Tropical bases by regular projections

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First session:

- Tropical hypersurface
- Newton polytope and its subdivision
- Properties of the tropical hypersurface
- Real valuations, Puiseux series
- Tropicalization of a polynomial
- Tropical variety and prevariety and their properties
- Tropical bases
- Linear spaces

Second session:

- The main theorem
- Construction of appropriate projections
- Preimage of a projection
- Example

The main algebraic structure in tropical geometry is the tropical semiring:

Definition. *The tropical semiring is the triple* $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ *with the following tropical addition and multiplication*

$$\begin{aligned} a \oplus b &:= \min(a, b) \\ a \odot b &:= a + b \end{aligned}$$

We observe:

- The compositions are commutative, associative und distributive.
- ∞ is the additive and 0 is the multiplicative neutral element
- There is NO TROPICAL SUBTRACTION.

Tropical polynomial ring in the unknowns $x_1, \ldots, x_n$:

Monomials:

$$x_1^{\odot i_1} \odot x_2^{\odot i_2} \odot \ldots \odot x_n^{\odot i_n}$$
$$i_1, \ldots, i_n \in \mathbb{Z}$$

Polynomials: Finite linear combinations of monomials

$$\begin{aligned} p(x_1, \ldots, x_n) &= a \odot x_1^{i_1} \odot x_2^{i_2} \odot \ldots \odot x_n^{i_n} \oplus \\ & \quad\; b \odot x_1^{j_1} \odot x_2^{j_2} \odot \ldots \odot x_n^{j_n} \oplus \ldots \\ &= \min\{a + i_1 x_1 + \ldots + i_n x_n, b + j_1 x_1 + \ldots + j_n x_n, \ldots\} \end{aligned}$$

Tropical hypersurface

Definition. *Let $p(x_1, \ldots, x_n)$ be a tropical polynomial. Then*

$$\mathcal{T}(p) := \{x \in \mathbb{R}^n \mid \textit{the minimum is attained at least twice in } x\}$$

is the *tropical hypersurface* *defined by p.*

Example. *Tropical curves in the plane*

$p := 0 \odot x_1 \oplus 1 \odot x_2 \oplus 2$ $\qquad$ $q := 1 \oplus 0 \odot x_1 \oplus 0 \odot x_2 \oplus 1 \odot x_1^2 \oplus 0 \odot x_1 \odot x_2 \oplus 1 \odot y^2$

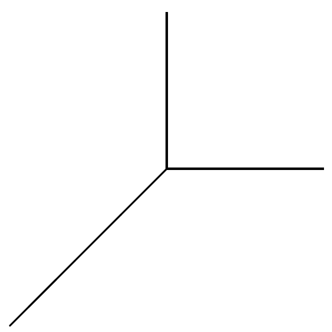


Figure: a tropical line

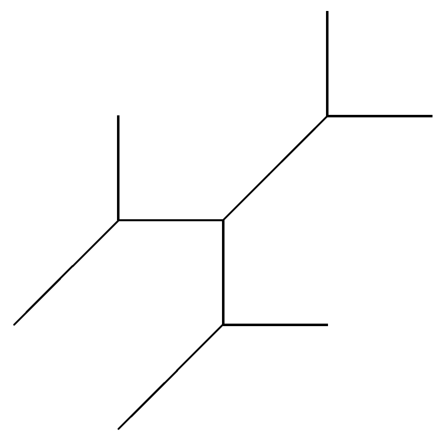


Figure: a tropical quadratic curve

Definition. *Let f be a tropical polynomial in the unknowns $x_1, \ldots, x_n$ with terms $x_1^{i_1} \odot \ldots \odot x_n^{i_n}$. Then the* Newton polytope *of f is the convex hull*

$$\mathrm{New}(f) := \mathrm{conv}\{(i_1, \ldots, i_n) \in \mathbb{R}^n \ : \ x_1^{i_1} \cdots x_n^{i_n} \ \textit{a term in } p\}$$

Example. $f = 2 \odot x^2 \ \oplus \ x^2 \odot y^3 \ \oplus \ 3 \odot x \odot y^2 \ \oplus \ 1$

