On colourings of projective planes

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The chromatic number of a hypergraph is the smallest number of colours needed to colour the points so that no edge is monochromatic. Projective planes of order greater than 2 have chromatic number 2. This simply means that projective planes of order greater than 2 have non-trivial blocking sets. We shall discuss general results on 2-colourability of n-uniform hypergraphs and their consequences for projective spaces. The 2-colouring can also be described by a function $f : V \to \{-1, +1\}$, where V is the ground set and the discrepancy of an edge E is just $|\sum_{x \in E} f(x)|$. Intuitively, this measures the difference of the sizes of the two colour classes along E. The discrepancy of a hypergraph is the maximum discrepancy of its edges. To determine the discrepancy of a projective plane is not trivial at all. Standard equations for one colour class give that it is at least $\sqrt{q}$ and Spencer proved in 1989 that this is the right order of magnitude. We will discuss other interesting questions (with some old answers) about 2-colourings of projective planes.

In the second part of the talk we shall discuss results about the upper chromatic number of projective planes. The notion comes from Voloshin's work on colourings of mixed hypergraphs. For a finite plane Π, the upper chromatic number $\overline{\chi}(\Pi)$ denotes the maximum number of colours in a colouring of the points such that each line has at least two points of the same colour. So, instead of excluding monochromatic lines we exclude "rainbow" ones. Few results are known for general planes, mainly due to Bacsó and Tuza. In the talk we focus on determining or bounding the upper chromatic number of $\mathrm{PG}(2, q)$. The following construction relates the upper chromatic number and the minimum size of a double blocking set: take a double blocking set and colour all of its points red. The remaining points get pairwise different colours. This shows that $\overline{\chi}(\Pi) \geq v - \tau_2(\Pi) + 1$, where v denotes the number of points and τ_2 is the minimum size of a double blocking set. In several cases we could prove that we actually have equality in this bound.

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