Problem #23 (Solved !)

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Summary: Must any termination ordering *used for proving termination of the* Battle of Hydra and Hercules-*system have the Howard ordinal as its order type?*

The following system [DJ90], based on the "Battle of Hydra and Hercules" in [KP82], is terminating, but not provably so in Peano Arithmetic:

$$
\begin{array}{rcl}
h(z, e(x)) & \to & h(c(z), d(z, x)) \\
d(z, g(0, 0)) & \to & e(0) \\
d(z, g(x, y)) & \to & g(e(x), d(z, y)) \\
d(c(z), g(g(x, y), 0)) & \to & g(d(c(z), g(x, y)), d(z, g(x, y))) \\
g(e(x), e(y)) & \to & e(g(x, y))
\end{array}
$$

Transfinite (ϵ_0-) induction is required for a proof of termination. Must any termination *ordering* have the Howard ordinal as its order type, as conjectured in [Cic90]?

Remark

If the notion of termination ordering is formalized by using ordinal notations with variables, then a termination proof using such orderings yields a slow growing bound on the lengths of derivations. If the order type is less than the Howard-Bachmann ordinal then, by Girard's Hierarchy Theorem, the derivation lengths are provably total in Peano Arithmetic. Hence a termination proof for this particular rewrite system for the Hydra game cannot be given by such an ordering [Andreas Weiermann, personal communication].

Remark

This has been answered to the negative by Georg Moser [Mos09], by giving a reduction order that is compatible with the above rewrite system, and whose order type is at most ϵ_0 (the proof theoretic ordinal of Peano arithmetic).

Bibliography

[Cic90] E. A. Cichon. Bounds on derivation lengths from termination proofs. Technical Report CSD-TR-622, Department of Computer Science, University of London, Surrey, England, June 1990.

[DJ90] Nachum Dershowitz and Jean-Pierre Jouannaud. Rewrite systems. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B: Formal Methods and Semantics, chapter 6, pages 243–320. North-Holland, Amsterdam, 1990.

[KP82] Laurie Kirby and Jeff Paris. Accessible independence results for Peano arithmetic. *Bulletin London Mathematical Society*, 14:285–293, 1982.

[Mos09] Georg Moser. The Hydra battle and Cichon's principle. *Applicable Algebra in Engineering, Communication and Computing*, 20(2):133–158, 2009. `doi:10.1007/s00200-009-0094-4`.