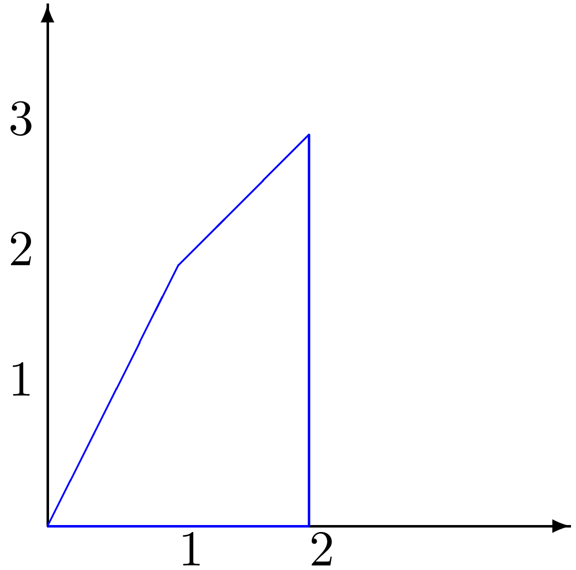


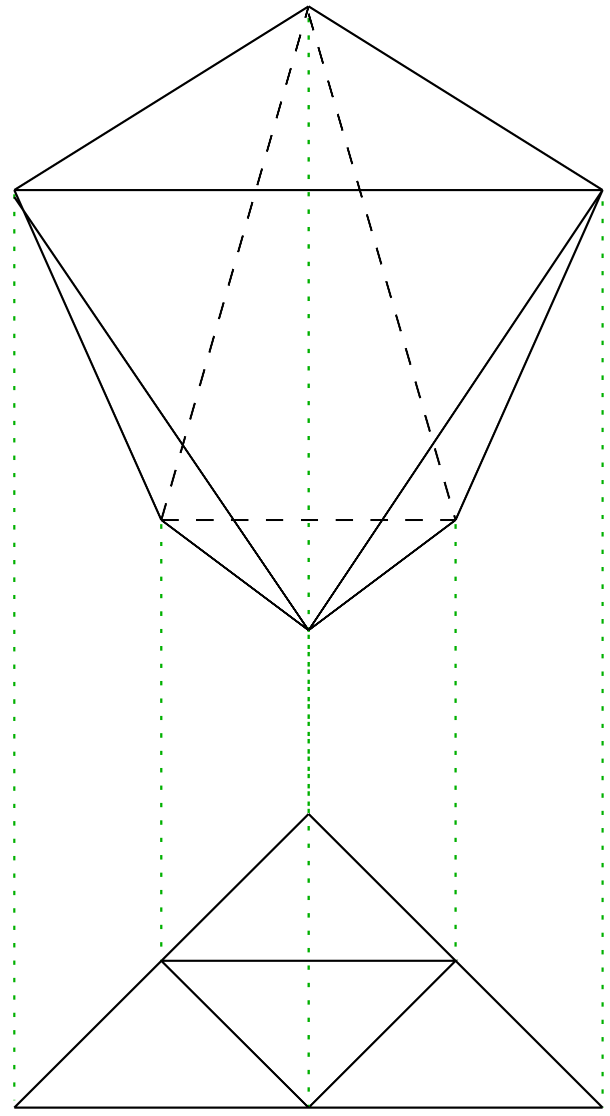
Figure: the Newton polytope of f

Definition. *For a tropical polynomial p the* *extended Newton polytope* *is:*

$$\mathrm{conv}\{(i_1, \ldots, i_n, c_i) : c_i \odot x_1^{i_1} \odot \ldots \odot x_n^{i_n} \text{ a monomial in } p\}$$

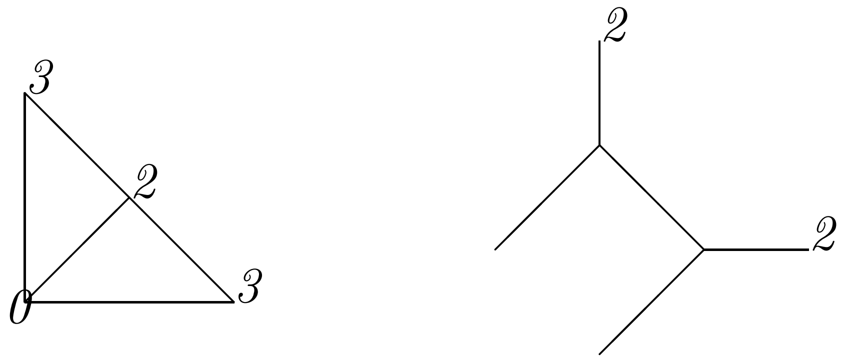
The projection of the lower bound onto the first n coordinates gives a subdivision of the Newton polytope.

The tropical hypersurface is dual to that subdivision.

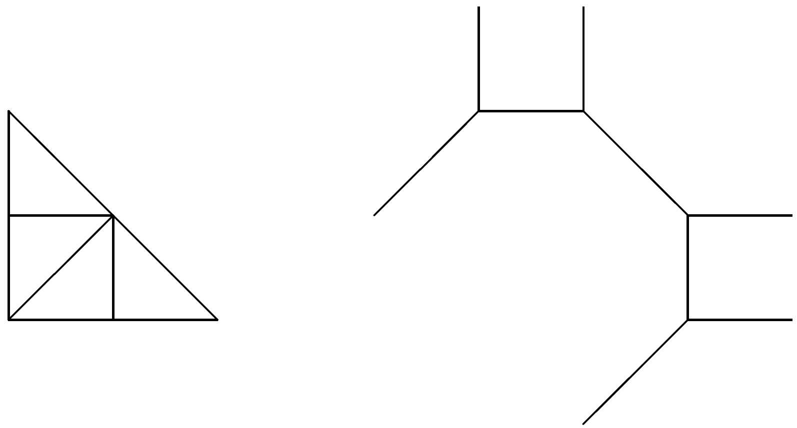


Example. *A subdivided Newton polytope and its dual.*

$$p := 3 \odot x_1^2 \oplus 2 \odot x_1 \odot x_2 \oplus 3 \odot x_2^2 \oplus 0$$



Example. *One can recognize the type of the tropical hypersurface only by analysing the subdivided Newton polytope.*



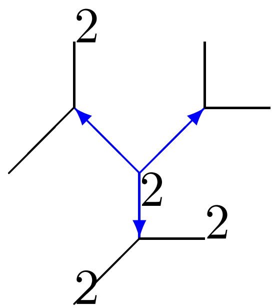
Let e be an edge of a tropical curve $\mathcal{T}(f)$ in $\mathbb{R}^2$.
Then one can define its multiplicity by the lattice length of the corresponding edge in the subdivided Newton polytope.

Let p be any vertex of $\mathcal{T}(f)$, $v_1, \ldots, v_r$ the primitive lattice vectors in the direction of the edges emanating from p and $m_1, \ldots, m_r$ the corresponding multiplicities.

