A Simplified Nonstandard Proof of the Spectral Theorem for Compact Self-adjoint Operators on Hilbert Space

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In this note we simplify Robinson's nonstandard proof of the spectral theorem for compact self-adjoint operators as given in [1, Chapter 5.4]. This simplification is obtained by using a standard compactness argument instead of a nonstandard one. Note that the proof of Lemma 7 of the above mentioned proof in [1] is incorrect.
For unexplained nonstandard terminology we refer to [1] or [2] .

Outline of proof (for notation, see statement of theorem) :

Step 1 : We construct a *finite-dimensional Hilbert space $\mathcal{F}$ that contains all standard elements of $^*\mathcal{H}$.
Step 2 : We consider the operator $\widetilde{T} := P{}^*T$, where P denotes the orthogonal projection onto $\mathcal{F}$. Elementary properties of $\widetilde{T}$ are derived.
Step 3 : A spectral decomposition of $\widetilde{T}$ is given and some properties of this decomposition are derived.
Step 4 : The spectral decomposition of $\widetilde{T}$ is used to find a spectral decomposition of T.

Theorem 1 *Let T be a compact self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there exist complex numbers ν_i $(i = 1, \ldots)$ and closed subspaces $\mathcal{H}_i$ of $\mathcal{H}$ $(i = 1, \ldots)$ such that $Tx = \sum_{i=1}^{\infty} \nu_i P_{\mathcal{H}_i} x$ for all $x \in \mathcal{H}$ ($P_{\mathcal{H}_i}$ denotes the orthogonal projection onto $\mathcal{H}_i$) .*

Proof:

Step 1.
Let $^*\mathcal{H}$ be a nonstandard model of $\mathcal{H}$ that is κ-saturated with $\kappa > \text{card}(\mathcal{H})$. For each $x \in \mathcal{H}$, let $\mathcal{F}_x$ denote the set of all finite-dimensional linear subspaces of $\mathcal{H}$ that contain x. It is obvious that the collection $\{\mathcal{F}_x : x \in \mathcal{H}\}$ is closed with respect to taking finite intersections. Hence, by the Saturation Principle, $\bigcap_{x \in {}^*\mathcal{H}} {}^*(\mathcal{F}_x) \neq \emptyset$. Let $\mathcal{F}$ be an arbitrary element of $\bigcap_{x \in {}^*\mathcal{H}} {}^*(\mathcal{F}_x)$. Then $\mathcal{F}$ is an internal *finite-dimensional linear subspace of $^*\mathcal{H}$ that contains $\{{}^*x : x \in \mathcal{H}\}$. In particular, $\mathcal{F}$ is a *Hilbert space, since it is a *finite-dimensional linear subspace of $^*\mathcal{H}$.

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Step 2
Let P denote the orthogonal projection from ${}^*\mathcal{H}$ onto $\mathcal{F}$ and define $\widetilde{T} : \mathcal{F} \to \mathcal{F}$ by $\widetilde{T} := P\,{}^*T$. Note that for $x \in \mathcal{H}$,

$$\widetilde{T}\,{}^*x = P\,{}^*T({}^*x) = P\,{}^*(Tx) = {}^*(Tx).$$

By the Transfer Principle, ${}^* \parallel P \parallel = 1$ and

$${}^* \parallel T \parallel \leq \parallel \widetilde{T} \parallel = {}^* \parallel P\,{}^*T \parallel \leq {}^* \parallel {}^*T \parallel = {}^* \parallel T \parallel .$$

Hence, we have $\parallel \widetilde{T} \parallel = {}^* \parallel T \parallel$. Moreover, $\widetilde{T}$ is self-adjoint on $\mathcal{F}$: if $x, y \in \mathcal{F}$ then

$$< \widetilde{T}x, y > = < P\,{}^*Tx, y > = < {}^*Tx, Py > = < {}^*Tx, y > = < x, {}^*Ty > = < x, P\,{}^*Ty > = < x, \widetilde{T}y >,$$

where we have used that orthogonal projections are self-adjoint.

