MANCHESTER
1824
The University of Manchester

A circulant preconditioned MINRES method for nonsymmetric Toeplitz problems

Jennifer Pestana and Andrew Wathen

June 17, 2015

M\crNA

Outline

- Introduction
  - Toeplitz and circulant matrices
- Symmetric Toeplitz problems
- Nonsymmetric Toeplitz problems
  - Alternative symmetrization
  - Adapting circulant preconditioners
- Results
- Extensions
- Conclusions

Toeplitz problems

$$A_n x = b, \quad A_n \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^n$$

$$A_n = \begin{bmatrix} a_0 & a_{-1} & \dots & a_{-n+2} & a_{-n+1} \\ a_1 & a_0 & a_{-1} & & a_{-n+2} \\ \vdots & a_1 & a_0 & \ddots & \vdots \\ a_{n-2} & & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{bmatrix}$$

- ODEs and PDEs
- Integral equations
- Time series
- Control theory
- Signal and image processing...

see, e.g., books by Chan and Jin (2007) and Ng (2004)

MANCHESTER 1824
The University of Manchester

Generating functions

Generating functions

Assume $\{a_k\}_{k=-n+1}^{n-1}$ are Fourier coefficients of a function f

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx.$$

- Real-valued $f \Rightarrow A_n$ Hermitian (for all n)
- Real-valued, even $f \Rightarrow A_n$ real symmetric (for all n)
- In many problems we have f and need to compute A_n, e.g., integral equations, time series,....

What properties do they have?

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

- Symmetric postive definite
- Real eigenvalues and orthogonal eigenvectors

$$\begin{bmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda & \\ & & & 1 & \lambda \end{bmatrix}$$

- Non-normal
- Non-diagonalizable

Circulants: special Toeplitz matrices

$$C_n = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

Eigenvalues and eigenvectors

$$C_n = F_n^{-1} \Lambda_n F_n$$

where

- $F_n = (f_{jk})$ with $f_{jk} = \frac{1}{\sqrt{n}} e^{-2(j-1)(k-1)\pi i/n}$, $j, k = 1, \ldots, n$,
- $\Lambda = \mathrm{diag}(F_n c)$, with c the first column of C_n

The University of Manchester

Circulants: special Toeplitz matrices

$$C_n = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

$$C_n = F_n^{-1} \Lambda_n F_n, \quad \Lambda = \mathrm{diag}(F_n c)$$

Matrix-vector products and solves with C_n can be computed in $O(n \log n)$ operations using fast Fourier transforms (FFTs)

What if A_n is Toeplitz but not circulant?

Matrix-vector products via circulant embedding, e.g.,

$$\begin{bmatrix} A_n & * \\ * & A_n \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A_n x \\ * \end{bmatrix}$$

Matrix-vector products can be computed at FFT speed with $O(2n\log(2n))$ complexity

Although superfast direct solvers exist with $O(n\log^2 n)$ cost, Krylov method are extremely effective for Toeplitz problems

Krylov subspace methods

Krylov subspace methods

- Choose $r_0 = b - A_n x_0$
- For $k = 1, 2, \ldots$ choose x_k such that
  $x_k - x_0 \in \mathcal{K}_k(A_n, r_0) = \mathrm{span}\{r_0, A_n r_0, \ldots, A_n^{(k-1)} r_0\}$

Coefficient matrix determines method

- SPD matrices: CG
- Symmetric indefinite matrices: MINRES or SYMMLQ
- Nonsymmetric matrices: GMRES, QMR/SQMR, Bi-CG, CG for the normal equations/LSQR,...

Divide between symmetric and nonsymmetric problems

Methods for symmetric (positive definite) problems

For SPD A_n the conjugate gradient method

- Minimizes $\|e_k\|_{A_n}$
- has short-term recurrences
- and has descriptive convergence rate bounds based on eigenvalues

CG convergence bound

$$\frac{\|e_k\|_{A_n}}{\|e_0\|_{A_n}} \leq \min_{\substack{p \in \Pi_k \\ p(0)=1}} \max_{\lambda \in \sigma(A_n)} |p(\lambda)|$$

Circulant preconditioners

- Idea: use circulant to capture most of A_n
  (Strang, 1986, Olkin, 1986)
- Quick to apply: $O(n \log n)$ complexity
- Generalized to discrete sine/cosine-based preconditioners, Hartley transform preconditioners, banded Toeplitz preconditioners, ...

$$C_n = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

Preconditioners for symmetric positive definite problems

Possible positive definite circulants:

