010); …

- **Wang's log-Harnack inequalities**
  Arnaudon-Wang-A.Th. (2014); …

- **Harnack type inequalities**
  Bakry-Gentil-Ledoux (2012); …

- …

Natural questions:

- How to characterize upper bounds for Ric^Z?
- How to characterize pinched bounds for Ric^Z?

Well-known:

Boundness of $|\mathrm{Ric}^Z|$, i.e.

$$|\mathrm{Ric}^Z| \leq K,$$

implies certain functional inequalities on path space,

e.g. Capitaine-Hsu-Ledoux (1997), Chen-Wu (2014), Driver (1992), Hsu (1994)

Boundedness of $|\mathrm{Ric}^Z|$

The problem of characterizing boundedness of Ric^Z has been solved by A. Naber and R. Haslhofer via analysis on path space:

Boundedness of $|\mathrm{Ric}^Z| \Longleftrightarrow$ functional inequalities on path space

- Aaron Naber, *Characterizations of bounded Ricci curvature on smooth and nonsmooth spaces*, arXiv:1306.6512v4 (2015)
- Robert Haslhofer and Aaron Naber, Characterizations of the Ricci flow, J. Eur. Math. Soc. (2016)

Our work:

- Li-Juan Cheng and A.Th.: *Characterization of pinched Ricci curvature by functional inequalities*, J. Geom. Anal. (2017)
- Li-Juan Cheng and A.Th.: *Spectral gap on Riemannian path space over static and evolving manifolds*, J. Funct. Anal. **274** (2018), 959-984

IV. Analysis on path space

- For fixed $T > 0$, let $W^T = C([0,T];M)$ and

$$\mathscr{F}C_{0,T}^{\infty} = \Big\{W^T \ni \gamma \mapsto f(\gamma_{t_1},\ldots,\gamma_{t_n}): \\ 0 < t_1 < \ldots < t_n \le T,\ f \in C_c^{\infty}(M^n)\Big\}.$$

  be the class of smooth cylindrical functions on W^T.

- Denote

$$X_{[0,T]} = \{X_t:\ 0 \le t \le T\}.$$

- For $F \in \mathscr{F}C_{0,T}^{\infty}$ with $F(\gamma) = f(\gamma_{t_1},\ldots,\gamma_{t_n})$, the intrinsic gradient is defined as

$$D_t^{//} F(X_{[0,T]}) = \sum_{i=1}^{n} \mathbb{1}_{\{t<t_i\}} //_{t,t_i}^{-1} \nabla^i f(X_{t_1},\ldots,X_{t_n}), \quad t \in [0,T],$$

  where ∇^i denotes the gradient with respect to the i-th component.

Theorem [A. Naber (2015) and R. Haslhofer and A. Naber (2016)]

The following conditions are equivalent ($K \geq 0$):

- $|\mathrm{Ric}^Z| \leq K$
- (Gradient inequality on path space) for $F \in \mathcal{F}C_0^\infty$,

$$\left|\nabla \mathbb{E}[F(X_{[0,T]})]\right| \leq \mathbb{E}\left[|D_0^{//}F| + K\int_0^T e^{Kr}|D_r^{//}F|\,dr\right]$$

- (Quadratic gradient inequality on path space) for $F \in \mathcal{F}C_0^\infty$,

$$\left|\nabla \mathbb{E}[F(X_{[0,T]})]\right|^2 \leq e^{KT}\,\mathbb{E}\left[|D_0^{//}F|^2 + K\int_0^T e^{Kr}|D_r^{//}F|^2\,dr\right].$$

Important observation

It is sufficient to check the estimates for very special $F \in \mathcal{F}C_0^\infty$.

