Off-diagonal low-rank preconditioner for PageRank problems

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Background

PageRank problem: ranking pages based on their adjacency structure.

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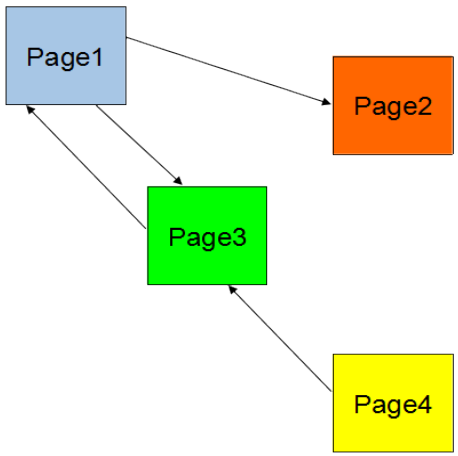
The structure of a network can be represented by a adjacency matrix G. If the j_{th} page points to the i_{th} page, $g_{ij} = 1$, otherwise, $g_{ij} = 0$.



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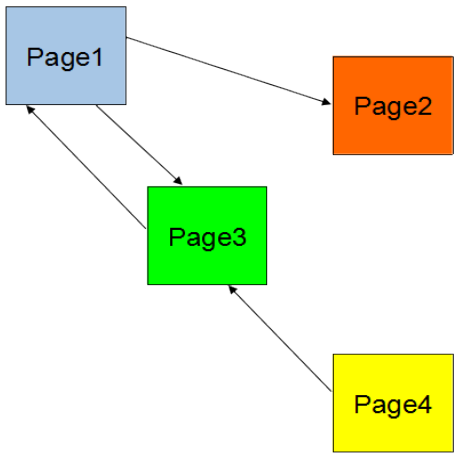

$$\longrightarrow \qquad G = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Background

Basic idea:

1) Consider a random walk on this structure
2) Use the stationary distribution to rank pages.

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The transition probability from page j to page i is defined as the reciprocal of page j's outlink amount.

$$\text{transition probability matrix } P = \begin{cases} p_{ij} = \frac{1}{\sum_{k=1}^{n} g_{kj}}, & \text{if } g_{ij} = 1 \\ p_{ij} = 0, & otherwise \end{cases} \tag{1}$$

The stationary distribution x satisfies:

$$Px = x,\ x > 0,\ ||x||_1 = 1 \tag{2}$$

To guarantee a unique solution of system (2), P must satisfy Perron-Frobenius(P.F.) theorem.

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Background

[**P.F. Theorem**] *If A is an irreducible non-negative $n \times n$ matrix with spectral radius $\rho(A) = r$, there is a unique positive eigenvector of eigenvalue r up to scale.*

As P may be reducible, system (2) is usually modified as

$$(\alpha P + (1-\alpha)ve^T)x = x, \tag{3}$$

to get irreducibility, where v is a distribution over all pages, e is a vector with all ones and $0 < \alpha < 1$ is a teleportation paramete.

Since $e^T x = 1$, system (3) can be reformulated as the final system:

$$(I - \alpha P)x = (1-\alpha)v. \tag{4}$$

$$\longrightarrow Ax = b,$$

where $A = I - \alpha P$ and $b = (1-\alpha)v$

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Off-diagonal low-rank property

From the definition of P: If $g_{ij} = 1$, $p_{ij} = \frac{1}{\sum_{k=1}^{n} g_{kj}}$, else $p_{ij} = 0$,

Property 1 Nonzero elements of columns all have the same value.

$$G = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \longrightarrow P = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{3} & 0 & 1 \end{pmatrix}$$

This property gives a benefit: rows are same if their patterns are same.

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$$G = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \longrightarrow I - \alpha P = \begin{pmatrix} 1-\alpha\frac{1}{4} & 0 & -\frac{1}{3} & 0 \\ -\frac{1}{4} & 1-\alpha\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1-\alpha\frac{1}{3} & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 0 & 1-\alpha 1 \end{pmatrix}$$

This property gives a benefit: rows are same if their patterns are same.

The final coefficient matrix A keeps this property except the diagonal elements, since $A = (I - \alpha P)$ destroy this property on diagonal elements.

Our object: construct efficient preconditioner for A by exploiting this property.