- Strang preconditioner (Strang 1986):

$$\text{If } n = 2m+1,\ c_n = \begin{cases} t_k, & 0 \le k \le m, \\ t_{k-n}, & m < k \le n-1, \\ c_{-k}, & -n+1 \le k < 0 \end{cases}$$

  ($n = 2m$ similar)

- Optimal preconditioner (T. Chan, 1988):
  Minimizes $\|A_n - C_n\|_F$ over all circulant matrices

- ...

see. e.g., Chan and Jin (2007) or Ng (2004)

Clustered eigenvalues

A real-valued $f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}$ is in the Wiener class if $\sum_{k=\infty}^{\infty} |a_k| < \infty$.

Circulant preconditioner decomposition

If f is positive and

- in the Wiener class for Strang preconditioner (Chan and Strang, 1989)
- $f \in \boldsymbol{C}_{2\pi}$ for the optimal preconditioner (Chan and Yeung, 1992)

then there exist M and $N > 0$ such that for all $n > N$

$$C_n^{-1} A_n = I + R + E,$$

where $\|E\|_2 \leq \epsilon$ and rank $R \leq M$.

MANCHESTER 1824
The University of Manchester

Clustered eigenvalues and convergence rates

Preconditioned CG convergence bound

$$\frac{\|e_k\|_{A_n}}{\|e_0\|_{A_n}} \leq \min_{\substack{p \in \Pi_k \\ p(0)=1}} \max_{\lambda \in \sigma(C_n^{-1}A_n)} |p(\lambda)|$$

Eigenvalues of circulant-preconditioned matrix

$C_n^{-1}A_n = I + R + E$, where $\|E\|_2 \leq \epsilon$ and rank $R \leq M$

- At most M eigenvalues lie outside $[1-\epsilon, 1+\epsilon]$
- Preconditioned CG converges quickly after (at most) M steps because convergence bounded by eigenvalues

Example

$$A_n = \begin{bmatrix} 1 & \frac{1}{2^\alpha} & \cdots & \frac{1}{(n-1)^\alpha} & \frac{1}{n^\alpha} \\ \frac{1}{2^\alpha} & 1 & \frac{1}{2^\alpha} & & \frac{1}{(n-1)^\alpha} \\ \vdots & \frac{1}{2^\alpha} & 1 & \ddots & \vdots \\ \frac{1}{(n-1)^\alpha} & & \ddots & \ddots & \frac{1}{2^\alpha} \\ \frac{1}{n^\alpha} & \frac{1}{(n-1)^\alpha} & \cdots & \frac{1}{2^\alpha} & 1 \end{bmatrix}$$

$\alpha = 1.1$:

n	Unpreconditioned	Strang	Optimal
100	26	7	7
1000	38	7	7
10000	45	8	8

Symmetric indefinite problems

MINRES Convergence bounds

$$\frac{\|r_k\|_2}{\|r_0\|_2} \leq \min_{\substack{p \in \Pi_k \\ p(0)=1}} \max_{\lambda \in \sigma(A_n)} |p(\lambda)|$$

MANCHESTER 1824
The University of Manchester

Symmetric indefinite problems

Preconditioned MINRES Convergence bound

$$\frac{\|C_n^{-1}r_k\|_2}{\|C_n^{-1}r_0\|_2} \leq \min_{\substack{p\in\Pi_k\\ p(0)=1}} \max_{\lambda\in\sigma(C_n^{-1}A_n)} |p(\lambda)|$$

- For certain generating functions there exist symmetric positive definite C_n for which

$$C_n^{-1}A_n = Q + R + E$$

  with Q real and orthogonal, $\text{rank}(R) \leq M$ and $\|E\|_2 \leq \epsilon$
- All but M eigenvalues lie in $[-1-\epsilon, -1+\epsilon] \cup [1-\epsilon, 1+\epsilon]$
- Preconditioned MINRES converges quickly after at most M steps because convergence bounded by eigenvalues

Summary of symmetric problems

- Can solve using preconditioned conjugate gradients or preconditioned MINRES
- Convergence bounded using eigenvalues
- Effective circulant preconditioners exist that cluster these eigenvalues ($C_n^{-1}A_n = I + R + E$)
- Guaranteed fast convergence

MANCHESTER 1824
The University of Manchester

Nonsymmetric problems

- For certain generating functions C_n exist for which

$$C_n^{-1} A_n = I + R + E$$

  with $\text{rank}(R) \le M$ and $\|E\|_2 \le \epsilon$
- This guarantees clustered singular values but not necessarily clustered eigenvalues
- Clustered eigenvalues don't guarantee fast convergence

Methods for nonsymmetric problems

GMRES, QMR/SQMR, Bi-CG, IDR,...