Namely:

- for $F(X^x_{[0,T]}) = f(X^x_t)$, and
- for 2-point cylindrical functions of the form

$$F(X^x_{[0,T]}) = f(x) - \frac{1}{2}f(X^x_t)$$

From this observation, equivalence of the following two items follows:

(i) $|\mathrm{Ric}^Z| \le K$ for $K \ge 0$;

(ii) for $f \in C_c^\infty(M)$ and $t > 0$,

$$|\nabla P_t f|^2 \le e^{2Kt} P_t |\nabla f|^2 \quad \text{and}$$
$$\left|\nabla f - \frac{1}{2}\nabla P_t f\right|^2 \le e^{Kt}\mathbb{E}\left[\left|\nabla f - \frac{1}{2}//_{0,t}^{-1}\nabla f(X_t)\right|^2 + \frac{1}{4}\left(e^{Kt}-1\right)|\nabla f(X_t)|^2\right].$$

Remark The inequalities in (ii) can be combined to the single condition:

$$|\nabla P_t f|^2 - e^{2Kt} P_t |\nabla f|^2 \le 4\Big((e^{Kt}-1)|\nabla f|^2 + \langle \nabla f, \nabla P_t f\rangle - \left\langle \nabla f, e^{Kt}\mathbb{E}[//_{0,t}^{-1}\nabla f(X_t)]\right\rangle\Big) \wedge 0.$$

Theorem (Characterization of pinched Ricci curvature; Cheng-A.Th. 2017)

Let k_1, k_2 be two real constants such that $k_1 \leq k_2$. The following conditions are equivalent:

(i) $k_1 \leq \mathrm{Ric}^Z \leq k_2$

(ii) (Gradient inequalities) *for $f \in C_c^\infty(M)$ and $t > 0$,*

$$|\nabla P_t f|^2 - e^{-2k_1 t} P_t |\nabla f|^2 \leq 4\left[\left(e^{\frac{k_2-k_1}{2}t} - 1\right)|\nabla f|^2 + \langle \nabla f, \nabla P_t f\rangle - e^{-k_1 t}\mathbb{E}\left\langle \nabla f, //_{0,t}^{-1}\nabla f(X_t)\right\rangle\right] \wedge 0$$

(ii') *for $f \in C_c^\infty(M)$ and $t > 0$,*

$$|\nabla P_t f|^2 - e^{-2k_1 t} P_t |\nabla f|^2 \leq 4\left(e^{\frac{k_2-k_1}{2}t}|\nabla P_t f|^2 - e^{-k_1 t}\mathbb{E}\left\langle \nabla P_t f, //_{0,t}^{-1}\nabla f(X_t)\right\rangle\right) \wedge 0$$

Theorem (continuation)

(iii) (Poincaré type inequality) *for* $f \in C_c^\infty(M)$, $p \in (1,2]$, $t > 0$,

$$\frac{p(P_t f^2 - (P_t f^{2/p})^p)}{4(p-1)} - \frac{1-e^{-2k_1 t}}{2k_1} P_t|\nabla f|^2$$
$$\leq 4\int_0^t \left(e^{\frac{k_2-k_1}{2}(t-r)} - 1\right) P_r|\nabla f|^2$$
$$+\mathbb{E}\left\langle \nabla f(X_r), \nabla P_{t-r} f(X_r) - e^{-k_1(t-r)} //_{r,t}^{-1} \nabla f(X_t) \right\rangle dr \wedge 0$$

(iv) (Log-Sobolev inequality) *for* $f \in C_c^\infty(M)$, $t > 0$,

$$\frac{1}{4}\left(P_t(f^2 \log f^2) - P_t f^2 \log P_t f^2\right) - \frac{1-e^{-2k_1 t}}{2k_1} P_t|\nabla f|^2$$
$$\leq 4\int_0^t \left(e^{\frac{k_2-k_1}{2}(t-r)} - 1\right) P_r|\nabla f|^2$$
$$+\mathbb{E}\left\langle \nabla f(X_r), \nabla P_{t-r} f(X_r) - e^{-k_1(t-r)} //_{r,t}^{-1} \nabla f(X_t) \right\rangle dr \wedge 0$$

The proof uses probabilistic formulas for calculating Ric^Z, e.g. Bakry (1994), von Renesse-Sturm (2005), Wang (2014).

