AN INVESTIGATION OF CERTAIN COMBINATORIAL PROPERTIES OF PARTIALLY BALANCED INCOMPLETE BLOCK EXPERIMENTAL DESIGNS AND ASSOCIATION SCHEMES, WITH A DETAILED STUDY OF DESIGNS OF LATIN SQUARE AND RELATED TYPES

By

Dale Marsh Mesner

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Outline of Studies

Major subject: Mathematical statistics
Minor subjects: Algebra, analysis, geometry

Biographical Items

Born, April 13, 1923, Central City, Nebraska

Undergraduate Studies, Nebraska Central College, 1941-43, University of Nebraska, 1946-48

Graduate Studies, Northwestern University, 1948-49, Michigan State College and Michigan State University, 1950-56

Experience: Instructor, Candler College, Marianao, Cuba, 1949-50, Graduate Teaching Assistant, Michigan State College, 1951-52, Temporary Instructor, Michigan State College, 1952-54, Instructor, Purdue University, 1954-56

Member of Society of the Sigma Xi, Pi Mu Epsilon; member of Institute of Mathematical Statistics, American Mathematical Society, Mathematical Association of America

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AN ABSTRACT

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A partially balanced incomplete block (PBIB) design is an arrangement of a set of experimental treatments into smaller subsets, or blocks, in accordance with a certain definition. Except for an introductory section in which the role of PBIB designs in the statistical analysis of experiments is discussed, this thesis is concerned with the combinatorial problems that arise in the construction of the designs. The definition states requirements for a relation of association between any two treatments, and the term "association scheme" is used for any method by which a relation of the kind specified can be set up. A considerable portion of the thesis is devoted to the study of association schemes rather than actual designs. Incidence matrices are used throughout the thesis to study the properties of designs and association schemes by algebraic methods.

A method of enumerating combinatorially possible PBIB designs with two classes of associates is outlined, based on both new and old methods. While tables of known designs have been published, no exhaustive tables of all possible PBIB designs have appeared heretofore. An extensive table of the possible parameter values of association schemes is compiled, along with tables of possible parameter values of the designs themselves in the cases of special interest in this study.

Known PBIB designs with two classes of associates have been classified according to the nature of their association schemes, and designs of Latin square type with g constraints, in which the number of treatments is a square n^2 and the association relation may be defined by a set of g mutually orthogonal $n \times n$ squares, are singled out for special study here. A related class of new designs is introduced and given the name "negative Latin square". While association schemes for the new designs cannot be constructed from Latin squares, a method based on finite fields is developed and used to construct some schemes of both types, including four in the new series. A fifth is constructed by other methods. Several new designs are constructed from the new association schemes.

Some examples are given to show the possibility of association schemes which have exactly the same parameter values as those of Latin square type with g constraints but in which the association relation cannot be defined by a set of g orthogonal squares. It is then proved that for a fixed value of g, this can be the case only for n less than a certain value, which is expressed as a function of g, and that for larger values of n the Latin square type association scheme is unique. The proof is based on a series of theorems on the structure of incidence matrices, some pertaining only to association schemes and others applying more generally. Some other applications of the methods are suggested.

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PREFATORY NOTE

Chapter I is introductory in nature. The next three chapters, tak-up special aspects of this study, are long and somewhat complex. For this reason a detailed summary or synopsis has been included as Chapter V. The reader may find it useful to appraise the scope and methods of Chapters II, III and IV by a preliminary reading of the summary.

I. GENERAL PROPERTIES OF PARTIALLY BALANCED DESIGNS AND ASSOCIATION SCHEMES

1.1 Introduction

Statistical analysis of many types of experimental data may be facilitated by proper planning of the experiment. Partially balanced incomplete block (PBIB) designs are a particular class of arrangements for this purpose. A definition of PBIB designs will be preceded by a simple example which illustrates the concepts involved.

An Illustrative Example with Historical Remarks. The average yields of seven new varieties of hybrid corn are to be compared in a field experiment. A possible plan is to divide the available land into seven plots and to plant one variety in each plot, as indicated in the following figure. (Throughout this example, varieties will be indicated by numbers from 1 to 7.)