For general nonsymmetric problems

- Minimize relevant quantity or have short-term recurrences
- Convergence rates can be tricky to describe and typically don't depend on eigenvalues alone

CG on normal equations (LSQR)

$$A_n^T A_n x = A_n^T b$$

For general nonsymmetric problems

- Minimizes $\|r_k\|_2$ and has short-term recurrences
- Has convergence bounds based on singular values of A_n
- Can be slow (square condition number)

Options

- Use solver without convergence theory
- Use LSQR and expect slower convergence

$$A_n = \begin{bmatrix} 1 & 1 & 1 & 1 & & & \\ -1 & 1 & 1 & 1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

	n	GMRES	LSQR	SQMR
Unpreconditioned	100	80	64	100
	1000	299	66	340
	10000	302	66	327
Strang	100	4	18	4
	1000	4	18	4
	10000	4	18	4
Optimal	100	7	22	8
	1000	6	20	6
	10000	5	20	5

Symmetrizing a real nonsymmetric Toeplitz problem

$$\begin{pmatrix} a_0 & a_{-1} & \dots & a_{-n+2} & a_{-n+1} \\ a_1 & a_0 & a_{-1} & & a_{-n+2} \\ \vdots & a_1 & a_0 & \ddots & \vdots \\ a_{n-2} & & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{pmatrix}$$

Symmetrizing a real nonsymmetric Toeplitz problem

$$\begin{pmatrix} a_0 & a_{-1} & \dots & a_{-n+2} & a_{-n+1} \\ a_1 & a_0 & a_{-1} & & a_{-n+2} \\ \vdots & a_1 & a_0 & \ddots & \vdots \\ a_{n-2} & & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{pmatrix}$$

Symmetrizing a real nonsymmetric Toeplitz problem

$$
\begin{pmatrix}
a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\
a_{n-2} & & \iddots & \iddots & a_{-1} \\
\vdots & a_1 & a_0 & \iddots & \vdots \\
a_1 & a_0 & a_{-1} & & a_{-n+2} \\
a_0 & a_{-1} & \cdots & a_{-n+2} & a_{-n+1}
\end{pmatrix}
$$

Symmetrizing a real nonsymmetric Toeplitz problem

$$\begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ a_{n-2} & & \iddots & \iddots & a_{-1} \\ \vdots & a_1 & a_0 & \iddots & \vdots \\ a_1 & a_0 & a_{-1} & & a_{-n+2} \\ a_0 & a_{-1} & \cdots & a_{-n+2} & a_{-n+1} \end{pmatrix}$$

Expressing this mathematically....

When applied on the left

$$Y_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

exchanges the rows of A_n so that $Y_n A_n$ is symmetric.

$$Y_n A_n x = Y_n b$$

can be solved by MINRES which

- has short-term recurrences and minimizes $\|r_k\|_2$
- has convergence bounds based on eigenvalues

If $A_n \in \mathbb{C}^{n\times n}$ then $Y_n A_n$ is complex symmetric

Preconditioning

$$Y_n A_n x = Y_n b$$

- Convergence may still be poor
- We know what our preconditioner should do: cluster eigenvalues
- For MINRES need symmetric positive definite preconditioner
- Most existing circulant preconditioners are nonsymmetric

Symmetric positive definite preconditioners

Idea: use absolute value

$$|C_n| = (C_n^T C_n)^{\frac{1}{2}} = (C_n C_n^T)^{\frac{1}{2}} = F_n^* |\Lambda_n| F_n,$$

$$\Lambda_n = \mathrm{diag}(|\lambda_i|)$$

- Can still apply using FFTs
- Connected to C_n by circulant, orthogonal matrix $\widetilde{C}_n$:

$$\begin{aligned} C_n &= F_n^* \Lambda_n F_n \\ &= (F_n^* \mathrm{diag}(|\lambda_i|) F_n)(F_n^* \mathrm{diag}(\lambda_i/|\lambda_i|) F_n) \\ &= |C_n| \widetilde{C}_n, \end{aligned}$$

- Approach also works with discrete sine or cosine preconditioners

(Related idea in Vecharynski and Knyazev (2013))