Then the following balancing condition holds:

$$m_1 \cdot v_1 \ + \ m_2 \cdot v_2 \ + \ldots + m_r \cdot v_r \ = \ 0$$

(This holds because of the duality)



$$2 \cdot (0, -1) + (1, 1) + (-1, 1) \ = \ 0$$

For a field K, a real valuation is a map $\mathrm{ord} : K \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with

- $K \setminus \{0\} \to \mathbb{R}$ and
- $0 \mapsto \infty$
- $\mathrm{ord}(ab) = \mathrm{ord}(a) + \mathrm{ord}(b)$ and
- $\mathrm{ord}(a + b) \geq \min\{\mathrm{ord}(a), \mathrm{ord}(b)\}$.

Example. $K = \mathbb{Q}$ *with the* p*-adic valuation: Every* $q \in Q$ *with*

$$q = p^s \frac{m}{n},\ p \not| m, p \not| n, s \in \mathbb{Z}$$

has the valuation

$$\mathrm{ord}(q) = v_p(q) := s.$$

We can extend the valuation map to an algebraic closure $\bar{K}$ and then to $\bar{K}^n$ via

$$\mathrm{ord}\ :\ \bar{K}^n \ \to\ \bar{\mathbb{R}}^n, \quad (a_1, \ldots, a_n) \ \mapsto\ (\mathrm{ord}(a_1), \ldots, \mathrm{ord}(a_n))\,.$$

Another interesting field is $\mathbb{C}\{\{t\}\}$, the field of puiseux series:

The elements are formal power series

- with coefficients in $\mathbb{C}$.
- with rational exponents, bounded below with a common denominator

$\mathbb{C}\{\{t\}\}$ is an algebraic closed field.

A valuation is given by the order map *ord*:

$$ord : \mathbb{C}\{\{t\}\} \to \mathbb{R}$$
$$\sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha \mapsto \min\{\alpha \mid c_\alpha \neq 0\}$$

Let $f = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in $K[x_1, \ldots, x_n]$ where K is a field with a valuation ord.

Definition. *The tropicalization of f is defined as*

$$\begin{aligned} \operatorname{trop}(f) &:= \bigoplus_\alpha \operatorname{ord}(c_\alpha) \odot x^\alpha \\ &= \bigoplus_\alpha \operatorname{ord}(c_\alpha) \odot x_1^{\alpha_1} \odot \ldots \odot x_n^{\alpha_n} \\ &= \min_\alpha \{\operatorname{ord}(c_\alpha) + \alpha_1 x_1 + \cdots + \alpha_n x_n\} \end{aligned}$$

and the tropical hypersurface of f is

$$\begin{aligned} \mathcal{T}(f) &:= \mathcal{T}(trop(f)) \\ &= \{w \in \mathbb{R}^n \,:\, \textit{the minimum in } \operatorname{trop}(f) \\ &\qquad \textit{is attained at least twice in } w\} \end{aligned}$$

Example. *Let* $f = 2x + 4y - x^2 + 3y^2$ *a polynomial in* $\mathbb{Q}[x,y]$ *with the* 2*-adic valuation. Then*

$$\begin{aligned} trop(f) &= 1 \odot x \;\oplus\; 2 \odot y \;\oplus\; 0 \odot x^2 \;\oplus\; 0 \odot y^2 \\ &= \min\{1+x, 2+y, 2x, 2y\} \end{aligned}$$

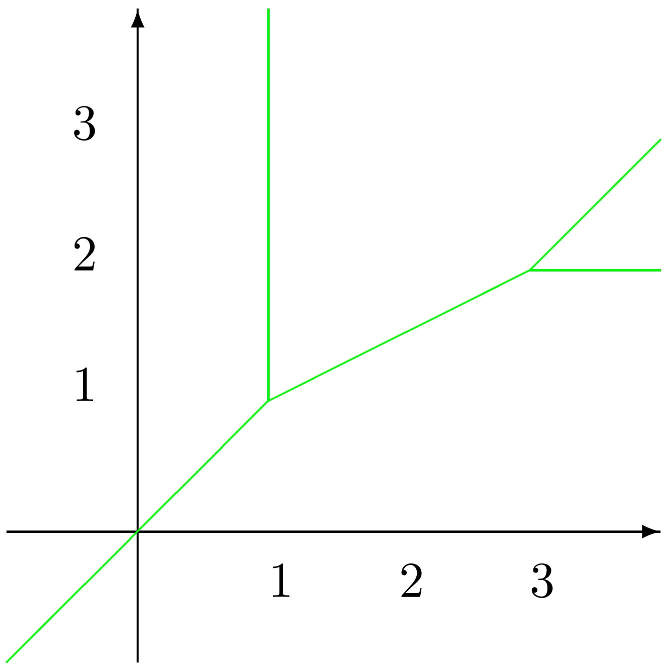


Figure: The tropical hypersurface $\mathcal{T}(f)$

We generalize that to ideals:

Definition. *For an ideal $I \lhd K[x_1, \dots, x_n]$, the tropical variety of I is defined by*

$$\mathcal{T}(I) \;=\; \bigcap_{f \in I} \mathcal{T}(f)$$

There is another description of the tropical variety of an ideal:

Proposition (Kapranov). *If the valuation is nontrivial, then the tropical variety is the topological closure*

$$\mathcal{T}(I) = \overline{\operatorname{ord} \mathcal{V}(I)}$$

where $\mathcal{V}(I) \subset (\bar{K}^*)^n$ is the variety of I.