Lemma

Let $v \in T_x M$ with $|v| = 1$. Let $f \in C_0^\infty(M)$ such that $\nabla f(x) = v$ and $\mathrm{Hess}_f(x) = 0$. Then,

(i) *for $p > 0$,*

$$\mathrm{Ric}^Z(v, v) = \lim_{t \to 0} \frac{P_t |\nabla f|^p(x) - |\nabla P_t f|^p(x)}{pt}$$

(ii) $\mathrm{Ric}^Z(v, v)$ *is also given by the following two limits:*

$$\begin{aligned} \mathrm{Ric}^Z(v, v) &= \lim_{t \to 0} \frac{\left\{\left\langle \nabla f, \mathbb{E} //_{0,t}^{-1} \nabla f(X_t) \right\rangle - \langle \nabla f, \nabla P_t f \rangle \right\}(x)}{t} \\ &= \lim_{t \to 0} \frac{\left\{\left\langle \nabla P_t f, \mathbb{E} //_{0,t}^{-1} \nabla f(X_t) \right\rangle - |\nabla P_t f|^2 \right\}(x)}{t} \end{aligned}$$

The theorem can be extended in various ways:

- to characterize variable curvature bounds

$$K_1(x) \le \mathrm{Ric}^Z(x) \le K_2(x), \quad x \in M,$$

  with functions K_1, K_2 on M
- to manifolds with boundary (reflecting diffusions generated by $L = \Delta + Z$) to characterize

$$K_1(x) \le \mathrm{Ric}^Z(x) \le K_2(x), \quad x \in M,$$
$$\sigma_1(x) \le \mathrm{II}(x) \le \sigma_2(x), \quad x \in \partial M,$$

  in terms of semigroups with Neumann boundary conditions.

  The second fundamental form of ∂M is given by

$$\mathrm{II}(X, Y) = -\langle \nabla_X N, Y \rangle, \quad X, Y \in T_x\partial M,\ x \in \partial M,$$

  where N is the inward normal unit vector field on ∂M.
- The theorem allows to characterize
  - Einstein manifolds (Ric is a multiple of the metric g)
  - Ricci solitons ($\mathrm{Ric} + \mathrm{Hess} f = c\, g$)
  - manifolds such that $\mathrm{Ric} = \nabla Z$
  - etc

V. Spectral gap on Riemannian path space

For $F \in \mathscr{F}C^\infty_{0,T}$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, let

$$D_t^{//} F(X^x_{[0,T]}) = \sum_{i=1}^n \mathbb{1}_{\{t<t_i\}} //_{t,t_i}^{-1} \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n})$$

be the *intrinsic gradient*, and

$$D_t F(X^x_{[0,T]}) = \sum_{i=1}^n \mathbb{1}_{\{t<t_i\}} Q_{t,t_i} //_{t,t_i}^{-1} \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n})$$

the *damped gradient*, where $Q_{t,r}$ takes values in the linear automorphisms of $T_{X_t^x} M$ satisfying for fixed $t \geq 0$:

$$\frac{dQ_{t,r}}{dr} = -Q_{t,r}\,\mathrm{Ric}^Z_{//_{t,r}}, \quad Q_{t,t} = \mathrm{id}.$$

- Let $\mathcal{L}$ be the Ornstein-Uhlenbeck operator defined as generator associated to the Dirichlet form

$$\mathcal{E}(F,F) = \mathbb{E}\left[\int_0^T |D_t^{//} F|^2(X_{[0,T]})\, dt\right].$$

Goal is estimating the spectral gap of $\mathcal{L}$ on manifolds of pinched curvature.

- For constants $k_1 \leq k_2$ let

$$\hat{D}_t^{//} F(X_{[0,T]}^x) = \sum_{i=1}^n \mathbb{1}_{\{t \leq t_i\}}\, e^{-\frac{k_1+k_2}{2}(t_i - t)} //_{t,t_i}^{-1} \nabla_i f(X_{t_1}^x, \ldots, X_{t_n}^x)$$

Theorem (Path space characterization of pinched curvature)

The following conditions are equivalent:

(i) $k_1 \le \mathrm{Ric}^Z \le k_2$;

(ii) *for any* $F \in \mathscr{F}C^\infty_{0,T}$,

$$\left|\nabla_x \mathbb{E} F(X^x_{[0,T]})\right| \le \mathbb{E}|\hat{D}^{//}_0 F| + \frac{k_2-k_1}{2}\int_0^T e^{-k_1 s}\,\mathbb{E}|\hat{D}^{//}_s F|\,ds;$$

(iii) *for any* $F \in \mathscr{F}C^\infty_{0,T}$ *and* $t_1 < t_2$ *in* $[0,T]$,