MINRES convergence

Solve $Y_n A_n x = Y_n b$ by preconditioned MINRES with $|C_n|$

Convergence bounds

$$\frac{\| |C_n|^{-1} r_k \|_2}{\| |C_n|^{-1} r_0 \|_2} \leq \min_{\substack{p \in \Pi_k \\ p(0)=1}} \max_{\lambda \in \sigma(|C_n|^{-1} Y_n A_n)} |p(\lambda)|$$

MINRES convergence

Solve $Y_n A_n x = Y_n b$ by preconditioned MINRES with $|C_n|$

Convergence bounds

$$\frac{\||C_n|^{-1} r_k\|_2}{\||C_n|^{-1} r_0\|_2} \leq \min_{\substack{p \in \Pi_k \\ p(0)=1}} \max_{\lambda \in \sigma(|C_n|^{-1} Y_n A_n)} |p(\lambda)|$$

For some existing circulants $C_n^{-1} A_n = I + R + E \Rightarrow$ clustered singular values.

Does this imply that $|C_n|^{-1} Y_n A_n$ has clustered eigenvalues?

Clustered eigenvalues

Proposition (P. & Wathen, 2015)

Let

$$C_n^{-1}A_n = I + R + E,$$

where rank$(R) \leq M$ *and* $\|E\| \leq \epsilon$, *where* $\|\cdot\|$ *is a unitarily invariant norm. Then,*

$$|C_n|^{-1}Y_nA_n = Q + \widehat{R} + \widehat{E},$$

where rank$(\widehat{R}) =$ rank(R) *and* $\|\widehat{E}\| = \|E\|$. *Moreover, the eigenvalues of* Q *are* 1 *or* -1.

- All but M eigenvalues lie in $[-1-\epsilon, -1+\epsilon] \cup [1-\epsilon, 1+\epsilon]$
- Preconditioned MINRES converges quickly after at most M steps because convergence bounded by eigenvalues

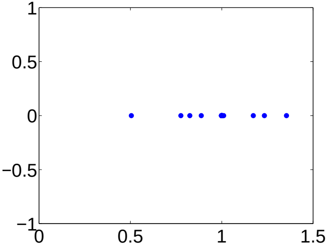
Existing preconditioners that can be adapted for MINRES

- Optimal preconditioner of Chan (1988)
- Black-box preconditioner of Oseledets and Tyrtyshnikov (2006)
- Improved circulants of Tyrtyshnikov, Yeremin and Zamarashkin (2006)

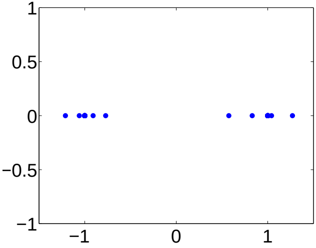
Example

Grcar matrix:

Singular values of $C_n^{-1}A_n$



Eigenvalues of $|C_n|^{-1}Y_nA_n$



MANCHESTER 1824
The University of Manchester

Clustered eigenvalues

Proposition (P. & Wathen, 2015)

Let the eigenvalues of $(C_n^{-1}A_n)^T(C_n^{-1}A_n)$ *be contained in the interval* $[1-\epsilon, 1+\epsilon]$ *with the exception of* M *outliers. Then the eigenvalues of* $|C_n|^{-1}Y_nA_n$ *are contained in*

$$\left[\sqrt{\kappa}(-1-\epsilon), \frac{-1+\epsilon}{\sqrt{\kappa}}\right] \cup \left[\frac{1-\epsilon}{\sqrt{\kappa}}, \sqrt{\kappa}(1+\epsilon)\right],$$

where $\kappa = \kappa_2(C_n)$ *is the 2-norm condition number of* C_n*, with the exception of* M *outliers.*

- Strang preconditioner satisfies this
- Bound can be pessimistic

MANCHESTER 1824
The University of Manchester

This bound may be pessimistic

$$\left[\sqrt{\kappa}(-1-\epsilon), \frac{-1+\epsilon}{\sqrt{\kappa}}\right] \cup \left[\frac{1-\epsilon}{\sqrt{\kappa}}, \sqrt{\kappa}(1+\epsilon)\right]$$

Example

If A_{100} is generated by $f(x) = |x|e^{ix}$, $x \in [-\pi, \pi]$ and C_n is the Strang preconditioner then the eigenvalues of $(C_n^{-1}A_n)^T(C_n^{-1}A_n)$ lie in

$$[0.9, 1.1] \cup \{5.5 \times 10^{-8}, 0.27, 0.63, 0.88, 978\}$$

Although $\kappa(C_n) = 2.5 \times 10^3$, the eigenvalues of $|C_n|^{-1}Y_nA_n$ lie in

$$[-1.1, -0.9] \cup [0.9, 1.1] \cup \{-1.22, -0.70, -5 \times 10^{-4}, 1.3, 5.3\}$$