Let I be generated by

$$f_1 := 2 + y - 4x^2y + x^2y^2 + 2xy^2$$

$$f_2 := xyz - 2z + 4xyz^2 - 2 + z^2$$

Then we get the tropical variety in $\mathbb{R}^3$:

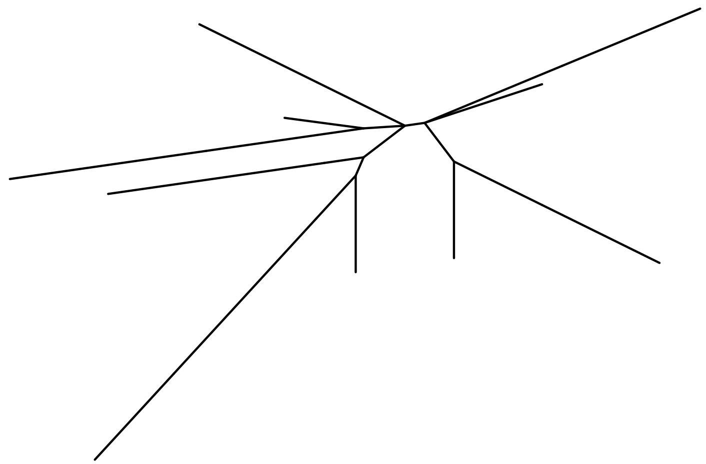
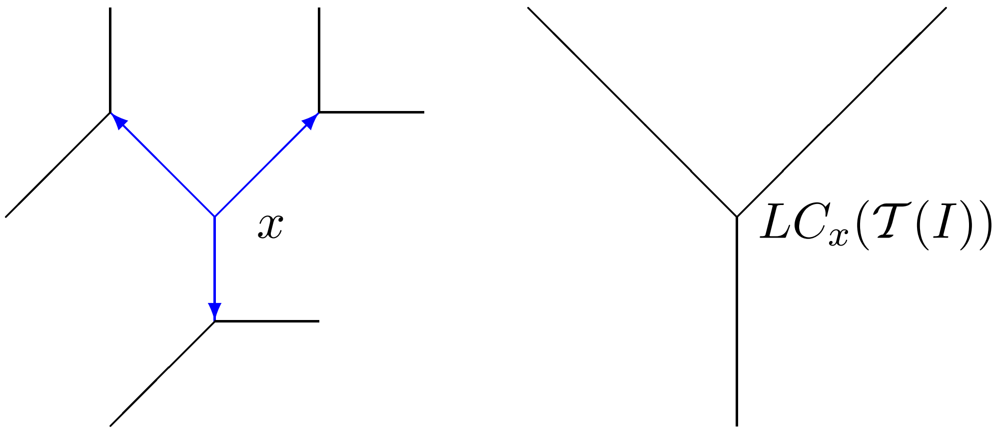


Figure: Tropical variety $\mathcal{T}(I)$

A tropical variety has several properties:

Proposition (Bieri-Groves 1984). *Let I be a prime ideal. Then $\mathcal{T}(I)$ is a pure m-dimensional complex where $m = \dim(I)$ is the Krull dimension of the ideal..*

Proposition (Bieri-Groves 1984). *$\mathcal{T}(I)$ is totally concave, which means that each convex hull of a local cone of a point x is an affine subspace.*



Speyer showed in 2005 that the balancing condition, which is stronger than the concavity condition, holds for tropical varieties in general.

Definition. *Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ be polynomials. Then the tropical prevariety defined by $f_1, \ldots, f_s$ is the intersection of the tropical hypersurfaces*

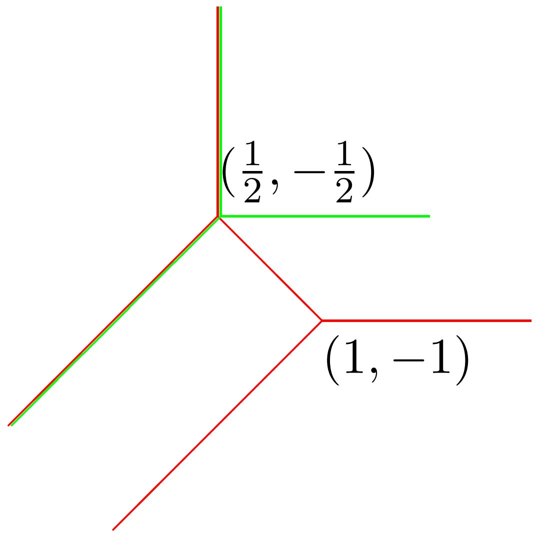
$$\mathcal{T}(f_1, \ldots, f_s) := \bigcap_{i=1}^{s} \mathcal{T}(f_i)$$

This is in general not a tropical variety:

Example.

$$f_1 := t + (t+1)x^2 + (2t^3 - t^4)y^2 + txy$$

$$f_2 := t + (t^{\frac{1}{2}} + t^{\frac{3}{2}})x + t^{\frac{3}{2}}y$$



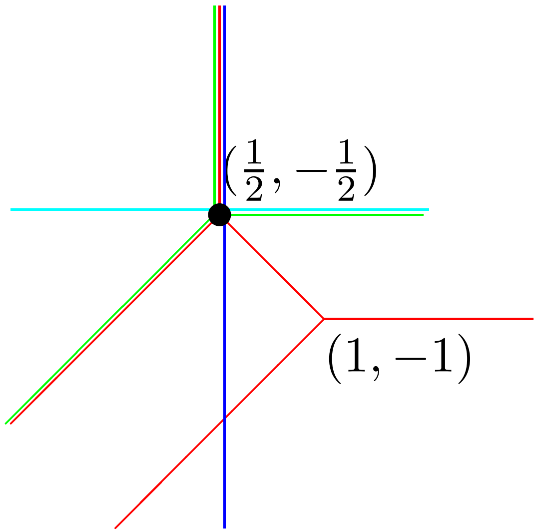
Definition. *A basis $\mathcal{F} = \{f_1, \dots, f_s\}$ of I is a tropical basis, if*

$$\mathcal{T}(I) = \bigcap_{i=1}^{s} T(f_i)$$

Proposition. *Let $I \lhd K[x_1, \dots, x_n]$ be an ideal. Then there exists a tropical basis.*

