WHAT ARE THE FOCI OF A PLANAR CURVE?

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ABSTRACT. There is a classical notion of *foci* for real algebraic curves which encompasses the usual foci of a quadric. Such foci appear, for instance, in *Kippenhahn's Theorem* on the eigenvalues and numerical range of an $n \times n$ complex matrix and in *Siebeck's Theorem* on the relative positions of roots and critical points of a complex polynomial (or positions of vortices and stagnation points of ideal planar fluid flows). However, the general definition gives an entirely different geometric interpretation of foci which is not very familiar—even as it applies to an ellipse!

I will discuss the general notion of focus of a curve C in an elementary context and show many graphical examples. It will be seen that foci are intimately related to Schwarzian reflection in C as well as analytic continuation of arclength parametrization of C. In the same context, I will explain how *quadrature formulas* arise for integration of harmonic functions over certain planar domains by finite sums. The unified approach to these topics will emphasize the *Schwarz function* of a real curve and leads to the study of singularities of certain meromorphic functions and differentials on the corresponding Riemann surface.