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Off-diagonal low-rank property

With a block partition dividing A into diagonal part D and off-diagonal part B_{off},

$$A = \begin{pmatrix} \boxed{D_1} & off_{12} & off_{13} \\ off_{21} & \boxed{D_2} & off_{23} \\ off_{31} & off_{23} & \boxed{D_2} \end{pmatrix} \rightarrow D = \begin{pmatrix} D_1 & & \\ & D_2 & \\ & & D_3 \end{pmatrix}, B_{off} = \begin{pmatrix} & off_{12} & off_{13} \\ off_{21} & & off_{23} \\ off_{31} & off_{32} & \end{pmatrix}$$

where B_{off} has some same row patterns, we can easily implement low-rank factorization of B_{off}.

$$B_{off} = \begin{pmatrix} 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & * & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & 0 & * \\ * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 \end{pmatrix} \rightarrow H = \begin{pmatrix} 0 & 0 & * & * & * & * \\ 0 & * & 0 & 0 & 0 & * \\ * & 0 & 0 & * & 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$B_{off} = F * H,$$

$$\rightarrow A = D + F * H$$

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$$B_{off} = F * H,$$

$$\rightarrow A = D + F * H$$

Then we can construct a preconditioner as

$$A^{-1} = D^{-1}(I - F(I + HD^{-1}F)^{-1}HD^{-1}). \quad (5)$$

based on Woodbury formulation.

Off-diagonal low-rank property

- same rows in B_{off} are compressed into one row of H
- F is an index matrix which can be represented by an integer vector

We define the total compression factor (tcf) as

$$tcf = nnz(B_{off}) - nnz(H) \tag{6}$$

If B_{off} has many same row patterns and the same row patterns are relatively dense, we can get a large tcf.

$\longrightarrow$ Expect memory and computing time saving.

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Exploit Internet structure

Does there exist a partition guarantee such a B_{off} which brings large *tcf*?

The web internet has a block structure according to the internet address.

$$\underbrace{\text{www.rug.nl}}_{1_{\text{st}}\ \text{level}}\underbrace{/\text{education}}_{2_{\text{nd}}\ \text{level}}\underbrace{/\text{find}-\text{out}-\text{more}}_{3_{\text{rd}}\ \text{level}} \qquad /\text{parents}/$$

several levels
each level has domains like
"parents" and "find-out-more".

- Pages has more connections to the other pages in the same domain at each level
- Pages in a domain may get some similar behaviour on connecting with other domains

These characters guarantee a possibility to generate such a partition with large *tcf*.

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How to generate it?

Try to get as large *tcf* as possible

We only look at the sparsity pattern matrix S of A. Denote set R as $\{1, 2 \ldots n\}$.

For a group N, the off-diagonal part between this group is defined as $S(N, R \backslash N)$.

$$
\begin{array}{c} \\ i \\ j \\ k \\ l \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & 2 & \cdots & i & \cdots & j & \cdots & k & \cdots & l & \cdots & n \end{array} \\
\begin{pmatrix}
* & * & \cdots & * & \cdots & * & \cdots & * & \cdots & * & \cdots & * \\
* & * & \cdots & * & \cdots & * & \cdots & * & \cdots & * & \cdots & * \\
* & * & \cdots & * & \cdots & * & \cdots & * & \cdots & * & \cdots & * \\
* & * & \cdots & * & \cdots & * & \cdots & * & \cdots & * & \cdots & *
\end{pmatrix}
\end{array}
$$

blue represents the off-diagonal part, black represents the diagonal part

We call the row patterns of this off-diagonal part as off-diagonal patterns of N
Basic idea:

gather rows if their off-diagonal patterns are the same.

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Try to get as large *tcf* as possible

First we permute rows of S in order of decreasing density.
Initially set the group of row i as: $i.group = \{i\}$.

Algorithm 1

1) **for** $i = 1 : n$, if row i is unmarked
2) **for** $j = (i + 1) : n$, if row j is unmarked
3) **if** the off-diagonal patterns between row j and the rows in $i.group$ are same.
4) $i.group \leftarrow \{i.group \cup j\}$; mark row j; back to 1) still at current i.

output: block ordering

Example.1

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0 & * & * & * & * & * & * & * & 0 \\ * & 0 & 0 & * & * & * & * & * & 0 \\ 0 & * & 0 & * & * & * & * & * & 0 \\ * & 0 & 0 & * & * & * & * & 0 & * \\ * & 0 & 0 & * & * & * & * & 0 & * \\ * & 0 & 0 & * & * & * & * & 0 & * \\ * & 0 & 0 & * & * & * & * & 0 & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 \end{pmatrix} \end{array}$$

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Try to get as large *tcf* as possible

This process will result in a blocked matrix.