1	2	3	4	5	6	7

Under conditions of strict control of soil fertility, water supply and drainage, and other extraneous factors, this might furnish the desired information on the varietal differences, but in experiments in the biological and social sciences such control is not usually possible. It will be impossible with this arrangement to know whether an observed difference between two plots can be attributed to differences in the two varieties or whether it is due to differences between the plots of ground. If the effects of extraneous factors cannot be controlled, the next best thing is to estimate their importance. This can be done by planting several plots to each variety and observing the variation among them. It is intuitively reasonable and proves to simplify analysis of the data to plant the same number of plots to each variety so that in effect we have a number of repetitions, or replications, of the original experiment. Three replications will be used in this example. Comparison of varieties grown under similar conditions will be easier if the 21 plots are grouped into blocks of seven plots, each block to contain a complete replication. Soil conditions are likely to be more homogeneous within a block than over the entire experimental area and will have a correspondingly smaller effect on comparisons made within a block. The blocks may or may not be contiguous in the field. This design is indicated in the following diagram, with blocks inclosed by heavy lines.

1	2	3	4	5	6	7
1	2	3	4	5	6	7
1	2	3	4	5	6	7

A defect of this plan is that the same arrangement of varieties is used in each block, so that effects of location within blocks may be impossible to distinguish from differences between varieties. For instance, an observed difference between varieties 1 and 7 could have been caused by a gradient in soil fertility from left to right. Other extraneous sources of variation which are less obvious may introduce a similar bias in favor of certain varieties. To insure that no variety or group of varieties will be systematically favored in all replications of the experiment, a device known as randomization may be used. In our example this would mean assigning the numbers from 1 to 7 to each block in such a way that each of the 7! possible arrangements is equally likely to result. In addition, the three blocks might be assigned to the three positions in the field in a random manner. The effect is that in each replication, each variety has an equal chance of being tested under favorable conditions. While the results of any particular randomization may favor certain treatments, this happens only to an extent that can be allowed for in the analysis and interpretation of the data.

The plan that results is called a randomized block design. It might appear as follows.

6	2	3	5	4	7	1
2	5	7	4	1	6	3
3	6	5	7	1	4	2

R. A. Fisher was the first to realize the importance of randomization as a scientific technique and to introduce it into designs for experiments. It is discussed in his book "The Design of Experiments" [19] with some illuminating examples.

It frequently happens that, within a block which includes an entire replication of an experiment, there is too much variability of conditions to allow useful measurements to be made. This may make it necessary to arrange the experimental plots in blocks of smaller size, with direct comparisons to be made only between varieties in the same block. In our example we shall suppose that it is necessary to cut down the block size to three plots. There is some loss of information here, as suggested by the fact that the number of possible direct comparisons is reduced from $3\binom{7}{2} = 63$ to $7\binom{3}{2} = 21$, but the gain in precision of comparisons may more than offset this. If some of the comparisons are less important than others, it may be possible to arrange the blocks so that the unimportant information is lost and the important information is mostly retained. However, in many situations all comparisons may be considered equally important; it will be assumed in this example that information is desired on the comparative yields of each pair of varieties. The term <u>incomplete block design</u> covers any experimental design in which the blocks are of size smaller than the number of treatments, while the term <u>balanced incomplete block</u> (BIB) <u>design</u> is used for the important special case in which an equal amount of information is retained on each pair of treatments. A BIB design may be defined as an arrangement of v varieties or treatments into b blocks each containing k distinct varieties, each variety being used the same number of times r, and each pair of distinct varieties occurring together in the same block the same number λ of times. It is easily verified that the following arrangement of our example satisfied these requirements, with $v = b = 7$, $r = k = 3$, $\lambda = 1$. (Blocks are enclosed by heavy lines.)

1	2	3
1	4	5
1	6	7
2	4	6
2	5	7
3	4	7
3	5	6

Randomization would be applied to this design by assigning the numbers 1, 2, ... , 7 to the varieties at random, assigning the three numbers in each block to the three plots in a random way, and assigning the blocks to the seven positions in the field by a third random procedure.

Balanced incomplete block designs were introduced by Yates in 1936 [38]. The construction of a BIB design for a given set of values of v, b, r, k, λ, is a combinatorial problem which may be considered apart from the analysis of experimental data. It is clear that the five parameters are not all independent. Considering the total number of plots we have

$$(1.1) \qquad vr = bk ,$$

and by counting pairs of varieties two ways we obtain

$$\lambda\binom{v}{2} = b\binom{k}{2}$$

These two results may be combined to give a more useful form of the latter.

$$\lambda = r\frac{k-1}{v-1} \tag{1.2}$$

Other necessary conditions for the existence of these designs have been obtained, along with some methods for constructing large classes of them. In 1938, Fisher and Yates [21] published all the BIB designs then known, with a list of the possible parameters of other designs of practical interest. (A design is of practical interest if it does not require more experimental material than the experimenter can afford: for a given number of treatments, this means "for r sufficiently small.") The construction of many of these designs was made possible by methods introduced by R. C. Bose in 1939 [4].