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\log \mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\Big] \\
&\qquad -\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\log \mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\Big] \\
&\le 2\int_{t_1}^{t_2}\left(1+\frac{k_2-k_1}{2}\int_t^T e^{-k_1(s-t)}ds\right) \\
&\qquad \times\left(\mathbb{E}|\hat{D}^{//}_t F|^2 + \frac{k_2-k_1}{2}\int_t^T e^{-k_1(s-t)}\mathbb{E}|\hat{D}^{//}_s F|^2 ds\right)dt.
\end{aligned}$$

Theorem

Assume $k_1 \le \mathrm{Ric}^Z \le k_2$. *Then*

$$\mathrm{gap}(\mathcal{L})^{-1} \le C(T, k_1, |k_1| \vee |k_2|) \wedge \left[C\left(T, k_1, \frac{k_2 - k_1}{2}\right) \times C\left(T, \frac{k_1 + k_2}{2}, \frac{|k_1 + k_2|}{2}\right) \right]$$

where

$$C(T, K_1, K_2) = \begin{cases} 1 + K_2 T + \frac{K_2^2 T^2}{2}, & K_1 = 0; \\ (1+\beta)^2 - \beta \sqrt{(2+\beta)\left(2 + 2\beta - \beta e^{-K_1 T}\right)} e^{-K_1 T/2}, & K_1 > 0; \\ \frac{1}{2} + \frac{1}{2}\left(1 + \beta(1 - e^{-K_1 T})\right)^2, & K_1 < 0. \end{cases}$$

with $\beta = K_2 / K_1$.

VI. Geometries evolving in time

- Let $g(t)_{t\in I}$ be a C^1 family of Riemannian metrics on M and let $\mathbb{M} := M \times I$ be space time.
- For $s \in I$, suppose that

$$(X_t, t)_{t\geq s}$$

is Brownian motion on $\mathbb{M}$ based at (x, s), i.e., $X_s = x$ and $\forall f \in C^\infty(M)$,

$$d(f(X_t)) - \Delta_{g(t)} f(X_t)\, dt = 0 \quad \text{(mod loc mart).}$$

- We call X_t a g(t)-Brownian motion on M, and write

$$X_t = X_t^{(x,s)}, \quad t \geq s.$$

- **(Semigroup)** $P_{s,t} f(x) := \mathbb{E}[f(X_t^{(x,s)})]$ for $s \leq t$ in I.

- Let $(M, g_t)_{t\in I}$ be a smooth family of Riemannian metrics. Write Ric_t, ∇^t for the Ricci tensor, Levi-Civita connection with respect to g_t, respectively.
- Let $(Z_t)_{t\in I}$ be a smooth family of vector fields on M and

$$L_t = \Delta_t + Z_t.$$

- Define

$$\mathcal{R}_t^Z(X,Y) := \mathrm{Ric}_t(X,Y) - \left\langle \nabla_X^t Z_t, Y \right\rangle_t - \frac{1}{2}(\partial_t g_t)(X,Y)$$

- Finally, for $f \in C_b(M)$,

$$P_{s,t}f(x) := \mathbb{E}\left[f(X_t^{(x,s)})\right] = \mathbb{E}^{(x,s)}\left[f(X_t)\right], \quad 0 \le s \le t \text{ in } I,$$

where $X_t^{(x,s)}$ is a L_t-diffusion starting from x at time s.

Theorem (Cheng-A.Th. 2017)

Let $(t,x)\mapsto K_1(t,x)$ and $(t,x)\mapsto K_2(t,x)$ be two continuous functions on $I\times M$ such that $K_1\le K_2$ (satisfying some weak integrability conditions).

The following statements are equivalent:

(i) *the curvature $\mathcal{R}_t^Z$ satisfies*

$$K_1(t,x)\le \mathcal{R}_t^Z(x)\le K_2(t,x),\quad (t,x)\in I\times M;$$

(ii) *for $f\in C_0^\infty(M)$ and $0\le s\le t$ in I,*

$$\begin{aligned}&|\nabla^s P_{s,t}f|_s^2-\mathbb{E}^{(x,s)}\Big[e^{-2\int_s^t K_1(r,X_r)\,dr}|\nabla^t f|_t^2(X_t)\Big]\\ &\le 4\bigg[\Big(\mathbb{E}^{(x,s)}e^{\frac{1}{2}\int_s^t (K_2(r,X_r)-K_1(r,X_r))\,dr}-1\Big)|\nabla^s f|_s^2+\langle\nabla^s f,\nabla^s P_{s,t}f\rangle_s\\ &\qquad -\Big\langle\nabla^s f,\mathbb{E}^{(x,s)}\Big[e^{-\int_s^t K_1(r,X_r)\,dr}//_{s,t}^{-1}\nabla^t f(X_t)\Big]\Big\rangle_s\bigg]\wedge 0;\end{aligned}$$

Theorem–cont.