- All the off-diagonal blocks have rank 1 or 0.
- The *tcf* is 11.

We found that it doesn't achieve the largest *tcf*

For example, consider the group corresponding to rows {4-7}

Example.1

	1	2	3	4	5	6	7	8	9
1	0	*	*	*	*	*	*	*	0
2	*	0	0	*	*	*	*	*	0
3	0	*	0	*	*	*	*	*	0
4	0	0	0	*	*	*	*	0	0
5	0	0	0	*	*	*	*	0	0
6	0	0	0	*	*	*	*	0	0
7	0	0	0	*	*	*	*	0	0
8	0	0	*	0	0	0	0	*	*
9	0	0	*	0	0	0	0	*	0

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Try to get as large *tcf* as possible

$$
\begin{array}{c}
4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{pmatrix}
0 & 0 & 0 & \boxed{*} & * & * & * & 0 & 0 \\
0 & 0 & 0 & * & \boxed{*} & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & \boxed{*} & * & 0 & 0 \\
0 & 0 & 0 & * & * & * & \boxed{*} & 0 & 0
\end{pmatrix}
$$

These rows have same patterns, leading them all in a group finally.
During this process, the compression factor reaches a local maximum after gathering row 5 to row 4 as a group, then it will be decreased through the following process of adding row 6, 7.

We add a criterion: each gathering must increase the *tcf*. Then row 6 and 7 will form another group instead of being added to {4,5}

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left(\begin{array}{ccc|cc|cc|cc}
0 & * & * & * & * & * & * & * & 0 \\
* & 0 & 0 & * & * & * & * & * & 0 \\
0 & * & 0 & * & * & * & * & * & 0 \\
\hline
0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 \\
\hline
0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 \\
\hline
0 & 0 & * & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & * & 0 & 0 & 0 & 0 & * & 0
\end{array}\right)
\end{array}
$$

with $tcf = 15$ which is larger than the previous $tcf = 11$.

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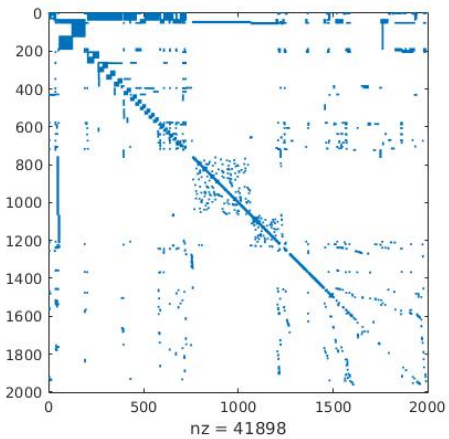
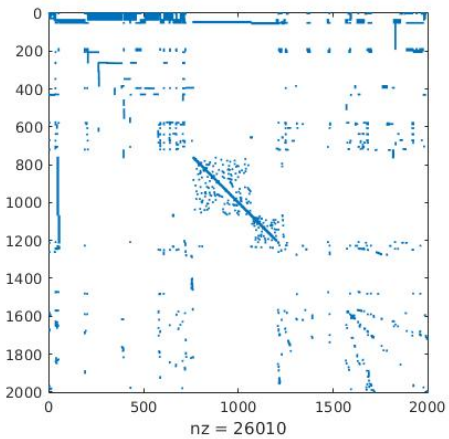
Try to get as large *tcf* as possible

We take a small web crawl of .in domain with size=2000, nnz=59,528.
Show the total compression factor(*tcf*) we can get,

and the difference between with and with out the criterion.





the off-diagonal part(B_{off}) with and without the criterion.

We can see more compressible blocks in B_{off} with this criterion.

methed	p	*tcf*	$\frac{tcf}{nnz}$	inne_size
`Ag1 with criterion`	1008	28801	48.38%	970
`Ag1 without criterion`	857	13812	23.20%	817

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p is the partition number, *inner_size* is denoted as the row size of *H*.