Example. *Adding the following polynomials to $\{f_1, f_2\}$ we get a tropical basis:*

$$\begin{aligned} f_3 :=\ & (3t - t^2)x^2 + (-t^{\frac{1}{2}} + 4t^{\frac{3}{2}} \\ & -2t^{\frac{5}{2}})x + (t + 2t^2 - t^3); \\ f_4 :=\ & (3t^3 - t^4)y^2 + (t^{\frac{3}{2}} + 2t^{\frac{5}{2}})y \\ & +(2t + t^2) \end{aligned}$$



A tropical d-plane in $\mathbb{R}^n$ is a tropical variety $\mathcal{T}(I)$ where

$$I = \langle \sum_{j=1}^{n} a_{i,j} x_j : i = 1, \ldots, n-d \rangle.$$

Here $(a_{i,j}) \in K^{(n-d)\times n}$ and $rank(a_{i,j}) = n - d$.
The Plücker coordinates are given by

$$P_{i_1 \ldots i_d} := (-1)^{i_1 + \ldots + i_d} \det \begin{pmatrix} a_{1,j_1} & \cdots & a_{1,j_{n-d}} \\ \vdots & \ddots & \vdots \\ a_{n-d,j_1} & \cdots & a_{n-d,j_{n-d}} \end{pmatrix} \neq 0$$

Here $1 \leq i_1 < \ldots < i_d \leq n$ and $1 \leq j_1 < \ldots < j_{n-d} \leq n$ is the complement.

Lemma. *A tropical basis for I is given by the circuits*

$$C_{i_0 \ldots i_d} = \sum_{r=0}^{d} (-1)^r P_{i_0 \ldots \hat{i}_r \ldots i_d} x_{i_r}$$

Bogart, Jensen, Speyer, Sturmfels and Thomas showed:

Proposition. *For any* $1 \leq d \leq n$*, there is a linear ideal* I *in* $\mathbb{C}[x_1, \ldots, x_n]$ *such that any tropical basis of linear forms in* I *has size at least* $\frac{1}{n-d+1}\binom{n}{d}$.

(Here $\mathbb{C}$ has the trivial valuation)

Summary:

- We know the tropical semiring,
- real valuation, tropicalization $trop(f)$, tropical hypersurface $\mathcal{T}(f)$
- tropical variety of an ideal $\mathcal{T}(I)$.
- There exist tropical bases.
- There are linear ideals where every tropical basis is big if we do not allow higher degrees of the polynomials.

But we can show that there are indeed small tropical bases if we drop this assumption.

Theorem (Theobald-H.). *Let* $I \lhd K[x_1, \ldots, x_n]$ *be a prime ideal generated by the polynomials* $f_1, \ldots, f_r$. *Then there exist* $g_0, \ldots, g_{n-\dim I} \in I$ *with*

$$\mathcal{T}(I) \;=\; \bigcap_{i=0}^{n-\dim I} \mathcal{T}(g_i) \qquad (1)$$

and thus $\mathcal{G} := \{f_1, \ldots, f_r, g_0, \ldots, g_{n-\dim I}\}$ *is a tropical basis for* I *of cardinality* $r + \operatorname{codim} I + 1$.

Because every algebraic set in n-space is the intersection of n hypersurfaces we have the corollary:

Corollary. *Let* $I \lhd K[x_1, \ldots, x_n]$ *be a prime ideal. Then there is a tropical basis for* I *of cardinality at most* $n + \operatorname{codim} I + 1$.

To prove the result we need the following steps:

- Proposition of Bieri and Groves:
  Let $I \triangleleft K[x_1, \ldots, x_n]$ be a prime ideal. Then there exist $\operatorname{codim} I + 1$ projections $\pi_0, \ldots, \pi_{\operatorname{codim} I} : \mathbb{R}^n \to \mathbb{R}^{\dim(I)+1}$ such that
$$\mathcal{T}(I) = \bigcap_{i=0}^{\operatorname{codim} I} \pi_i^{-1} \pi_i(\mathcal{T}(I)) \,.$$
- Then we want to proof that $\pi^{-1}\pi(\mathcal{T}(I))$ is a tropical hypersurface.
- Computation of an example

First we want to find $m := \mathrm{codim}(I)$ projections

$$\pi_1, \ldots, \pi_{\mathrm{codim}(I)} : \mathbb{R}^n \longrightarrow \mathbb{R}^{m+1}$$

with the property, that

$$\mathcal{X} = \bigcap_{i=\mathrm{codim}(I)}^{n} \pi_i^{-1}\pi_i(\mathcal{T}(I))$$

is m-dimensional. Clearly it contains $\mathcal{T}(I)$.

This is done inductively:
Let $\mathcal{X}_0 = \{\mathbb{R}^n\}$. Let $\pi_{\mathrm{codim}(I)}$ be an arbitrary projection with m-dimensional image. Then

$$\mathcal{X}_1 := \{\text{cells in the preimage } \pi^{-1}_{\mathrm{codim}(I)}\pi_{\mathrm{codim}(I)}(\mathcal{T}(I))\}.$$

So all maximal cells have dimension $n-1$ and $\mathcal{T}(I) \subset |\mathcal{X}_1|$.

