On linear subspaces of the Hilbert nullcone and polarization in invariant theory

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Consider the usual representation of SL_n on the symmetric bilinear forms Sym_n by means of $g \cdot A \mapsto (g^{-1})^t A g^{-1}$. Let $H \subset Sym_5$ be a subspace on which the determinant vanishes identically.

Question: Is H equivalent to a subspace of either

$$\begin{bmatrix} * & * & * & * & \\ * & * & * & * & \\ * & * & * & * & \\ * & * & * & * & \\ & & & & \end{bmatrix} \text{ or } \begin{bmatrix} * & * & * & * & * \\ * & * & * & & \\ * & * & * & & \\ * & & & & \\ * & & & & \end{bmatrix} \text{ or } \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & & & \\ * & * & & & \\ * & * & & & \end{bmatrix}$$

under the above operation of SL_5?

This question has the following background in classical invariant theory: For a reductive group G and a complex representation V we denote by $\mathcal{O}(V)^G$ the ring of invariant polynomial functions. The Hilbert nullcone $\mathcal{N}_V \subset V$ is the zero set of all non- constant homogeneous elements of $\mathcal{O}(V)^G$. Even when V is irreducible, finding the generators of $\mathcal{O}(V)^G$ is usually very difficult. Even more so, if we are looking for the generators of $\mathcal{O}(V^{\oplus k})^G$, where the operation of G on $V^{\oplus k}$ is given by $g(v_1, \ldots, v_k) = (gv_1, \ldots, gv_k)$. In this talk I will explain an interesting connection between the structure of the linear subspaces of the nullcone $\mathcal{N}_V$ on one hand, and the question, whether a certain set of invariants of $\mathcal{O}(V^{\oplus k})^G$ (obtained by the classically known *polarization process*) defines the nullcone $\mathcal{N}_{V^{\oplus k}} \subset V^{\oplus k}$ on the other hand.

By a result of HILBERT, finding invariants that define the nullcone $\mathcal{N}_{V^{\oplus k}} \subset V^{\oplus k}$ is an important step in finding a complete set of generators for $\mathcal{O}(V^{\oplus k})^G$.

For the representation of SL_n on Sym_n the invariant ring is generated by the determinant and hence the nullcone $\mathcal{N}_{Sym_n}$ is the set of all forms on which the determinant vanishes. For $n = 5$ the above mentioned connection leads exactly to the question posed above. Its answer is 'no', however.