Gathering blocks to decrease the size of the inner-system

Our preconditioner: $M \approx A^{-1} = D^{-1}(I - F(I + HD^{-1}F)^{-1}HD^{-1})$.

Application of M requires: inverse D and the inner-system $Inn = (I + HD^{-1}F)$

The previously-discussed partition method doesn't guarantee a satisfied performance of M, which will be shown latter.

We thought the unsatisfactory performance is caused by the large Inn, because Inn is usually much more dense than A.

large $Inn \longrightarrow$ large memory and time for computing $Inn^{-1}v$ when using ilu factorization

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Control the size of Inn below a user-defined upper bound.

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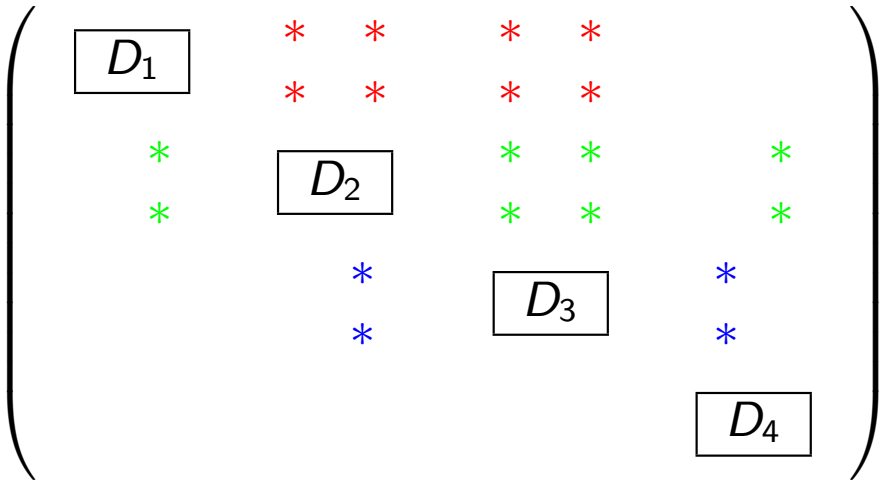
Gathering blocks to decrease the size of the inner-system

As $Inn = (I + HD^{-1}F)$, the size of the Inn is equal to the row amount in H which is equal to the amount of different row patterns in B_{off}.

After the previously-discussed process, each group of rows must have the same off-diagonal patterns.

Example.2



contribute 1 to the size

contribute 1 to the size

contribute 1 to the size

contribute 0 to the size

Each group's off-diagonal part contributes 0 or 1 to the size of Inn. We call the groups which have contribution to the size of Inn as contribution groups.

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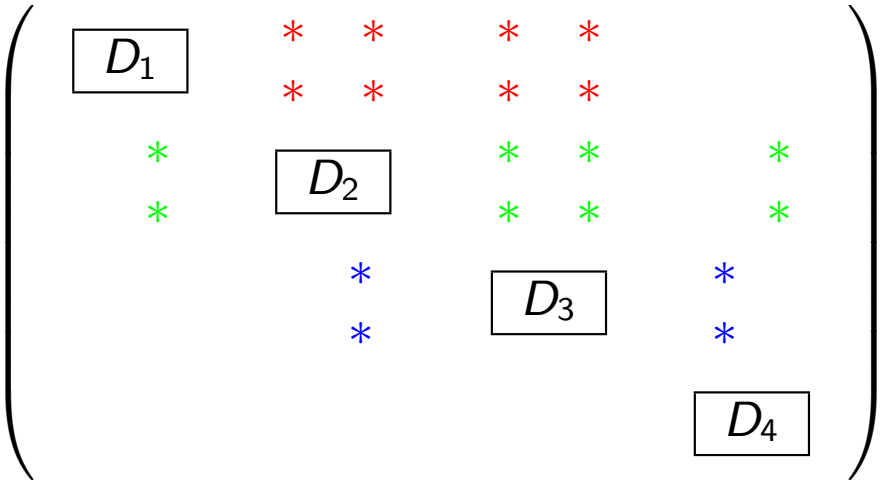
Gathering blocks to decrease the size of the inner-system

We consider to decrease the size of inner-system by making some group's contribution from 1 to 0.