The set of constructible BIB designs was soon found to be inadequate for the needs of experimenters. A simple case in which no convenient balanced design is available is obtained from the first example by considering eight varieties of hybrid corn instead of seven, again to be planted in blocks of three plots. With $v = 8$ ~~and~~ $k = 3$, the smallest value of r which can be used in (1.1) and (1.2) to give integral values of b and λ is found to be 21, and the blocks of the design are all the combinations of the eight varieties three at a time. It was to provide useful designs for such values of v and k that arrangements like the following were introduced.

1	2	3
1	4	6
1	7	8
2	4	7
2	5	8
3	4	5
3	6	8
5	6	7

Figure 1. Example of PBIB design.

This is not a balanced design because the pairs of distinct varieties do not all occur equally often. Every pair occurs once with the exceptions (1,5), (2,6), (3,7), (4,8), which do not appear at all in the same block. The remaining requirements for a balanced design are satisfied. This is an example of a <u>partially balanced incomplete block</u> (PBIB) design.

Partially balanced incomplete block designs were introduced by R. C. Bose and K. R. Nair in 1939 [8]. They are a generalization of balanced incomplete block designs and include them as a special case, along with certain other incomplete block designs which were already known. Their analysis is somewhat more difficult than that of balanced designs, though conditions are specified (paragraph iii, c, of the definition which follows) which simplify it greatly. They have not been studied as extensively as balanced designs. Some of the literature on the subject will be discussed in later chapters.

Combinatorial properties of partially balanced incomplete block designs will be the principal subject of this thesis. The problems of analysis and interpretation of experimental data will not be taken up. For our purposes from now on, a PBIB design is an arrangement of objects known as varieties or treatments into blocks according to certain rules. A definition of PBIB designs will now be given.

An incomplete block design is said to be partially balanced if it satisfies the following conditions:

(i) The treatments or varieties being tested are grouped into b blocks, each consisting of k distinct treatments.

(ii) There are v treatments, each of which occurs in r blocks.

(iii) There can be established a relation of association between any two treatments satisfying the following requirements:

(a) Two treatments are either 1st, 2nd, . . . , or mth associates.

(b) Each treatment has exactly n_i, i^{th} associates.

(c) Given any two treatments which are i^{th} associates, the number of treatments common to the j^{th} associates of the first and the k^{th} associates of the second is p^i_{jk} and is independent of the pair of treatments with which we start. Also $p^i_{jk} = p^i_{kj}$.

(iv) Two treatments which are ith associates occur together in exactly λ_i blocks.

It should be noted from (iii) that the association relation is symmetric but not necessarily transitive.

It was proved by Bose and Nair [8] that the following relations hold among the parameters.

$$(1.3) \qquad bk = vr \,,$$

$$(1.4) \qquad n_1 + n_2 + \cdots + n_m = v - 1 \,,$$

$$(1.5) \qquad n_1\lambda_1 + n_2\lambda_2 + \cdots + n_m\lambda_m = r(k-1) \,,$$

$$(1.6) \qquad \sum_{k=1}^{m} p^i_{jk} = n_j \quad (\text{if } i \neq j) \,,$$

$$= n_{j-1} \quad (\text{if } i = j) \,,$$

$$(1.7) \qquad n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij} \,.$$

For fixed i the parameters p^i_{jk} are conveniently displayed in an $m \times m$ matrix with j and k as row and column indices, denoted by P_i. By the final remark of (iii)(c), these matrices are symmetric.

It is easily verified that the example given in Figure 1 is a PBIB design with two associate classes and the parameters

$$v = b = 8, \quad r = k = 3, \quad n_1 = 1, \quad n_2 = 6, \quad \lambda_1 = 0, \quad \lambda_2 = 1,$$

$$P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 6 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}.$$

It may also be verified that these parameters satisfy relations (1.3) to (1.7).

1.2 Association Schemes and Incidence Matrices. The definition of the previous section is not in the same form as originally given by Bose and Nair, but follows closely several papers around 1952, notably Bose and Shimamoto, "Classification of PBIB designs with two associate classes" [10]. The definitions are logically very similar,[1] but as pointed out by Bose and Shimamoto, the association relations among the treatments do not depend on how they are distributed in blocks. In this form of the definition the association relations are completely specified in paragraph (iii). They may be taken up without considering the parameters b, r, k, λ_i.

An association scheme is a convenient device for describing the association relations of a design. It is a table or other arrangement listing for each treatment its 1^{st}, 2^{nd}, . . . , m^{th} associates. The treatments may be assigned the numbers from 1 to v in any convenient order for such a table. Bose and Shimamoto found it possible to classify the association schemes of all known designs with $m = 2$ association schemes into five types, some of which can be set down very concisely. Perhaps the simplest type of scheme is the group divisible (GD), in which $v = mn$ and the treatments are divided into m groups of n each, treatments in the same group are first associates and treatments in different groups are second associates. A compact form for the association scheme is an m by n rectangle, with the n treatments in a row constituting a group. The example given in Figure 1 is a GD design with the following association scheme.