(iii) for $f \in C_0^\infty(M)$, $p \in (1,2]$ and $0 \le s \le t$ in I,

$$\frac{p(P_{s,t}f^2 - (P_{s,t}f^{2/p})^p)}{4(p-1)} - \mathbb{E}^{(x,s)}\left[\int_s^t e^{-2\int_r^t K_1(\tau,X_\tau)\,d\tau} dr \times |\nabla^t f|_t^2(X_t)\right]$$
$$\le 4\int_s^t \left[\mathbb{E}^{(x,s)} e^{\frac{1}{2}\int_r^t (K_2(\tau,X_\tau)-K_1(\tau,X_\tau))d\tau} - 1\right] P_{s,r}|\nabla^r f|_r^2$$
$$+ \mathbb{E}^{(x,s)}\left\langle \nabla^r f(X_r), \nabla^r P_{r,t}f(X_r) - e^{-\int_r^t K_1(\tau,X_\tau)d\tau} //_{r,t}^{-1}\nabla^t f(X_t)\right\rangle_r dr \wedge 0;$$

(iv) for $f \in C_0^\infty(M)$ and $0 \le s \le t$ in I,

$$\frac{1}{4}\left(P_{s,t}(f^2 \log f^2) - P_{s,t}f^2 \log P_{s,t}f^2\right)$$
$$- \mathbb{E}^{(x,s)}\left[\int_s^t e^{-2\int_r^t K_1(\tau,X_\tau)\,d\tau} dr \times |\nabla^t f|_t^2(X_t)\right]$$
$$\le 4\int_s^t \left[\mathbb{E}^{(x,s)} e^{\frac{1}{2}\int_r^t (K_2(\tau,X_\tau)-K_1(\tau,X_\tau))d\tau} - 1\right] P_{s,r}|\nabla^r f|_r^2$$
$$+ \mathbb{E}^{(x,s)}\left\langle \nabla^r f(X_r), \nabla^r P_{r,t}f(X_r) - e^{-\int_r^t K_1(\tau,X_\tau)d\tau} //_{r,t}^{-1}\nabla^t f(X_t)\right\rangle_r dr \wedge 0.$$

Corollary [Cheng-A.Th. 2017]

Let $(t,x) \mapsto K(t,x)$ be some continuous function on $I \times M$. The following statements are equivalent to each other:

(i) the family $(M, g_t)_{t\in I}$ evolves by

$$\frac{1}{2}\partial_t g_t = \mathrm{Ric}_t - \nabla^t Z_t - K(t,\cdot) g_t, \quad t \in I;$$

(ii) for $f \in C_0^\infty(M)$ and $0 \le s \le t$ in I,

$$|\nabla^s P_{s,t} f|_s^2 - \mathbb{E}^{(x,s)}\Big[e^{-2\int_s^t K(r,X_r)dr} |\nabla^t f|_t^2 (X_t)\Big]$$
$$\le 4\Big[\langle \nabla^s f, \nabla^s P_{s,t} f\rangle_s - \Big\langle \nabla^s f, \mathbb{E}^{(x,s)}\Big[e^{-\int_s^t K(r,X_r)dr} //_{s,t}^{-1} \nabla^t f(X_t)\Big]\Big\rangle_s\Big] \wedge 0;$$

(iii) version of a Poincaré inequality

(iv) version of a log-Sobolev inequality

- If $Z_t \equiv 0$ and $K \equiv 0$, the results characterize solutions to the Ricci flow; see Haslhofer and Naber (2016) for functional inequalities on path space characterizing Ricci flow.

