Preconditioning saddle point systems with approximate null space factorizations

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Abstract

We wish to solve the linear equation

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix}, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times n}$. Given $\hat{\boldsymbol{x}}$ that satisfies $B\hat{\boldsymbol{x}} = \boldsymbol{g}$ and a matrix Z whose columns span the null-space of B, we can solve (1) by solving the equivalent problem

$$Z^T AZ\boldsymbol{z} = Z^T(\boldsymbol{f} - A\hat{\boldsymbol{x}}),$$

since $\boldsymbol{x} = Z\boldsymbol{z} + \hat{\boldsymbol{x}}$. This is null-space method, and it has long been used as a way to reduce the coefficient matrix to a symmetric (often positive definite) matrix of dimension $n - m$, that is correspondingly easier to solve.

Recently a number of factorizations have been proposed for systems of the form (1) that can be shown to be equivalent to the null space method; see, e.g., [1] for an overview.

In this talk I will describe a number of preconditioning paradigms that are based upon (block-wise) incomplete versions of a null-space factorization. Such methods are particularly attractive in cases where $n - m$ is small, but also often performs well when applied to problems with small m. Such preconditioners require the user to provide an invertible subset of the columns of B, and so they are most applicable to problems that either have a structure where this comes at little or no cost (e.g., PDE-constrained optimization or resistor network problems) or problems where families of equations with the same B block have to be solved (e.g., interior point methods in optimization). I will give numerical results from a range of problems demonstrating the strengths of this approach.

References

[1] Tyrone Rees and Jennifer Scott, *The null-space method and its relationship with matrix factorizations for sparse saddle point systems*, Tech. Report RAL-TR-2014-016, STFC Rutherford Appleton Laboratory, 2014.