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1. The original definition contained the specification that the λ_i be distinct. This was dropped in a 1942 paper by Nair and Rao [28], generalizing the class of PBIB designs somewhat. Their definition is equivalent to the one given here.

1	5
2	6
3	7
4	8

Figure 2. Example of Group Divisible association scheme.

The first associate of treatment 1 is treatment 5, etc. It is natural to attempt to generalize this by taking two treatments as first associates if they appear in the same row or the same column, but if $m \neq n$ it is easy to see that condition (iii)(c) of the definition is violated. For example, the number p^1_{11} of treatments common to the first associates of treatments 1 and 5 would be 0; the number for treatments 1 and 2 would be 2. If $m = n$ so that $v = n^2$, this generalization leads to an association scheme described by Bose and Shimamoto as of Latin square type. The following array is given as an illustration.

1	2	3
4	5	6
7	8	9

Figure 3. Example of Latin square association scheme.

This array defines a GD association scheme in which treatment has as its first associates treatments 2 and 3; it also defines a Latin square type scheme in which the first associates of treatment 1 are 2, 3, 4, 7.

The various types of association schemes will be discussed further in Section 2.1.

It may be noted that for a group divisible scheme the relation for first associates is transitive as well as symmetric; that is, the first associates of a treatment are pairwise first associates. This is a sufficient condition for the scheme to be GD, for it implies that the treatments may be divided into groups such that two treatments in the same group are first associates and two treatments not in the same group are second associates, while the condition that each treatment have n_1 first associates requires that the groups be of equal size. The Latin square scheme described above illustrates that in general two treatments which are first associates of the same treatment may not be first associates of each other.

There may be several designs for any one association scheme. The following is another design using the GD association scheme of Figure 2.

$v = 8$, $r = 3$, $k = 4$, $b = 6$, $\lambda_1 = 3$, $\lambda_2 = 1$.

1	5	2	6
1	5	3	7
1	5	4	8
2	6	3	7
2	6	4	8
3	7	4	8

Figure 4. Another PBIB design.

A number of possible designs for the Latin square association scheme of Figure 3 will be enumerated in Table IV of the Appendix.

The portion of an association scheme corresponding to i^{th} associates, $i = i, \ldots, m$, may be represented by a $v \times v$ matrix

$$A_i = \left(a^i_{\mu\nu}\right)$$

where $a^i_{\mu\nu}$ has the value 1 or 0 according as treatments μ and ν are or are not i^{th} associates. A_i will be called the <u>incidence matrix</u> for i^{th} associates, or simply the i^{th} association matrix. It follows from paragraph (iii) of the definition that it is a symmetric matrix with exactly n_i 1's in each row and column. Before further properties of the A_i are derived, a connection will be pointed out with another incidence matrix pertaining to the design.

The incidence matrix for treatments and blocks of a PBIB design is a $v \times b$ matrix

$$N = \left(n_{\mu\nu}\right)$$

where

(1.8) $\quad n_{\mu\nu} = 1$ if treatment μ occurs in block ν,
$\quad\quad = 0$ otherwise.

That is, positions of 1's in row μ of the matrix indicate the blocks of the design which contain treatment ν. We shall consider the product of N on the right by its transpose N'. This product NN' will be a symmetric $v \times v$ matrix. Let

(1.9) $\quad NN' = (b_{\mu\nu}).$

The diagonal element $b_{\mu\mu}$ of NN' is equal to the number of 1's in row μ of N, or the number of blocks of the design which contain treatment μ. For $\mu \neq \nu$, $b_{\mu\nu}$ is equal to the inner product of rows μ and ν taken as vectors, or the number of blocks which contain both of treatments μ and ν. For a PBIB design we have by paragraphs (ii) and (iv) of the definition,

(1.10) $$b_{\mu\mu} = r,$$

$$b_{\mu\nu} = \lambda_i \text{ when } \mu \neq \nu \text{ and treatments } \mu \text{ and } \nu \text{ are } i^{th} \text{ associates.}$$

That is,

(1.11) $$NN' = r\, I_v + \sum_{i=1}^{m} \lambda_i A_i,$$ where I_v is the $v \times v$ identity matrix.

The matrices N and NN' have been used extensively since about 1950 in various studies of balanced and partially balanced designs.[1] The matrices A_i do not seem to have received much attention.