Assume $\mathcal{X}_t$ is constructed and is $(n-t)$-dimensional with

$$\mathcal{T}(I) \subset |\mathcal{X}_t| = \bigcap_{i=\operatorname{codim}(I)-t+1}^{\operatorname{codim}(I)} \pi_i^{-1}\pi_i(\mathcal{T}(I)).$$

Let $\mathcal{B}$ be the finite set of all $(n-t)$-dimensional subspaces of $\mathbb{R}^n$ parallel to at least one of the affine subspaces supporting cells of $\mathcal{X}_t$.
Then choose $\pi_{\operatorname{codim}(I)-t} : \mathbb{R}^n \to \mathbb{R}^{m+1}$ as a projection with

$$\mathbb{R}^n = \ker \pi_{\operatorname{codim}(I)-t} + V \text{ for all } V \in \mathcal{B}.$$

Let $\mathcal{Y}$ be the set of all $(n-1)$-dimensional subspaces of $\pi_{\operatorname{codim}(I)-t}^{-1}\pi_{\operatorname{codim}(I)-t}(\mathcal{T}(I))$ and let

$$\mathcal{X}_{t+1} = \{Y \cap X \mid Y \in \mathcal{Y}, X \in \mathcal{X}_t\}$$

Now $\mathcal{X}_{t+1}$ has dimension $n-t-1$ and $\mathcal{T}(I) \subset |\mathcal{X}_{t+1}|$.
Choose $\mathcal{X} := \mathcal{X}_{\operatorname{codim}(I)}$.

Definition. *Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{R}^n$. A projection $\pi : \mathbb{R}^n \to \mathbb{R}^{m+1}$ is called geometrically regular if the following two conditions hold.*

1. *For any k-face σ of $\mathcal{C}$ we have $\dim(\pi(\sigma)) = k$, $0 \le k \le \dim \mathcal{C}$.*
2. *If $\pi(\sigma) \subseteq \pi(\tau)$ then $\sigma \subseteq \tau$ for all $\sigma, \tau \in \mathcal{C}$.*

Secondly choose π_0 to be a geometrically regular projection with respect to $\mathcal{X}$. Then the m-dimensional cells of $\mathcal{X}$ are mapped to m-dimensional cells of $\mathbb{R}^{m+1}$ and two different m-dimensional cells have different images.
So a m-dimensional cell C is in $\mathcal{T}(I)$ iff $\pi_0^{-1}\pi_0(C)$ is in $\pi_0^{-1}\pi_0(\mathcal{T}(I))$.
$\mathcal{T}(I)$ is a pure m-dimensional polyhedral complex so

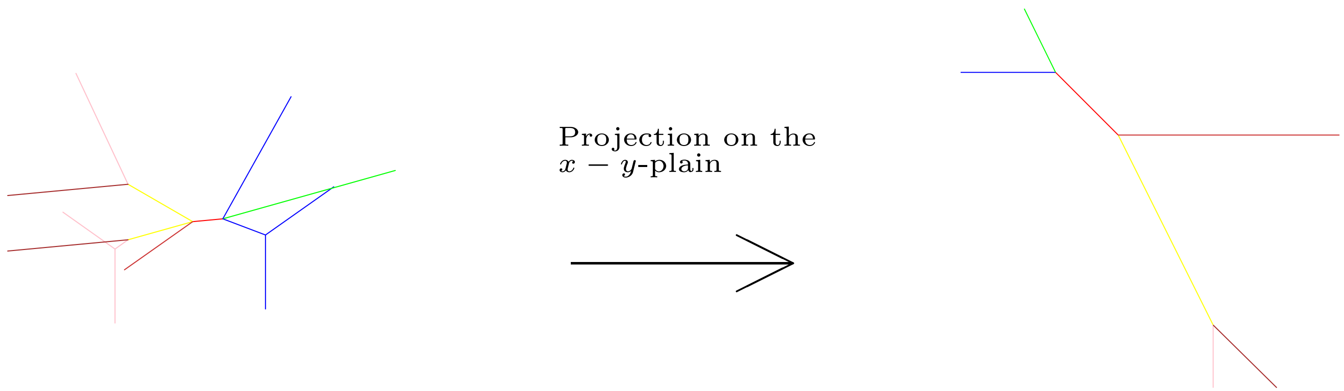
$$\mathcal{T}(I) = \bigcap_{i=0}^{\operatorname{codim} I} \pi_i^{-1}\pi_i(\mathcal{T}(I)) .$$

Let $m = dim(I)$ and the projection be described by

$$\begin{array}{rcl} \pi : \mathbb{R}^n & \longrightarrow & \mathbb{R}^{m+1}, \\ x & \longmapsto & Ax \end{array}$$

with a regular rational matrix A whose rows are denoted by $a^{(1)}, \ldots, a^{(m+1)}$.

Example. *Here:* $n = 3, m = 1, l = 1$



Let $u^{(1)}, \ldots, u^{(l)} \in \mathbb{Q}^n$ with $l := n - (m + 1)$ be a basis of the orthogonal complement of span$\{a^{(1)}, \ldots, a^{(m+1)}\}$.

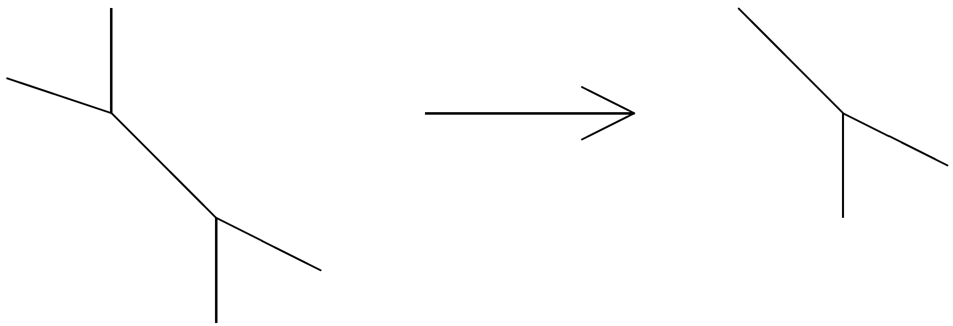
Set $R = K[x_1, \dots, x_n, \lambda_1, \dots, \lambda_l]$, and define the ideal $J \lhd R$ by

$$J \quad = \quad \langle g \in R \, : \, g = f(x_1 \prod_{j=1}^{l} \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^{l} \lambda_j^{u_n^{(j)}}) \text{ for some } f \in I \rangle \, .$$

