Uniform Schur complement preconditioning for XFEM applied to time dependent Stokes problems

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This is joint work with S. Groß, T. Ludescher, M. Olshanskii.

Abstract

In this contribution we treat the following Stokes problem on a connected polygonal domain Ω in d-dimensional Euclidean space ($d = 2, 3$): Find a velocity u, with $u = 0$ on $\partial\Omega$, and a pressure p such that

$$\begin{aligned} \tau u - \operatorname{div}(\mu(x)D(u)) + \nabla p = f \quad \text{in } \Omega, \\ \operatorname{div} u = 0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

with $D(u) := \frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$, $\tau \geq 0$ a given constant and μ a given strictly positive viscosity function. This model problem is motivated by time dependent two-phase Stokes flow problems, which after time discretization result in a quasi-stationary problem of the form (1). In such two-phase flow applications there is an interface Γ, which separates the domain into two subdomains, denoted by Ω_i, $i = 1, 2$. The viscosity typically has different constant values $\mu_i > 0$ in the two subdomains. In the source term f there are surface tension forces acting only at the interface Γ. These result in a *discontinuity in the pressure* across the interface Γ. Hence, (1) has to be replaced by a suitable weak formulation. In applications the interface is unknown and finding its location is part of the numerical simulation. Very often the fluid dynamics problem is coupled with an interface capturing technique, e.g. the level set method. In such a setting, typically in the discretization of the flow equations the interface is *not* aligned with the grid. This causes difficulties with respect to an accurate discretization of the flow variables. Recently, extended finite element techniques (XFEM) have been developed to obtain accurate finite element discretizations. We consider one particular XFEM, in which the pressure variable is approximated in a conforming P_1-XFE space and the velocity is approximated in the standard conforming P_2-FE space. This is a popular pair of spaces for the discretization of two-phase incompressible flows. The pair is not LBB stable and therefore a stabilization technique is needed. This stabilized discretization is analyzed for the *stationary variant* of (1), i.e., $\tau = 0$, in [1]. An inf-sup stability result is derived with the key property that the stability constant is *uniform with respect to h, the viscosity quotient μ_1/μ_2 and the position of the interface in the triangulation*. Based on this result, optimal discretization error bounds are derived in [1]. We use this discretization for the quasi stationary problem (1). This results in a symmetric saddle point system. Optimal preconditioners for the velocity block in the stiffness matrix are known (e.g., multigrid). In [1] a robust Schur complement preconditoner for the *stationary case* ($\tau = 0$) is presented. In this presentation we present a new Schur complement preconditioner for the quasi-stationary case ($\tau \geq 0$). We restrict to the case $\mu_1 = \mu_2$ and propose a preconditioner that is *robust with respect to h, τ and the position of the interface in the triangulation.*

[1] M. Kirchhart, S. Groß, A. Reusken, *Analysis of an XFEM Discretization for Stokes Interface Problems*, IGPM Preprint 420, RWTH Aachen University (2015). Submitted.