There follow as examples matrices N and NN' for the design given in Figure 4, preceded by the matrix A_1 for the association scheme of this design, given in Figure 2.

1. The following papers referred to in this dissertation make substantial use of N, NN' and related matrices: [7], [15], [16], [17], [30].

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$NN' = \begin{bmatrix} 3 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 & 3 \\ 3 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 & 3 \end{bmatrix}$$

1.3 Applications and Algebraic Properties of the Matrices A_i .

Consider the product of two of the association incidence matrices, not necessarily distinct.

$$(1.12) \qquad A_j A_k = (c^{jk}_{\mu\nu}) ,$$

where

$$(1.13) \qquad c^{jk}_{\mu\nu} = \sum_{\sigma=1}^{v} a^{j}_{\mu\sigma} \; a^{k}_{\sigma\nu} .$$

In (1.13), each term of the sum has the value 0 or 1 and is equal to 1 only if treatment σ is a j^{th} associate of treatment μ and a k^{th} associate of treatment ν . Thus $c^{jk}_{\mu\nu}$ is equal to the number of treatments which have this property. From (iii)(c) of the definition, page 8, we have

$$(1.14) \qquad c^{jk}_{\mu\nu} = p^{i}_{jk} \text{ when } \mu \neq \nu \text{ and } \mu \text{ and } \nu \text{ are } i^{th} \text{ associates.}$$

A diagonal element $c^{jk}_{\mu\mu}$ of the produce is equal to the number of treatments which are simultaneously j^{th} associates and k^{th} associates of treatment μ . Therefore

$$(1.15) \qquad c^{jk}_{\mu\mu} = \delta_{jk} \; n_j ,$$

where δ_{jk} is the Kronecker function defined as 1 if $j = k$ and 0 if $j \neq k$. Statements (1.12 to 1.15) lead to the following theorem.

Theorem 1.1. If A_i denotes the incidence matrix for i^{th} associates in a PBIB design with m associate classes, then

$$(1.16) \qquad A_j A_k = A_k A_j = \delta_{jk} \, n_j I_v + \sum_{i=1}^{m} p^{i}_{jk} \, A_i ,$$

where δ_{jk} is the Kronecker delta function and I_v is the $v \times v$ identity matrix.

Proof: Statements (1.12) to (1.15) show that $A_j A_k$ has the indicated form. The statement, in (iii)(c) of the definition, that $p^i_{jk} = p^i_{kj}$ then implies that the product is commutative.

The statement that products of the A_i are commutative is equivalent to the statement that the products are symmetric, for if A and B are symmetric matrices, then

$$BA = B'A' = (AB)'$$

and $(AB)'$ is equal to AB if and only if AB is symmetric.

Formula (1.16) for forming products is almost a sufficient as well as a necessary condition that the matrices A_i satisfy the conditions of partial balance. The sufficient conditions are stated in the following theorem.

Theorem 1.2. If A_i , $i = 1, 2, \ldots, m$, are a set of symmetric $v \times v$ incidence matrices whose sum is the matrix with 0's on the main diagonal and 1's elsewhere, and if there exist non-negative integers n_i and p^i_{jk} such that (1.16) holds for $j, k = 1, 2, \ldots, m$, then the A_i are the association matrices of an association scheme satisfying the conditions of partial balance.

Proof:

It must be verified that parts (iii) (a) , (b) and (c) of the definition on page 8 are satisfied. The statement that the sum of the incidence matrices is a matrix with 1's in all off-diagonal positions shows that every pair of distinct treatments are i^{th} associates for some i, which is equivalent to (iii)(a). The number of j^{th} associates of treatment μ is equal to the number of 1's in row μ of A_j, which is in turn equal to the element in the μ, μ position of the product matrix $A_j A_j'$. By symmetry of A_j, this is identical with A_j^2 and may be computed by (1.16), which shows that all diagonal elements of A_j^2 are equal to n_j. Therefore each treatment has n_j j^{th} associates and (iii)(b) is satisfied. The set of j^{th} associates of treatment μ is determined by the positions of the 1's of row μ of A_j, and the set of k^{th} associates of ν is determined in the same way by row ν of A_k. The number of treatments common to these sets is equal to the inner product of these two rows taken as vectors and appears as the element in the μ, ν position of the product $A_j A_k'$, which by symmetry of A_k is identical with $A_j A_k$ and has the form of (1.16). The only term of the sum $\sum_{i=1}^{m} p_{jk}^{i} A_i$ which contributes to the element in the μ, ν position is the term with i such that μ and ν are i^{th} associates. Therefore the number is equal to p_{jk}^{i} when μ and are any pair of i^{th} associates, proving most of (iii)(c). The final statement follows from the fact that $A_j A_k = A_k A_j$, and the proof is complete.