Example: Ricci flow

- Consider the heat equation under Ricci flow:

$$\begin{cases} \frac{\partial}{\partial t}u - \Delta_{g_t} u = 0 \\ \frac{\partial}{\partial t}g_t - 2\,\mathrm{Ric}_{g_t} = 0 \end{cases}$$

  The case of forward Ricci flow we get by reparametrizing the metric:

$$\hat{g}_t := g_{T-t}$$

- Let

$$u(x,t) = (P_{s,t}f)(x), \quad 0 \le s \le t \text{ in } I.$$

- We have

$$P_{s,t}f(x) = \mathbb{E}\left[f(X_t^{(x,s)})\right], \quad 0 \le s \le t,$$

  and

$$\nabla^s P_{s,t}f(x) = \mathbb{E}\left[Q_{s,t}//_{s,t}^{-1}\nabla^t f(X_t^{(x,s)})\right], \quad 0 \le s \le t,$$

  where $Q_{s,t} \in \mathrm{Aut}(T_{X_s}M)$ is constructed as solution to the equation:

$$\frac{dQ_{s,t}}{dt} = -Q_{s,t}\mathcal{R}_{//_{s,t}}, \quad Q_{s,s} = \mathrm{id}.$$

- Recall

$$\mathcal{R}_{//_{s,t}} = //_{s,t}^{-1}\left(\mathrm{Ric}_{g_t} - \frac{1}{2}\partial_t g_t\right)//_{s,t}, \quad Z = 0,\ K_1 = K_2 = K = 0.$$

- We see that $Q_{s,t} = \text{identity}$ if and only if the metric evolves by (backward) Ricci flow.
- This explains why Riemannian manifolds evolving under Ricci flow share many properties of Ricci flat static manifolds.

Supersolutions to the Ricci flow

For a smooth family $(M, g(t))_{t\in I}$ of Riemannian metrics are equivalent:

- $(M, g(t))_{t\in I}$ is a supersolution to the Ricci flow, i.e.

$$2\,\mathrm{Ric}_{g(t)} - \frac{\partial}{\partial t} g(t) \geq 0.$$

- For each $f \in C_c^\infty(M)$ the heat flow on $(M, g(t))_{t\in I}$ satisfies

$$|\nabla^s P_{s,t} f|_{g(s)} \leq P_{s,t} |\nabla^t f|_{g(t)}, \quad 0 \leq s < t \text{ in } I.$$

- For each $f \in C_c^\infty(M)$ the heat flow on $(M, g(t))_{t\in I}$ satisfies

$$|\nabla^s P_{s,t} f|^2_{g(s)} \leq P_{s,t} |\nabla^t f|^2_{g(t)}, \quad 0 \leq s < t \text{ in } I.$$

- Let $\mathcal{P}^{(x,s)}M$ be the space of continuous paths on M, starting in x at time s and $\mathbb{P}^{(x,s)}$ the probability measure on it, induced by the (inhomogeneous) BM

$$X_t^{(x,s)}, \quad t \geq s.$$

- For a cylindrical function F on $\mathcal{P}^{(x,s)}M$ with

$$F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_r}), \quad s \leq t_1 < \ldots < t_r \leq t,$$

consider again the intrinsic gradient defined as

$$D_s^{//} F(X_{[s,t]}) = \sum_{i=1}^{r} //_{s,t_i}^{-1} (\nabla^i_{g(t_i)} f)(X_{t_1}, \ldots, X_{t_r}),$$

where ∇^i denotes the gradient with respect to the i-th component.

Characterization of solutions to the Ricci flow

For a smooth family $(M, g(t))_{t\in I}$ of Riemannian metrics are equivalent:

- $(M, g(t))_{t\in I}$ is a solution to the Ricci flow, i.e.

$$\frac{\partial}{\partial t} g(t) - 2\,\mathrm{Ric}_{g(t)} = 0.$$

- For each cylindrical function $F \colon \mathcal{P}^{(x,s)} M \to \mathbb{R}$,

$$|\nabla_x^s \mathbb{E}^{(x,s)} F| \leq \mathbb{E}^{(x,s)} |D_s^{//} F|.$$

- For each cylindrical function $F \colon \mathcal{P}^{(x,s)} M \to \mathbb{R}$,

$$|\nabla_x^s \mathbb{E}^{(x,s)} F|^2 \leq \mathbb{E}^{(x,s)} |D_s^{//} F|^2.$$

  Here $|\cdot| = |\cdot|_{g(s)}$.