Connection to SQMR

- QMR uses two starting vectors r_0 and $\widetilde{r}_0$
- SQMR (Freund and Nachtigal, 1995) uses Y_n to reduce work of QMR by setting $\widetilde{r}_0 = \psi_0 Y_n r_0$
- Can use with indefinite preconditioner but need two preconditioner solves per iteration
- Does not minimize residual or error with respect to easily identified norm
- Breakdown may still occur

MANCHESTER 1824
The University of Manchester

Results: Jordan block

Jordan block with eigenvalue 1.1:

	n	GMRES	LSQR	MINRES	SQMR
Unpre	100	95 (0.12)	200 (0.12)	100 (0.062)	99 (0.066)
	1000	182 (0.43)	370 (0.3)	366 (0.31)	199 (0.18)
	10000	185 (3.7)	376 (2.2)	376 (2.3)	205 (1.3)
Strang	100	2 (0.0047)	6 (0.0092)	4 (0.0068)	2 (0.0039)
	1000	2 (0.0088)	6 (0.011)	4 (0.0079)	2 (0.0042)
	10000	2 (0.25)	6 (0.049)	4 (0.036)	2 (0.013)
Optimal	100	7 (0.013)	16 (0.02)	14 (0.017)	7 (0.0083)
	1000	4 (0.012)	12 (0.019)	10 (0.016)	5 (0.0081)
	10000	3 (0.26)	10 (0.073)	8 (0.062)	4 (0.021)

Results: Grcar matrix

$$A_n = \begin{bmatrix} 1 & 1 & 1 & 1 & & & \\ -1 & 1 & 1 & 1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

	n	GMRES	LSQR	MINRES	SQMR
Unpre	100	80 (0.097)	64 (0.038)	62 (0.039)	100 (0.066)
	1000	299 (0.98)	66 (0.054)	66 (0.058)	340 (0.31)
	10000	302 (8.1)	66 (0.41)	66 (0.42)	327 (2.1)
Strang	100	4 (0.011)	18 (0.022)	10 (0.013)	4 (0.0053)
	1000	4 (0.012)	18 (0.027)	10 (0.016)	4 (0.0068)
	10000	4 (0.27)	18 (0.12)	10 (0.075)	4 (0.021)
Optimal	100	7 (0.011)	22 (0.025)	16 (0.019)	8 (0.0094)
	1000	6 (0.015)	20 (0.031)	14 (0.021)	6 (0.0096)
	10000	5 (0.27)	20 (0.13)	12 (0.088)	5 (0.026)

Results: dense matrix

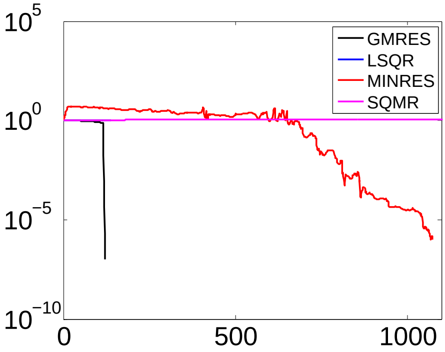
Generating function $f(x) = |x|e^{ix}$, $x \in [-\pi, \pi]$:

	n	GMRES	LSQR	MINRES	SQMR
Unpre	100	100 (0.13)	—	—	—
	1000	922 (7.7)	—	—	—
	10000	8157 (4.2×10^3)	—	—	—
Strang	100	9 (0.018)	26 (0.03)	16 (0.018)	9 (0.01)
	1000	11 (0.021)	32 (0.045)	18 (0.027)	12 (0.052)
	10000	12 (4.3)	48 (0.57)	22 (0.23)	14 (3.9)
Optimal	100	11 (0.018)	30 (0.034)	18 (0.021)	11 (0.012)
	1000	13 (0.026)	42 (0.06)	24 (0.036)	15 (0.065)
	10000	16 (4.9)	62 (0.61)	35 (0.33)	18 (4.6)

Convergence and SQMR

Generating function $f(x) = |x|^3 e^{ix}$, $x \in [-\pi, \pi]$, optimal preconditioner

$$\kappa_2(A_{1000}) = 9.9 \times 10^8,\ \kappa_2(C_{1000}) = 4.4 \times 10^4$$



Summary of nonsymmetric problems

- Can solve linear systems with nonsymmetric A_n using preconditioned MINRES
- Convergence bounded using eigenvalues
- Effective circulant preconditioners exist that cluster these eigenvalues ($|C_n|^{-1} Y_n A_n = Q + R + E$)
- Guaranteed fast convergence