The stipulation that the A_i are incidence matrices is necessary in Theorem 1.2. It is possible to construct matrices having elements other than 0's and 1's which satisfy all the other hypotheses of the theorem, but are of course not association matrices. An example with $m = 2$ is the following.

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

A typical product is $A_2^2 = 3I + 3A_1 + A_2$.

Next consider matrices which are linear combinations of the identity I_v and the association matrices A_i, say

$$(1.17) \qquad \lambda_0 I_v + \lambda_1 A_1 + \ldots + \lambda_m A_m ,$$

where the λ_i are scalars. A product of two such matrices will be a linear combination of terms of the form I_v, A_i and $A_i A_j$, and by Theorem 1.1 will reduce to the form (1.17). Thus the set of matrices of this form is closed under multiplication. Some consequences of this are mentioned below. Among the products of matrices which are readily computed by application of (1.16) are integral powers of the A_i matrices and of the matrix NN'. The square and cube of the matrix A_1 for first associates in a design with $m = 2$ associate classes will now be given as illustrations.

$$(1.18) \qquad \begin{aligned} A_1^2 &= n_1 I_v + p_{11}^1 A_1 + p_{11}^2 A_2 , \\ A_1^3 &= n_1 A_1 + p_{11}^1 A_1^2 + p_{11}^2 A_1 A_2 \end{aligned}$$

$$(1.19) \quad A_1^3 = n_1 p_{11}^1 I_v + (n_1 + p_{11}^{1\,2} + p_{12}^1 p_{11}^2) A_1 + (p_{11}^1 p_{11}^2 + p_{11}^2 p_{12}^2) A_2 \, .$$

It was pointed out that the set of matrices of the form (1.17) is closed under multiplication. It is obviously also closed under addition, and if negative coefficients are allowed, under subtraction. It follows from these remarks and from general properties of matrices that the set forms a commutative ring of matrices with a unit element. This has a number of interesting consequences, of which one may be mentioned. The matrices I_v , A_1 , ... , A_m are easily seen to be linearly independent and form a basis of $m + 1$ elements for the ring. For any matrix C in the ring, the set I_v , C , C^2 , ... , C^{m+1} contains $m + 2$ elements which must be linearly dependent. Therefore C satisfies an equation with scalar coefficients of degree at most $m + 1$. This means that the minimum equation of C has degree at most $m + 1$, or that any matrix of the form (1.17) has at most $m + 1$ distinct characteristic roots. In particular, this applies to NN'. It is possible to use methods based on the commutative ring to find the characteristic roots and their multiplicities. The same results on the number of distinct characteristic roots of NN' , together with a computation of the values and multiplicities of the roots, appear in a paper of Connor and Clatworthy [17] which does not use the A_i matrices. This paper was published before the present work on the association matrices was completed. Several theorems of [17] will be used in Chapters II and III.

Credit is also due to R. C. Bose for some work on the association matrices A_i , including the equivalent of Theorem 1.1, which has not been published but was presented at a meeting of the Institute of Mathematical Statistics at Ann Arbor, Michigan on September 2, 1955. The portions of this research making use of the association matrices had already been completed at that time.

Another possible interpretation of association schemes is by means of linear graphs. (A linear graph may be defined for our purposes as a finite set of points, certain pairs of which are joined by non-directed lines.) The association scheme for i^{th} associates in a PBIB design may be identified with a linear graph on v points by identifying points with treatments and joining points which are i^{th} associates. Since each treatment has n_i i^{th} associates, each point of the graph will lie on n_i lines. Since any two treatments which are i^{th} associates have as common i^{th} associates p^i_{ii} other treatments, each line of the graph will lie on p^i_{ii} triangles. More generally, if an arbitrary line of the i^{th} graph joins points A and B then there are just p^i_{jk} other points which are joined to A by a line of the j^{th} graph and to B by a point of the k^{th} graph. In the case of PBIB designs with two associate classes the graphs may be described more simply. Each of the two graphs is the complement of the other and it is sufficient to describe the one for first associates. In this graph there are n_1 lines on each point, each line lies on p^1_{11} triangles, and each pair of points not joined by a line is joined by p^2_{11} chains of two lines.

The incidence matrix A_i of i^{th} associates may also be interpreted as the incidence matrix of the i^{th} graph, a 1 in the μ, ν position of the matrix indicating that points μ and ν of the graph are joined by a line. Incidence matrices are useful in analysis of the structure of graphs. The terminology of linear graphs will be used in parts of Chapter IV for the investigation of the structure of association schemes.