Step 3
By the non-standard characterization of compact operators, ${}^*T$ maps finite points of ${}^*\mathcal{H}$(i.e. those $x \in {}^* \mathcal{H}$ for which there exists an $M \in \mathbb{R}$ such that ${}^* \parallel x \parallel < {}^*M$) to near-standard points of ${}^*\mathcal{H}$ (i.e., those $y \in {}^*\mathcal{H}$ for which there exists a $z \in \mathcal{H}$ such that ${}^* \parallel y - {}^*z \parallel \approx 0$). Hence, $\widetilde{T}$ maps finite points of $\mathcal{F}$ to near-standard points of $\mathcal{F}$, because for near-standard $x \in {}^*\mathcal{H}$ we have $Px \approx P^*({}^{\mathrm{st}}(x)) = {}^*({}^{\mathrm{st}}(x)) \approx x$.
It follows from standard linear algebra and the Transfer Principle that $\widetilde{T}$ has eigenvalues $\lambda_1, \ldots, \lambda_\nu$ in ${}^*\mathbb{C}$ ($\nu \in {}^*\mathbb{N}$; we permit repetitions among the $\lambda_i's$) with corresponding orthonormal eigenvectors $r_1, \ldots, r_\nu$ in $\mathcal{F}$. By the Transfer Principle, we may rearrange these *-finitely many eigenvalues as follows: ${}^*|\lambda_1| \geq {}^*|\lambda_2| \geq \ldots \geq {}^*|\lambda_\nu|$. We define $V = \{j \in {}^*\mathbb{N} \mid 1 \leq j \leq \nu$ and $\lambda_j \not\approx 0\}$.
Each λ_i $(1 \leq i \leq \nu)$ is finite, since

$$|\lambda_i| = \parallel \lambda_i r_i \parallel = \parallel \widetilde{T} r_i \parallel \leq \parallel \widetilde{T} \parallel \parallel r_i \parallel = \parallel \widetilde{T} \parallel .$$

It follows that if $j \in V$, then the corresponding r_j is near-standard, because $(\lambda_j)^{-1}$ is finite and $r_j = (\lambda_j)^{-1}\, \widetilde{T} r_j$ (the near-standardness of $\widetilde{T} r_j$ follows from the compactness of T) . Moreover, for $j \in V$,

$${}^*(T^{\mathrm{st}}(r_j)) = P^*(T^{\mathrm{st}}(r_j)) = P^*T({}^*({}^{st}(r_j))) \approx P^*Tr_j = \widetilde{T}r_j = \lambda_j r_j \approx^{\mathrm{st}} (\lambda_j)^{\mathrm{st}}(r_j).$$

It follows that if $j \in V$, then ${}^{\mathrm{st}}(r_j)$ is an eigenvector of T with corresponding eigenvalue ${}^{\mathrm{st}}(\lambda_j)$.
We now make an excursion to the standard world by considering the eigenvalues ${}^{\mathrm{st}}(\lambda_j)$ $(j \in V)$ of the operator T. Since T is self-adjoint, the eigenvectors ${}^{\mathrm{st}}(r_j)$ are orthogonal. Hence,

$$\parallel^{\mathrm{st}} (\lambda_i)\, {}^{\mathrm{st}}(r_i) -^{st} (\lambda_j)^{\mathrm{st}}(r_j) \parallel^2 = ({}^{\mathrm{st}}(\lambda_i))^2 + ({}^{\mathrm{st}}(\lambda_j))^2 \tag{1}$$

The standard definition of compact operators implies that the set $\{{}^{\mathrm{st}}(\lambda_j)\, {}^{\mathrm{st}}(r_j) \mid j \in \mathbb{N} \cap V\}$ is relatively sequentially compact. Thus (1) implies that if $\mathbb{N} \cap V$ is infinite, then ${}^{\mathrm{st}}(\lambda_j)$ converges to zero if j goes to infinity through the standard set $\mathbb{N} \cap V$. We now return to the non-standard world. Using the last result and the monotonicity of the numbers λ_j we see that

$$\lambda_k \not\approx 0, \text{ then } k \in \mathbb{N} \tag{2}$$

$$\text{if } \lambda_{r+1} = \ldots = \lambda_{r+k} \not\approx 0, \text{ then } k \in \mathbb{N}. \tag{3}$$