If the contribution of a group is 1, there must be some nonzero off-diagonal blocks at this block row and define the corresponding groups as the row neighbours.



row neighbours : $2, 3$

row neighbours : $1, 3, 4$

row neighbours : $2, 4$

none row neighbours

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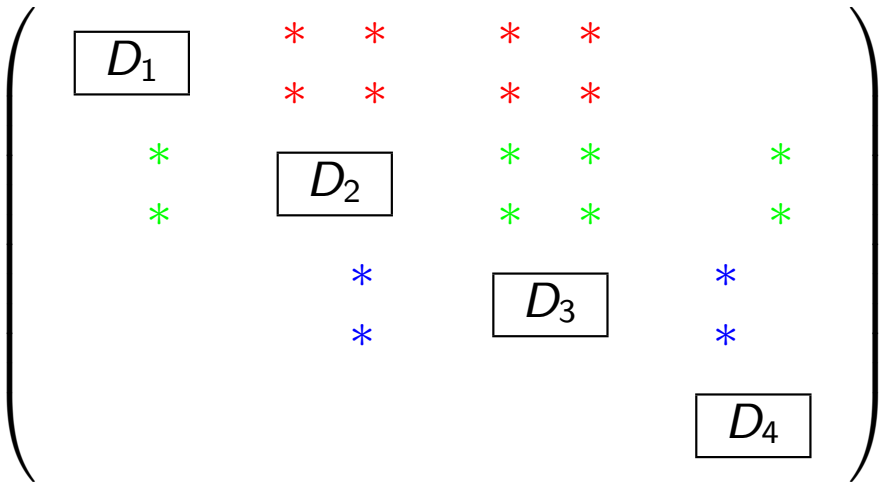
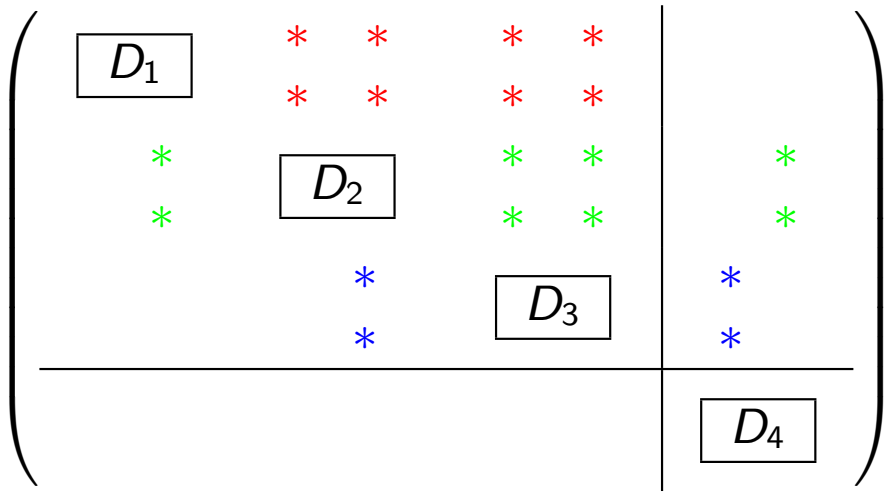
Gathering blocks to decrease the size of the inner-system

If we choose a group and gather it with its row neighbours, the size of the *Inn* will decrease at least by 1.
We call this gathering as Neighbourhood gathering.

The size of *Inn* is 3.
(red, green and blue)



Gathering group 1 with its row neighbours: group 2 and 3.



The size of the *Inn* becomes 2.
only green and blue.

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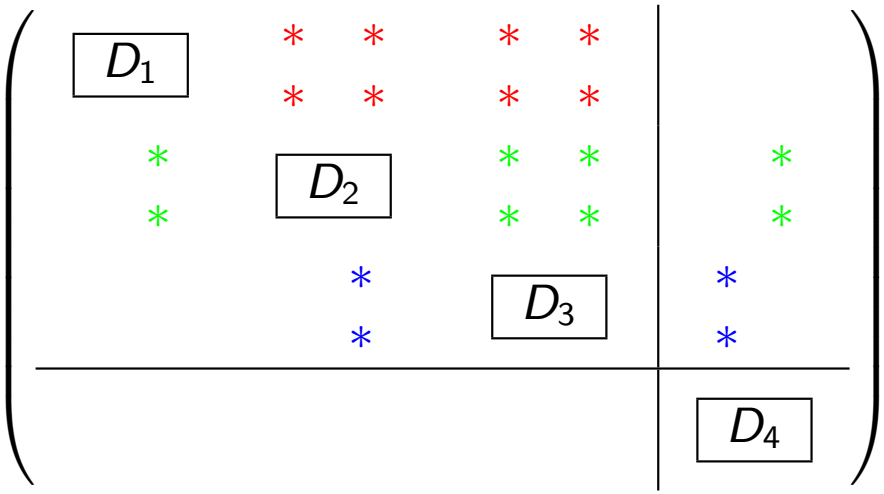
Gathering blocks to decrease the size of the inner-system