II. ENUMERATION OF POSSIBLE DESIGNS AND ASSOCIATION SCHEMES WITH TWO ASSOCIATE CLASSES

2.1 The Class of PBIB Designs with Two Associate Classes

In this chapter, attention will be confined to partially balanced incomplete block designs with two associate classes. Bose and Shimamoto discussed these designs thoroughly in 1952 [10] and introduced a classification of them into five types according to the form of the association scheme. An extensive set of tables of these designs was compiled by Bose, Clatworthy and Shrikhande and published in 1954 [6], following the classification of Bose and Shimamoto. Over 370 designs are listed, about three-fourths of them of group divisible type. The authors state that the compilation includes all designs that were known at that time, but do not claim that additional designs cannot be constructed. The classification of association schemes seems also to be a summary of known types, and is not represented as a listing of all possible schemes. Some new association schemes to be constructed in Chapter III fall outside the classification, showing that it is not exhaustive. The classification is described later in this section.

A computing procedure developed from some known necessary conditions on association schemes is introduced in Section 2.2 and used in Tables I and II of the Appendix to list the parameters of all possible association schemes with two associate classes and not of group divisible type, for all numbers of treatments $v \leq 100$. Several new necessary conditions are also proved in Section 2.2. Necessary and sufficient conditions for the existence of an association scheme are not known. Of 101 sets of parameters tabulated in the Appendix, 56 correspond to schemes which are already known or constructed in this dissertation, and 4 are proved impossible. These schemes are identified in Table II. There remain 41 sets of parameters for which the existence or non-existence of an association scheme is unknown. Such a list was frequently promised by the early writers on PBIB designs, but appears never to have been compiled and is offered here as an original contribution, along with the computing scheme and the necessary conditions of Section 2.2.

The next logical step is to list all combinatorially possible designs for each association scheme, identifying those known to exist or to be impossible. The counterpart of this list for BIB designs was mentioned on page 6; it was published by Fisher and Yates in 1938 and revised in 1943 and 1947 [21]. It includes 16 sets of parameters about which nothing was known in 1938 but which were subsequently attacked so assiduously by various writers that by now all but two (at most) have either been constructed or shown impossible. Such a list for PBIB designs would be much longer, even for the association schemes so far constructed, and its compilation has perhaps been deterred by the fact that enough PBIB designs are already available to satisfy most of the needs of experimenters. The parameters of all possible designs with $r \leq 10$, $k \leq 10$ will be listed for the schemes under special study in this dissertation, mostly of Latin Square type. The list appears in Tables III and IV of the Appendix and the method by which it is constructed is developed in Section 2.3, using known theorems for the most part. The parameters of schemes which have been constructed or proved impossible are identified in Table IV. Some new designs and impossibility proofs are included for reference in Section A3 of the Appendix. It is believed that this list has not appeared before.

The parameters for designs with $m = 2$ associate classes are

$$v,\ b,\ k,\ r,\ \lambda_1,\ \lambda_2,\ n_1,\ n_2,$$

$$P_1 = \begin{bmatrix} p^1_{11} & p^1_{12} \\ p^1_{12} & p^1_{22} \end{bmatrix},\quad P_2 = \begin{bmatrix} p^2_{11} & p^2_{12} \\ p^2_{12} & p^2_{22} \end{bmatrix}.$$

These parameters satisfy relations (1.3) to (1.7), which are restated here for this special case.

(2.1) $$bk = vr$$

(2.2) $$n_1 + n_2 = v-1$$

(2.3) $$n_1\lambda_1 + n_2\lambda_2 = r(k-1)$$

(2.4) $$1 + p^1_{11} + p^1_{12} = p^2_{11} + p^2_{12} = n_1$$

$$p^1_{12} + p^1_{22} = 1 + p^2_{12} + p^2_{22} = n_2$$

(2.5) $$n_1\, p^1_{12} = n_2 p^2_{11}$$

$$n_1\, p^1_{22} = n_2\, p^2_{12}$$

The classification by Bose and Shimamoto of association schemes for PBIB designs with two associate classes will now be taken up. This classification was introduced by Shimamoto in a master's thesis written under the direction of Bose, was first published in a joint paper in 1952 [10], and has been used, with minor changes, in later papers by the same authors and others. The five types of designs are Group Divisible (GD), Triangular, Simple, Cyclic and Latin Square (L_g) and will be described separately.