Theorem. *Let $I \lhd K[x_1, \dots, x_n]$ be an m-dimensional prime ideal and $\pi : \mathbb{R}^n \to \mathbb{R}^{m+1}$ be a rational projection. Then $\pi^{-1}(\pi(\mathcal{T}(I)))$ is a tropical variety with*

$$\pi^{-1}(\pi(\mathcal{T}(I))) \;=\; \mathcal{T}(J \cap K[x_1, \dots, x_n]) \, . \tag{2}$$

$\mathcal{T}(J \cap K[x_1, \dots, x_n])$ is a tropical hypersurface if the projection is m-dimensional. But there can occur some degenerations:



Lemma. *For any* $w \in \mathcal{T}(J \cap K[x_1, \ldots, x_n])$ *and* $u \in \operatorname{span}\{u^{(1)}, \ldots, u^{(l)}\}$ *we have* $w + u \in \mathcal{T}(J \cap K[x_1, \ldots, x_n])$.

Proof. Let w.l.o.g. $u = \sum_{i=1}^{l} \mu_j u^{(j)}$ with $\mu_1, \ldots, \mu_l \in \mathbb{Q}$.

Let $w \in \mathcal{T}(J \cap K[x_1, \ldots, x_n])$. Then we can assume w.l.o.g that there exists $z \in \mathcal{V}(J \cap K[x_1, \ldots, x_n])$ with $\operatorname{ord} z = w$. Define $y = (y', y'') \in (\bar{K}^*)^{n+l}$ by

$$y \;=\; (y', y'') \;=\; \left(z_1 t^{\sum_{j=1}^{l} \mu_j u_1^{(j)}}, \ldots, z_n t^{\sum_{j=1}^{l} \mu_j u_n^{(j)}}, t^{-\mu_1}, \ldots, t^{-\mu_l} \right) .$$

For any $f \in I$, the point y is a zero of the polynomial

$$f(x_1 \prod_{j=1}^{l} {\lambda_j}^{u_1^{(j)}}, \ldots, x_n \prod_{j=1}^{l} {\lambda_j}^{u_n^{(j)}}) \;\in\; R \,,$$

and thus $y \in \mathcal{V}(J)$. Hence, $y' \in \mathcal{V}(J \cap K[x_1, \ldots, x_n])$. Moreover,

$$\operatorname{ord} y' \;=\; (w_1 + \sum_{j=1}^{l} \mu_j u_1^{(j)}, \ldots, w_n + \sum_{j=1}^{l} \mu_j u_n^{(j)}) \;=\; w + \sum_{j=1}^{l} \mu_j u^{(j)} \;=\; w + u \,.$$

□

Lemma. *Let $I \lhd K[x_1, \ldots, x_n]$ be an ideal. Then $J \cap K[x_1, \ldots, x_n] \subseteq I$.*

Proof. Let $p = \sum_i h_i g_i$ be a polynomial in $J \cap K[x_1, \ldots, x_n]$ with

$$g_i \;=\; f_i(x_1 \prod_{j=1}^{l} \lambda_j^{u_1^{(j)}}, \ldots, x_n \prod_{j=1}^{l} \lambda_j^{u_n^{(j)}}) \;\in\; R \;\text{ and }\; f_i \in I\,.$$

Since p is independent of $\lambda_1, \ldots, \lambda_l$ we have

$$p \;=\; p|_{\lambda_1=1,\ldots,\lambda_l=1} \;=\; \sum_i h_i|_{\lambda_1=1,\ldots,\lambda_l=1} f_i \;\in\; I.$$

□

Definition. *We call a projection algebraically regular for I if for each $i \in \{1, \ldots, l\}$ the elimination ideal $J \cap K[x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_i]$ has a finite basis $\mathcal{F}_i$ such that in every polynomial $f \in \mathcal{F}_i$ the coefficients of the powers of λ_i (when considering f as a polynomial in λ_i) are monomials in $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{i-1}$.*

Theorem. *Let $I \lhd K[x_1, \ldots, x_n]$ be a prime ideal and $\pi : \mathbb{R}^n \to \mathbb{R}^{m+1}$ be an algebraically regular projection. Then $\pi^{-1}\pi(\mathcal{T}(I))$ is a tropical variety with*

$$\pi^{-1}\pi(\mathcal{T}(I)) \;=\; \mathcal{T}(J \cap K[x_1, \ldots, x_n])\,. \tag{3}$$

Proof. Idea:
Let $w \in \pi^{-1}\pi(\mathcal{T}(I))$.
w.l.o.g. it exists $z' \in \mathcal{V}(I)$ and $u \in \operatorname{span}\{u^{(1)}, \ldots, u^{(l)}\}$ with $\operatorname{ord} z' = w + u$.
For any $f \in I$, the point

$$z \;:=\; (z', 1)$$

is in the variety $\mathcal{V}(J)$. Hence,

$$z' \in \mathcal{V}(J \cap K[x_1, \ldots, x_n])$$

By Lemma 1, $w \in \mathcal{T}(J \cap K[x_1, \ldots, x_n])$ as well.