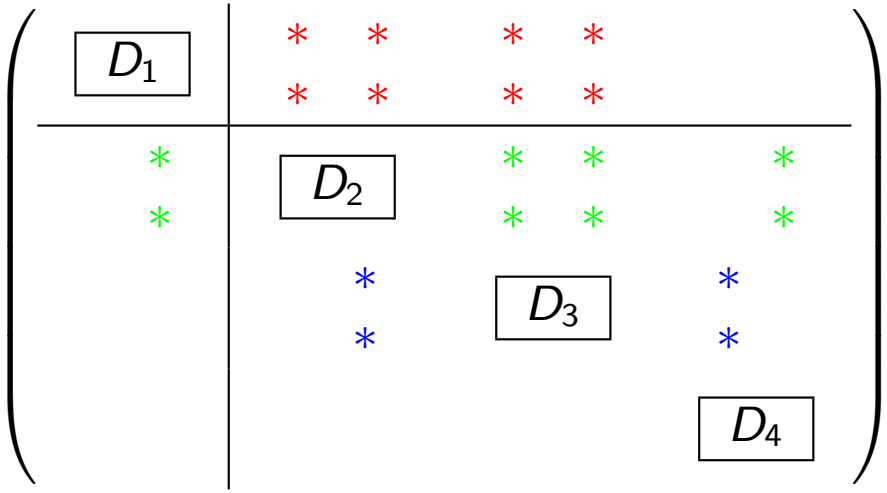
Each step of neighbourhood gathering decreases the size of *Inn*.
However, it also put more elements in the diagonal blocks which can't be compressed immediately. $\longrightarrow$ *tcf* $\downarrow$
We define a index named diagonal fill-in to record this of each group.

neighbourhood gathering on group 1.



its diagonal fill-in is 16.

neighbourhood gathering on group 3.



its diagonal fill-in is 10.

Both decrease the size of *Inn* by 1.

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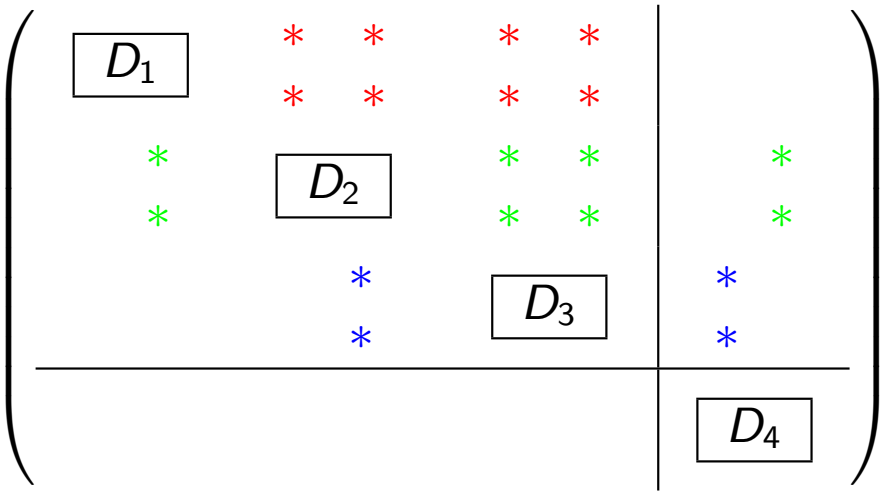

