Witt rings and their applications to the theorem of Brauer-Nesbitt

Rob Eggermont

Abstract

Let k be a field and A a unital, associative k-algebra. Let B be a subset of A that generates A as a k-vector space. Let M, N be (left) A-modules that are finite-dimensional as a k-vector space.

It has been known for some time that if k has characteristic 0, then one has the following: M and N are Jordan-Hölder isomorphic if and only if for all $b \in B$, the traces of the action of b on M and N are equal. This result is not necessarily true if k has positive characteristic p; there are in general non-zero A-modules P such that the trace of the action of any element of A on P equals 0. In 1937, Richard Brauer and Cecil Nesbitt showed that, regardless of the characteristic of k, the modules M and N are Jordan-Hölder isomorphic if and only if for all $b \in B$, the characteristic polynomials of the action of b on M and N are equal.

Taking a slightly non-standard definition of the characteristic polynomial, the characteristic polynomials of the actions of elements of A on M live in the ring $1+Tk[[T]]$ of power series with constant coefficient 1. In 1997, Arjeh Cohen, Gábor Ivanyos and David Wales showed that if k has positive characteristic p, the modules M and N are Jordan-Hölder isomorphic if and only if for all $b \in B$, the projections of the characteristic polynomials to $1 + kT + kT^p + kT^{p^2} + ...$ of the action of b on M and N are equal. However, the latter set is not a group.

By a slight shift in notation, one can identify $1 + Tk[[T]]$ with the so-called Witt ring $W(k)$ of k. This ring can be projected to a ring $W_p(k)$ that is in a way comparable to the set $1+kT+kT^p+kT^{p^2}+...$ in the theorem by Cohen, Ivanyos and Wales. We will use Witt rings to find an easier proof of this theorem, as well as find some further ways to describe when two modules are Jordan-Hölder isomorphic.