Block Toeplitz matrices

If

$$B_{nm} = S_n \otimes T_m$$

where S_n and T_m are Toeplitz matrices then

$$W_{nm} B_{nm} = (Y_n \otimes Y_m)(S_n \otimes T_m) = (Y_n S_n) \otimes (Y_m T_m)$$

- Kronecker products of circulant matrices possible preconditioners but these don't work so well
- Multigrid preconditioners also available

Extensions: Hamiltonian matrices

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & -H_{11}^* \end{bmatrix}, \quad H_{12} = H_{12}^*, \; H_{21} = H_{21}^*$$

is Hamiltonian, i.e., JH is Hermitian, where

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

- Arise in control theory, algebraic Ricatti equations, systems analysis, ...
- Since JH is symmetric can use MINRES to solve $JHx = Jb$
- Preconditioners generally required

Conclusions

- Introduced a simple preconditioner that symmetrizes Toeplitz matrices
- Can use MINRES to solve resulting system
- Existing circulant preconditioners easily adapted
  - Clustered eigenvalues
  - Guaranteed fast convergence
  - Also works with discrete sine/cosine preconditioners
- Extension to block Toeplitz and Hamiltonian matrices possible but need good preconditioners

Thank you!

References I

[Cha88] T. F. Chan, *An optimal circulant preconditioner for Toeplitz systems*, SIAM J. Sci. Stat. Comput. **9** (1988), 766–771.

[CJ07] R. H.-F. Chan and X.-Q. Jin, *An introduction to iterative Toeplitz solvers*, SIAM, 2007.

[FN95] R. W. Freund and N. M. Nachtigal, *Software for simplified Lanczos and QMR algorithms*, Appl. Numer. Math. **19** (1995), 319–341.

[KK93] T.-K. Ku and C.-C. J. Kuo, *Spectral properties of preconditioned rational Toeplitz matrices: the nonsymmetric case*, SIAM J. Mat. Anal. Appl. **14** (1993), 521–544.

[Ng04] M. K. Ng, *Iterative Methods for Toeplitz Systems*, Oxford University Press, 2004.

The University of Manchester

References II

[PW15] J. Pestana and A. J. Wathen, *A preconditioned MINRES method for nonsymmetric Toeplitz matrices*, SIAM J. Mat. Anal. Appl. **36** (2015), 273–288.

[Str86] G. Strang, *A proposal for Toeplitz matrix calculations*, Stud. Appl. Math. **74** (1986), 171–176.

[Tyr92] E. E. Tyrtyshnikov, *Optimal and superoptimal circulant preconditioners*, SIAM J. Matrix Anal. Appl. **13** (1992), 459–473.

Eigenvalues of symmetric A_n

Fast matrix-vector products

Symmetrizing on the right

Superlinear convergence

Results: eigenvalues

Block Toeplitz matrices

MANCHESTER 1824
The University of Manchester

Eigenvalues of symmetric A_n

Recall the generating function

Assume $\{a_k\}_{k=-n+1}^{n-1}$ are Fourier coefficients of a function f

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx.$$

MANCHESTER 1824
The University of Manchester

Eigenvalues for symmetric A_n

Let $\boldsymbol{C}_{2\pi}$ be the space of 2Π-periodic real-valued continuous functions defined on $[-\pi, \pi]$.

Theorem (Grenander-Szegö theorem)

Let A_n have generating function $f \in \boldsymbol{C}_{2\pi}$ with minimum value $f_{\min}$ and maximum value $f_{\max}$. If $\lambda_{\min}(A_n)$ and $\lambda_{\max}(A_n)$ are the smallest and largest eigenvalues of A_n then

$$f_{\min} \leq \lambda_{\min}(A_n) \leq \lambda_{\max}(A_n) \leq f_{max}$$

Additionally, if $0 < f_{\min}$ then A_n is positive definite for all n and the eigenvalues λ_j are distributed as $f(2\pi j/n)$ for large n.