Step 4
Now rewrite the numbers $\lambda_1, \ldots, \lambda_\nu$ without repetitions as $\kappa_1, \ldots, \kappa_\mu$ ($\mu \in {}^*\mathbb{N}$). Denote the eigenvectors corresponding to κ_j by $r_1^{(j)}, \ldots, r_{m_j}^{(j)}$. Write $E_j := {}^*\mathrm{span}\{r_1^{(j)}, \ldots, r_{m_j}^{(j)}\}$. It follows from standard linear algebra and the Transfer Principle that $\widetilde{T} = \kappa_1 P_{E_1} + \ldots + \kappa_\mu P_{E_\mu}$.
We now define $W := \{j \in *\mathbb{N} \mid 1 \leq j \leq \mu \text{ and } \kappa_j \not\approx 0\}$ and $\nu_j :=^{\mathrm{st}} (\kappa_j)$. Moreover, we define $\mathcal{H}_j := \mathrm{span}\left\{^{\mathrm{st}}\left(r_1^{(j)}\right), \ldots, ^{\mathrm{st}}\left(r_{m_j}^{(j)}\right)\right\}$ for $j \in W$. If $W = \emptyset$, then for each $x \in \mathcal{H}$ we have $\widetilde{T}^* x \approx 0$. Indeed,

$$\begin{aligned}
\| \widetilde{T}^* x \|^2 &= \| \kappa_1 P_{E_1}{}^* x + \ldots + \kappa_\mu P_{E_\mu}{}^* x \|^2 \\
&= \| \kappa_1 P_{E_1}{}^* x \|^2 + \ldots + \| \kappa_\mu P_{E_\mu}{}^* x \|^2 \\
&\leq |\kappa_1|^2 \left\{ \| P_{E_1} x \|^2 + \ldots + \| P_{E_\mu} x \|^2 \right\} \\
&= |\kappa_1|^2 \| {}^* x \|^2 \\
&\approx 0.
\end{aligned}$$

This means that $T = 0$. So we may assume that $W \neq \emptyset$. It follows from (2) and (3) that either $W = \mathbb{N}$ or $W = \{1, \ldots, k\}$ for some $k \in \mathbb{N}$.
We first consider the case $W = \mathbb{N}$. Define $s_n := Tx - \sum_{i=1}^n \nu_i P_{\mathcal{H}_i} x$ ($n \in \mathbb{N}$). If $n \in \mathbb{N}$ and $x \in \mathcal{H}$, then ${}^*\left(\sum_{i=1}^n \nu_i P_{\mathcal{H}_i} x\right) \approx \sum_{i=1}^n \nu_i P_{E_i} x$ and ${}^*T^* x = \widetilde{T}^* x = \sum_{i=1}^\mu \kappa_i P_{E_i}{}^* x$. Hence,

$$s_n^2 \approx \sum_{i=n+1}^{\mu} \kappa_i P_{E_i}{}^* x^2 = \sum_{i=n+1}^{\mu} |\kappa_i|^2 P_{E_i}{}^* x^2 \leq |\kappa_{n+1}|^2 \| {}^* x \|^2,$$

thus $\lim s_n = 0$.
If $W = \{1, \ldots, k\}$ for some $k \in \mathbb{N}$, then

$${}^*T^* x = \widetilde{T}^* x = \sum_{i=1}^{k} \kappa_i P_{E_i}{}^* x + \sum_{k=1}^{\mu} \kappa_i P_{E_i}{}^* x \approx \sum_{i=1}^{k} \kappa_i P_{E_i}{}^* x \approx \sum_{i=1}^{k} \nu_i P_{\mathcal{H}_i} x.$$

In both cases, we have $Tx = \sum_{i=1}^k \nu_i P_{\mathcal{H}_i} x$. $\square$

References

[1] M. Davis, *Applied Nonstandard Analysis*, Wiley, New York, 1977.

[2] A. Di Bucchianico, Introduction to Nonstandard Analysis, unpublished manuscript, `http://www.win.tue.nl/~sandro/publications.html#unpublished`.