Multilevel Krylov for Symmetric Singular Systems

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Define the two-level preconditioner [1]

$$P_N = I - AZE^{-1}Z^T + \lambda_n ZE^{-1}Z^T, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $E = Z^T AZ$, $Z \in \mathbb{R}^{n\times k}$, with $rank(Z) = k$, and $\lambda_n = \max_{0\le i\le n} |\lambda_i(A)|$. We consider solutions of the consistent system $Ax = b$, where A is a symmetric, singular matrix of coefficients, by solving either $P_N Au = P_N b$ (left precondition) or $AQ_N\tilde{u} = b$ (right precondition), with $Q_N = P_N^T$ and $u = Q_N\tilde{u}$. The coarse-grid matrix E in (1) is in general singular, but can be forced to be nonsingular by requiring $\mathcal{N}(A) \nsubseteq \mathcal{R}(Z)$.

Since $P_N A$ and AQ_N are not symmetric, the preconditioned systems must be solved by a Krylov method for nonsymmetric system, like GMRES. Convergence of GMRES, however, is not guaranteed since $P_N A$ or AQ_N is singular and, in general, not range-symmetric. They are range-symmetric if, for instance, columns of Z consists of orthogonal eigenvectors of A that are not associated with $\lambda(A) = 0$. In this case, the result of [2] guarantees that GMRES converges to the least-squares or pseudo-inverse solution. For general vectors used for columns of Z, one possible solution which can be extracted from a Krylov subspace is associated with the Drazin inverse of the singular matrix $P_N A$ or AQ_N, via DGMRES [3].

In this talk, we shall present theoretical aspects of the multilevel Krylov for singular systems and some numerical results, in the two-level and multilevel setting (i.e., when E^{-1} is computed approximately and iteratively). The numerical results demonstrate mesh-independent convergence of the method, applied to, e.g., the Poisson equation with pure Neumann boundary conditions.

Keywords: Krylov subspace, multilevel preconditioner, singular system.

References

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