Generalized quadrangles

1 Point-line geometries

A *point-line geometry* is a triple $(P, L, *)$ where P and L are disjoint sets (of objects called *points* and *lines*, respectively), and $* \subseteq (P \times L) \cup (L \times P)$ is a symmetric relation (called *incidence*) such that if $p, q \in P$ and $\ell, m \in L$ and $p * \ell * q * m * p$, then $p = q$ or $\ell = m$.

In other words: two distinct lines cannot have two distinct points in common. In other words: two distinct points lie on at most one line.

This concept is self-dual: it is invariant for an interchange of the concepts 'point' and 'line'.

Consider the bipartite graph (known as *incidence graph* or *Levi graph*) with as vertex set $P \cup L$, where p is adjacent to ℓ (written $p \sim \ell$) if and only if $p * \ell$.

The axiom for point-line geometry says that this graph does not have 4-cycles, so that its *girth* (the length of a shortest cycle) is at least 6.

2 Generalized polygons

A *generalized m-gon* is a point-line geometry such that the incidence graph is a connected, bipartite graph of diameter m and girth $2m$.

A *projective plane* is a generalized 3-gon.

A *generalized quadrangle* is a generalized 4-gon.

A generalized polygon is called *regular* of order (s, t) for certain (finite or infinite) cardinal numbers s, t if each line is incident with $s + 1$ points and each point is incident with $t + 1$ lines.

It is an easy exercise (done below for the case of generalized quadrangles) to show that a generalized m-gon is regular, except in certain easy-to-describe cases.

Theorem 2.1 (Feit-Higman) *A finite generalized m-gon of order (s, t) with $s > 1$ and $t > 1$ satisfies $m \in \{2, 3, 4, 6, 8\}$.*

3 Generalized quadrangles

A *weak generalized quadrangle* is a point-line geometry $(P, L, *)$ with the property that for each $p \in P$ and $\ell \in L$ if not $p * \ell$ then there is a unique sequence of incidences $p * m * q * \ell$.

The incidence graph of a generalized quadrangle is connected, unless $P = \emptyset$ or $L = \emptyset$.

Degenerate examples of generalized quadrangles are those with all lines on a single point, or all points on a single line.

The mentioned examples are precisely the weak generalized quadrangles that are not generalized quadrangles according to the definition given earlier (connected, diameter 4, girth 8).

Other nonregular examples of generalized quadrangles are

a) Complete bipartite graphs: the points fall into two sets P_1 and P_2 and each line has two points, one from each set P_i.

b) Grids: the lines fall into two sets L_1 and L_2, and each point is on two lines, one from each L_i.

In case a) all lines have two points, but the number of lines on a point is either $|P_1|$ or $|P_2|$ and takes two values if $|P_1| \neq |P_2|$.

In case b) all points are on two lines, but the number of points on a line is either $|L_1|$ or $|L_2|$ and takes two values if $|L_1| \neq |L_2|$.

In all other cases we have a $GQ(s, t)$, that is, a generalized quadrangle such that all lines have $s + 1$ points, and all points are on $t + 1$ lines. (Here s and t are finite or infinite cardinal numbers.)

Proof: Since the hypothesis is self-dual we need only prove that any two lines have the same cardinality.

If ℓ and m are disjoint lines, then the axiom for generalized quadrangles provides a 1-1 correspondence between both so that they are incident with the same number of lines.

If ℓ and m meet, but both are disjoint from a third line n, then ℓ and n and m all have the same number of points.

If ℓ and m meet (in a point p, say), and no line is disjoint from both, then every line not on p meets either ℓ or m, and through each point not on ℓ or m there passes a unique line of meeting ℓ and a unique line meeting m. We see that the lines form a grid, as desired. $\square$

4 Generalized quadrangles with lines of size 3

There are five types of generalized quadrangle with all lines of size 3.

The first case is where $L = \emptyset$. If there are no lines, then surely all lines have size 3. This is an infinite family, since we may choose $|P|$.

The second case is where all lines pass through a single point. Again a family, we may choose $|L|$.

The third case is that of the square grid 3×3 with 9 points and 6 lines, a GQ(2,1).

The fourth case is that of the generalized quadrangle on 15 points and 15 lines described by the pairs of a 6-set and the partitions of a 6-set into 3 pairs, with obvious incidence. A GQ(2,2).

The fifth and last case is that of a GQ(2,4).

A uniform description of these last three: take a quadric of Witt index 2 in a projective space over GF(2), the field with two elements. Let the points and lines of the generalized quadrangle be the points and lines on the quadric, with natural incidence.

That there are no other examples can be proved in many ways. In the lectures I gave Cameron's proof (which has the advantage that it also works in the infinite case).

An application of the classification of generalized quadrangles with lines of size 2 is that it leads to a classification of root lattices. Given an irreducible root lattice Λ, pick two roots a, b with inner product 1. (So $(a, a) = (b, b) = 2$ and $(a, b) = 1$.) Let $Q = \{r \in \Lambda \mid (r, r) = 2$ and $(r, a) = (r, b) = 1$. Then Q is a generalized quadrangle with lines of size 3 if we take the triples $\{x, y, z\}$ with $x + y + z = a + b$ as lines. (These are precisely the triples of mutually orthogonal points.) It turns out that Λ is spanned by $\{a, b\} \cup Q$ so that the structure of Q determines Λ.

Now the five types of generalized quadrangle correspond to the five types of root lattice: A_n $(n \geq 2)$ corresponds to $n - 2$ isolated points, no lines.
D_n $(n \geq 3)$ corresponds to $n - 3$ lines passing through a fixed point.
E_6, E_7, E_8 correspond to $GQ(2, 1)$, $GQ(2, 2)$, $GQ(2, 4)$.

The case A_1 was lost since we cannot pick a, b in that case. Note that $A_3 = D_3$ and that D_2 is not irreducible.

5 Strongly regular graphs

The collinearity graph of a finite $GQ(s, t)$ is strongly regular with parameters $v = (s+1)(st+1)$, $k = s(t+1)$, $\lambda = s-1$, $\mu = t+1$ and eigenvalues $r = s-1$, $s = -t - 1$.

The eigenvalues r and s have multiplicities $f = st(s + 1)(t + 1)/(s + t)$ and $g = s^2(st + 1)/(s + t)$ and if a $GQ(s, t)$ exists, these multiplicities must be integers. For $s = 2$ (three points per line) this condition says that $(t + 2)|(8t + 4)$, i.e., $(t + 2)|12$, so that $t \in \{1, 2, 4, 10\}$.

The Krein conditions imply that if $t \neq 1$ then $s \leq t^2$, and if $s \neq 1$ then $t \leq s^2$. This rules out $t = 10$ and leaves the parameters of the known examples $GQ(2, 1)$, $GQ(2, 2)$, $GQ(2, 4)$. There is a unique geometry in each of these three cases.