# ALGORITHM FOR SOLVING LEAST-SQUARES PROBLEMS WITH A HELMHOLTZ BLOCK AND MULTIPLE RIGHT HAND SIDES

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Solving large (1e6 - 1e9 grid points) Helmholtz equations is a challenging problem on its own. Recently, PDE-constrained optimization has inspired the development of a reduced-space quadratic penalty method for nonlinear constrained optimization. This method requires repeated solutions of linear least-squares sub-problems. The problem of interest is mildly overdetermined:

$$\bar{\mathbf{u}} = \operatorname*{argmin}_{\mathbf{u}} \left\| \begin{pmatrix} \lambda H(\mathbf{m}) \\ P \end{pmatrix} \mathbf{u} - \begin{pmatrix} \lambda \mathbf{q} \\ \mathbf{d} \end{pmatrix} \right\|_2,$$

where $H(\mathbf{m}) \in \mathbb{C}^{N\times N}$ is a sparse, full-rank discrete linear operator arising from the Helmholtz equation and it depends on the medium parameters $\mathbf{m} \in \mathbb{R}^N$; $P \in \mathbb{R}^{m\times N}$ is a sparse matrix which selects the field variables $\mathbf{u} \in \mathbb{C}^N$ at the receiver locations; $\mathbf{d} \in \mathbb{C}^m$ is the observed field; the source term is indicated by $\mathbf{q} \in \mathbb{C}^m$ and $\lambda$ is a scalar penalty parameter.

Typical number of rows in $P$ (receivers) is between $[1 - 100]$ and the number of right hand sides (sources) is between $[1 - 1000]$.

The proposed approach is related to [1]. We reformulate the problem, generating an identity + low-rank structure in the system matrix. To achieve this, we invest computational resources to compute explicitly $W = (\lambda H)^{-*}P^*$. The resulting preconditioned system is given by

$$(I + WW^*)\mathbf{y} = \lambda\mathbf{q} + W\mathbf{d}, \quad \text{with} \quad \lambda H\bar{\mathbf{u}} = \mathbf{y},$$

where $WW^*$ is a rank $m$ matrix. The computation of $W$ is done once and for all and can be used efficiently for a large number of right hand sides (sources). Posing the system as a low-rank correction of the identity matrix allows for a fast inversion procedure, which entails a modest memory requirement, capitalizing on the fact that $P$ typically has a small number of rows.

To solve systems with $H$, we use matrix-free solvers with very low memory consumption, such as the CGMN method (4 working vectors), CARP-CG [2] or methods based on a shifted-Laplacian preconditioning approach for Helmholtz specifically.

To further accelerate convergence, we apply inexact solves, and derive error bounds that relate the relative error of the solution to the relative residual of the PDE solves.

The proposed method can solve systems of dimensions up to $N = 1e9$ on one compute node with 20 cores and 128 GB. Our method is suitable for a small number of rows of $P$, but there are ways in which it can be extended to other settings. Current work involves randomized approximation of the data part of the system, $P$ and $\mathbf{d}$, by way of undersampling, practically reducing the number of rows in $P$.

## REFERENCES

[1] LM Delves and I Barrodale, *A fast direct method for the least squares solution of slightly overdetermined sets of linear equations*, IMA Journal of Applied Mathematics, 24 (1979), pp. 149–156.

[2] D. Gordon and R. Gordon, *CARP-CG: A robust and efficient parallel solver for linear systems, applied to strongly convection dominated pdes*, Parallel Computing, 36 (2010), pp. 495–515.