Let now $w \in \mathcal{T}(J \cap K[x_1, \ldots, x_n])$.
Again we can assume that there is a $z \in \mathcal{V}(J \cap K[x_1, \ldots, x_n]) \subseteq (K^*)^n$ with $w = \operatorname{ord}(z)$. The projection is algebraically regular so by the Extension Theorem, we can extend the root z inductively to a root $\tilde{z} \in \mathcal{V}(J)$ with the same first n entries. The definition of J says that

$$z' \;:=\; (z_1 \tilde{z}_{n+1}^{u_1^{(1)}} \cdots \tilde{z}_{n+l}^{u_1^{(l)}}, \ldots, z_n \tilde{z}_{n+1}^{u_n^{(1)}} \cdots \tilde{z}_{n+l}^{u_n^{(l)}})$$

is a root of I. Then

$$\operatorname{ord}(z') \;=\; \operatorname{ord}(z) + \sum_{i=1}^{l} \operatorname{ord}(\tilde{z}_{n+i}) u^{(i)}$$

which means that $\operatorname{ord}(z) = w \in \pi^{-1}\pi(\mathcal{T}(I))$. □

Remark. *We can show, that we can always lift a point in the tropical variety $\mathcal{T}(J \cap K[x_1, \ldots, x_n])$ to a point in the tropical variety $\mathcal{T}(J)$, so the theorem holds in general.*

Example.

Let $I \lhd \mathbb{Q}[x, y, z]$ be generated by

$$f_1 := 2 + y - 4x^2y + x^2y^2 + 2xy^2$$

$$f_2 := xyz - 2z + 4xyz^2 - 2 + z^2$$

and let ord be the 2-adic valuation.

For the first projection π_2 we take the one with kernel $(0, 0, 1)$, so it is the projection on the plane $z = 0$. Then $J \cap K[x, y, z]$ is generated by f_1 and the tropical variety is

$$V_1 := \{(x, y, z) \in \mathbb{R}^3 \mid x = 1/2 - y, 1 \leq y \leq 2, z \in \mathbb{R}\},$$

$$V_2 := \{(x, y, z) \in \mathbb{R}^3 \mid y = 1, -1/2 \leq x, z \in \mathbb{R}\},$$

$$V_3 := \{(x, y, z) \in \mathbb{R}^3 \mid x = 1, y \leq -2, z \in \mathbb{R}\},$$

$$V_4 := \{(x, y, z) \in \mathbb{R}^3 \mid y = -1 - 2x, x \leq -3/2, z \in \mathbb{R}\},$$

$$V_5 := \{(x, y, z) \in \mathbb{R}^3 \mid x = -1 - y, y \leq -2, z \in \mathbb{R}\},$$

$$V_6 := \{(x, y, z) \in \mathbb{R}^3 \mid y = -2x, -1/2 \leq x \leq 1, z \in \mathbb{R}\},$$

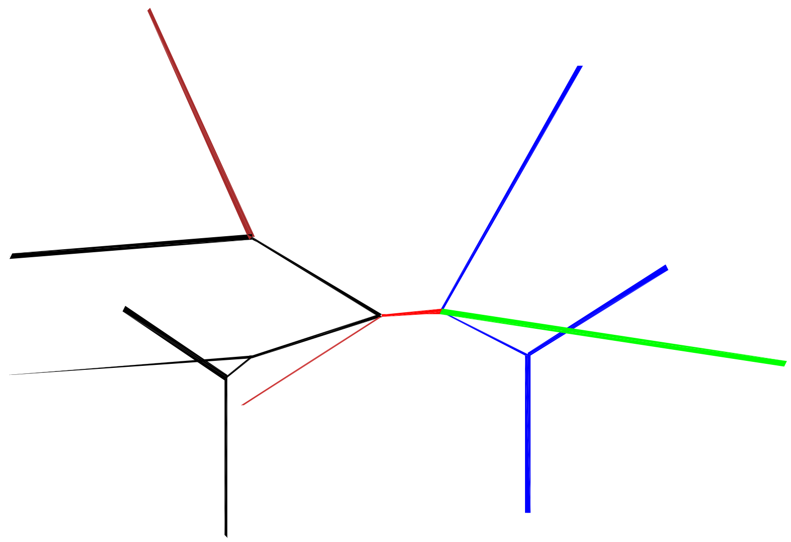
$$V_7 := \{(x, y, z) \in \mathbb{R}^3 \mid y = 2, x \leq -3/2, z \in \mathbb{R}\}.$$

The kernel of the second projection π_1 should not lie in the subspaces parallel to the supporting affine subspaces of the above sets. We can choose for example $\langle(1,1,0)\rangle$.

$$J \cap K[x,y,z] =$$
$$\langle 192xyz + 1008xyz^2 + 16xy + 2176xyz^3 + 1996xyz^4 + 448z^5xy$$
$$-260z^6xy + 1153xyz^8 + 712xyz^7 - 128x^2z - 32y^2z - 1728x^2z^3$$
$$-896x^2z^2 - 512x^2z^4 - 594y^2z^3 - 240y^2z^2 - 666y^2z^4 - 368x^2z^7$$
$$+288x^2z^6 + 1120x^2z^5 + 64x^2z^8 + 52y^2z^7 - 16y^2z^6 - 335y^2z^5 + 16y^2z^8\rangle$$

The ideal is generated by our third polynomial f_3. This gives us a 1-dimensional set $\mathcal{X}$, which consists of the supporting affine subspaces of the prevariety $\mathcal{T}(f_1) \cap \mathcal{T}(f_3)$.

To choose a geometrically regular projection with respect to $\mathcal{X}$ we can for example take $(2, 4, 1)$ as a kernel for π_0. Computing the polynomial in the elimination ideal (it has 63 terms) and the intersection of the tropical variety of all three polynomials we get:



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