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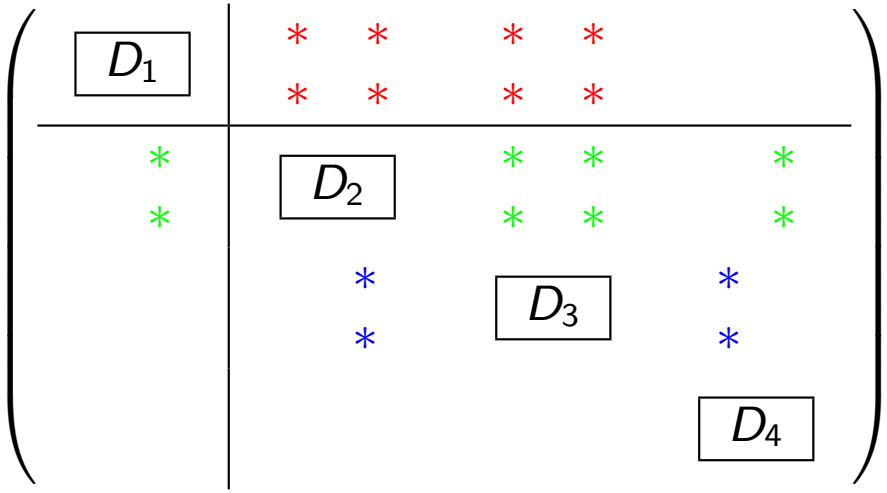
neighbourhood gathering on group 3.



its diagonal fill-in is 10.

Both decrease the size of *Inn* by 1.
We prefer the latter case which causes fewer diagonal fill-in.

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Gathering groups to decrease the size of the inner-system

Basic idea: Each step we choose a contribution group with minimal diagonal fill-in, implement neighbourhood gathering to this group.

Given an upper bound τ for the size(Inn).
Algorithm 2:

1) **while** size($inner$)$\geq \tau$
2) implement neighbourhood gathering to a contribution group which causes minimal diagonal fill-in.
3) update group information

The whole partition method combines Ag 1 and Ag 2.

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Gathering groups to decrease the size of the inner-system

Compare the whole partition method with the algorithm 1 alone.
Simply set $\tau = size(A)/20$, compare the performance of this preconditioner on $GMRES(20)$[1].

web crawl of .in with size 7000.

methed	size(Inn)	$\frac{nnz(D)}{nnz(A)}$	it	time	$\frac{nnz(M)}{nnz(A)}$
whole partition	346	48.84%	8	0.095s	1.17
Algorithm 1	3306	28.08%	15	0.135s	1.30

web crawl of .eu with size 7000.

methed	size(Inn)	$\frac{nnz(D)}{nnz(A)}$	it	time	$\frac{nnz(M)}{nnz(A)}$
whole partition	347	34.25%	9	0.096s	1.26
Algorithm 1	4026	13.04%	16	0.146s	1.86

- residual threshold is $1e-8$
- we use the *ilupack* as solver for $D^{-1}v$ and $Inn^{-1}v$
- all the parameters for *ilupack* keep the same

nnz of D increase little compared with the size decreased of Inn

memory saved; increased convergence rate; less time

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[1] we choose $\alpha = 0.999 \Rightarrow$ very accurate and difficult PageRank model

Compared with *ilupack*

As the solver for $D^{-1}v$ and $Inn^{-1}v$ is *ilupack* which is a state-of-the-art solver(preconditioner).
Some experiments of our preconditioner against the standalone *ilupack* preconditioner are presented to test the effect of our strategy[2].

Droptol for our preconditioner keeps *1e-1*. Tune the droptol of the ilupack to limit the memory cost($\frac{nnz(M)}{nnz(A)}$) nearly the same.

web crawl of .eu domain with size 7,000.

methed	it	time	memory
`our method`	9	0.096s	1.26
`ilupack`	38	0.195s	1.27
`Gmres(20)`	241	0.695s	#

web crawl of .in domain with size 7,000.

methed	it	time	memory
`our method`	8	0.095s	1.17
`ilupack`	40	0.221s	1.18
`Gmres(20)`	1031	3.140s	#

web crawl of stanford-berkeley with size 70,000.

methed	it	time	memory
`our method`	13	0.371s	1.19
`ilupack`	90	4.120s	1.19
`Gmres(20)`	>5000	>67.5s	#

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[2] we choose $\alpha = 0.999 \Rightarrow$ very accurate and difficult PageRank model

Conclusion and following work

Conclusion

- Both our preconditioner and ilupack preconditioners are efficient for PageRank problems
- Our method shows the potential for improving preconditioners' performance by exploiting the special property of PageRank problems

Following work

- Optimize the implementation of the partition method
- Test larger problems

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