Finite difference matrix

Example

$$A_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

$$f(x) = -\cos(-x) + 2 - \cos(x) = 4\sin^2\frac{x}{2}$$

$\lambda_j(A_n) = 4\sin^2\left(\frac{\pi(j+1)}{n+1}\right)$ which approaches $4\sin^2(\frac{2\pi j}{2n})$ as $n \to \infty$

Fast matrix-vector products

$$C_n = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

- $C_n = F_n^{-1} \Lambda_n F_n$ where $\Lambda = \mathrm{diag}(F_n c)$.
- $C_n x$ and $C_n^{-1} y$ can be quickly computed using fast Fourier transforms (FFTs) with $O(n \log n)$ complexity

Fast matrix-vector products

Matrix-vector products

$$y = C_n x = F_n^{-1} \Lambda_n F_n x$$

Fast matrix-vector products

Matrix-vector products

$$y = C_n x = F_n^{-1} \Lambda_n F_n x$$

1. FFT to compute $\tilde{x} = F_n x$

Fast matrix-vector products

Matrix-vector products

$$y = C_n x = F_n^{-1} \Lambda_n \tilde{x}$$

1. FFT to compute $\tilde{x} = F_n x$
2. FFT to compute $\tilde{c} = F_n c = \mathrm{diag}(\Lambda_n)$

Fast matrix-vector products

Matrix-vector products

$$y = C_n x = F_n^{-1} \Lambda_n \tilde{x}$$

1. FFT to compute $\tilde{x} = F_n x$
2. FFT to compute $\tilde{c} = F_n c = \mathrm{diag}(\Lambda_n)$
3. $z = \tilde{x} .* \tilde{c}$

Fast matrix-vector products

Matrix-vector products

$$y = C_n x = F_n^{-1} z$$

1. FFT to compute $\tilde{x} = F_n x$
2. FFT to compute $\tilde{c} = F_n c = \mathrm{diag}(\Lambda_n)$
3. $z = \tilde{x}. * \tilde{c}$
4. Inverse FFT to compute $y = F_n^{-1} z$

Fast matrix-vector products

Matrix-vector products

$$y = C_n x = F_n^{-1} \Lambda_n F_n x$$

1. FFT to compute $\tilde{x} = F_n x$
2. FFT to compute $\tilde{c} = F_n c = \mathrm{diag}(\Lambda_n)$
3. $z = \tilde{x} .* \tilde{c}$
4. Inverse FFT to compute $y = F_n^{-1} z$

So x is computed with 3 FFTs and a Hadamard product with complexity $O(n \log n)$.

Fast matrix-vector products

Solves

$$y = C_n^{-1} x = F_n^{-1} \Lambda_n^{-1} F_n x$$

1. FFT to compute $\tilde{x} = F_n x$
2. FFT to compute $\tilde{c} = F_n c = \mathrm{diag}(\Lambda_n)$
3. $z = \tilde{x} ./ \tilde{c}$
4. Inverse FFT to compute $y = F_n^{-1} z$

So x is computed with 3 FFTs and a "Hadamard quotient" with complexity $O(n \log n)$.

Symmetrizing on the right instead

In general, $A_n Y_n \neq Y_n A_n$. However, it doesn't matter if we symmetrize on the left or the right.

Lemma (P. & Wathen, 2015)

When the same starting vector $x_0 \in \mathbb{R}^n$ is used, the kth iterate of preconditioned MINRES applied to

$$Y_n A_n x = Y_n b$$

with preconditioner $|C_n|$ is equal to the kth iterate of preconditioned MINRES applied to

$$(A_n Y_n) z = b, \quad x = Y_n z,$$

with preconditioner $|C_n|$ for $k = 1, 2, \ldots, n$.

Superlinear convergence

Proposition (P. & Wathen, 2015)

Let $|C_n|^{-1}Y_nA_n$ *have*

- p *eigenvalues* $\lambda \in \{z : |1-z| \leq \epsilon\}$,
- q *eigenvalues* $\lambda \in \{z : |1+z| \leq \epsilon\}$,
- $d = n - p - q$ *outlying eigenvalues*

After the d*th iteration of preconditioned MINRES*

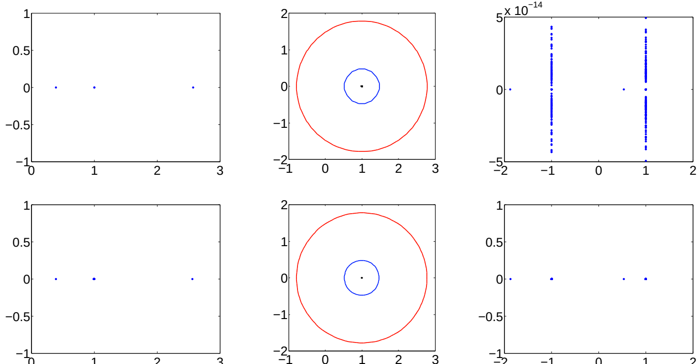
$$\frac{\||C_n|^{-1}r_{d+2k}\|_2}{\||C_n|^{-1}r_0\|_2} \leq C(2\epsilon + \epsilon^2)^k,$$

where C *is independent of* k

If $|C_n|^{-1}Y_nA_n = Q_n + \widehat{R} + \widehat{E}$ number of iterations depends on rank$(\widehat{R})$ and $\|\widehat{E}\|$

Eigenvalues: Jordan block

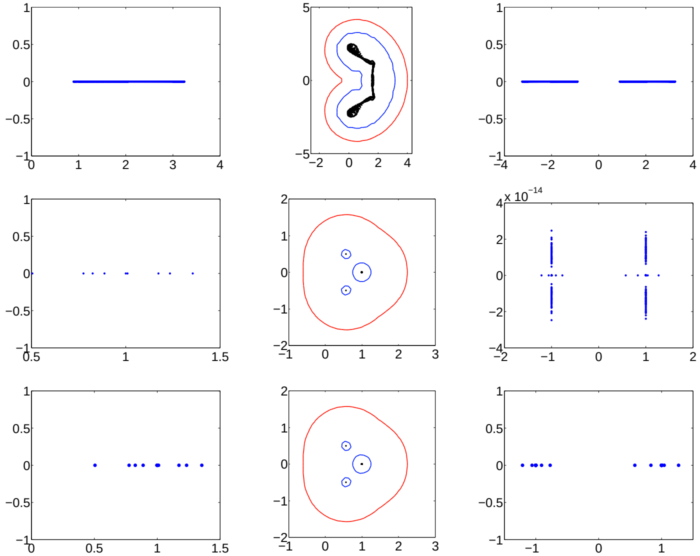
Jordan block with eigenvalue 1.1:



MANCHESTER
1824
The University of Manchester

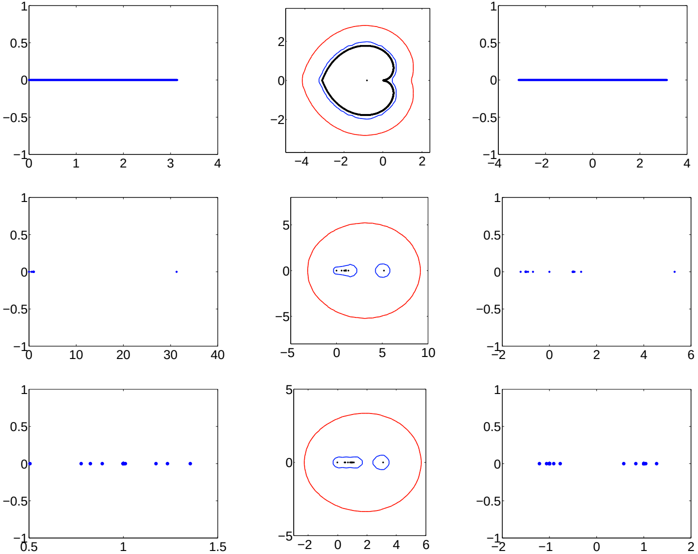
Eigenvalues: Grcar

Grcar matrix:



Eigenvalues: Dense

Generating function $f(x) = |x|e^{ix}$, $x \in [-\pi, \pi]$:



Block Toeplitz matrices

Proposition (P. & Wathen, 2015)

Let

$$J_n^{-1} S_n = I + R_1 + E_1 \textit{ and } K_m^{-1} T_m = I + R_2 + E_2$$

where $\operatorname{rank}(R_1) = M_1$, $\operatorname{rank}(R_2) = M_2$, $\|E_1\| \leq \epsilon_1$ *and* $\|E_2\| \leq \epsilon_2$, *where* $\|\cdot\|$ *is a unitarily invariant norm. Then,*

$$G_{nm}^{-1} B_{nm} = I + R + E,$$

where $\operatorname{rank}(R) \leq mM_1 + nM_2$ *and* $\|E\| \leq \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2$. *Moreover,*

$$|G_{nm}|^{-1} W_{nm} B_{nm} = Q_{mn} + \widehat{R} + \widehat{E},$$

where Q_{mn} *is orthogonal,* $\operatorname{rank}(\widehat{R}) \leq mM_1 + nM_2$ *and* $\|\widehat{E}\| \leq \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2$.