<u>Group Divisible</u> designs are defined when the number of treatments v may be expressed as a product mn. The treatments are divided into m groups of n treatments each, treatments in the same group being taken as first associates while those in different groups are second associates. GD designs have been mentioned with some examples in Chapter I. They form the most important class of partially balanced designs and the largest known class, and have been studied more extensively than any other partially balanced designs. In 1952 Bose and Connor [7] published several results on these designs, one of which will be generalized to Latin square type designs in Chapter IV. One feature of [7] was the division of the designs into three subclasses, essentially on the basis of the rank of the matrix NN', though the connection with the characteristic roots of NN' was not brought out clearly until 1954 in a paper by Connor and Clatworthy [17]. A similar classification of Latin square type designs will be mentioned in Section 2.3. Some other important publications on GD designs are [16], [10], [11].

Triangular designs are defined when v is equal to a triangular number $\frac{n(n-1)}{2}$. The association scheme is an array of n rows and n columns with the following properties:

(i) The positions in the main diagonal are left blank.

(ii) The $\frac{n(n-1)}{2}$ positions above the main diagonal are filled with the numbers $1, 2, \ldots, \frac{n(n-1)}{2}$ corresponding to the treatments.

(iii) The positions below the main diagonal are filled so that the array is symmetric.

(iv) For any treatment θ the first associates are those treatments which lie in the same row (or in the same column) as θ.

Simple designs include designs with various values of v. In the 1939 paper [8] in which PBIB designs were first introduced, Bose and Nair gave some examples of designs obtained by dualizing BIB designs, that is, interchanging the roles of treatments and blocks. The treatments in one particular block of the dual design then correspond to the blocks of the original design which contain a particular treatment. The duals of some BIB designs fail to satisfy the conditions of partial balance, but in any case the dual design will have the property that any two blocks will have the same number of treatments in common. This led to the designation "linked block" for such designs [39]. The duals of several classes of BIB designs were shown by Shrikhande [31] to be partially balanced with two associate classes. Some of these designs fall within the triangular class discussed above; the others have the property that

$\lambda_1 = 1$, $\lambda_2 = 0$. In the classification by Bose and Shimamoto in their 1952 paper [10], these were listed as a separate "linked block" type of designs. In the tables published by Bose, Clatworthy and Shrikhande in 1954 [6] this classification is enlarged somewhat to include some designs which are not obtained by dualization and do not have the linked block property, but which do have the property that $\lambda_1 \neq 0$, $\lambda_2 = 0$. They are referred to as simple designs. The classification was enlarged a little too much in the 1954 tables, as the three designs listed for $v = 19$ are not partially balanced. Table II and Theorem 2.2 will each show the impossibility of a PBIB design with two associate classes and $v = 19$.

In <u>Cyclic</u> designs the first associates of treatment θ are the treatments $\theta + d_1$, $\theta + d_2$, . . . , $\theta + d_{n_1}$, reduced modulo v, for a suitably chosen set of integers d_j. The association matrix A_1 thus has the special property that each row is a cyclic permutation of the first row. In every known design of this type v is a prime of the form $4t + 1$ and the set of d's may be taken either as the set of quadratic residues of v or as the set of quadratic non-residues.

<u>Latin Square</u> type designs are defined when v is equal to a square n^2. The association scheme consists of an $n \times n$ array of the numbers $1, 2, \ldots, n^2$, possibly with an orthogonal set of one or more $n \times n$ Latin squares superimposed. Two treatments are taken as first associates if they occur in the same row or column or if they coincide with the same letter in any of the Latin squares. A scheme of this type using $g-2$ Latin

squares is said to be of Latin square type with g constraints and is denoted briefly by the symbol L_g. This type of designs will be treated in some detail in this dissertation. Parameters of possible L_g designs will be enumerated in Section 2.3 and tabulated in Tables III and IV of the Appendix, and a number of properties of the association schemes will be investigated in Chapters III and IV.

An $n \times n$ Latin square is an arrangement of n letters into the cells of an $n \times n$ array in such a way that every row and every column of the array contains every letter exactly once. A Latin square may be constructed for every n, for example by taking each row as a cyclic permutation of the first, as in this example.

```
A B C D
B C D A
C D A B
D A B C
```

Two Latin squares are said to be orthogonal if, when one is superimposed on the other, every ordered pair of letters occurs exactly once in the resulting square. Thus the following 3×3 Latin squares are orthogonal.

```
A B C   A B C
B C A   C A B
C A B   B C A
```

On the other hand, there exists no 4×4 Latin square orthogonal to the example above. It has been shown that at most $n-1$ mutually orthogonal $n \times n$ Latin squares can be constructed, and that the construction of such a set, called a complete orthogonal set, can actually be accomplished if n is a prime or a power of a prime [26], [32], [5]. The following three squares form a complete orthogonal set for the case $n = 4$.

A	B	C	D		A	B	D	C		A	B	C	D
B	A	D	C		C	D	A	B		D	C	B	A
C	D	A	B		D	C	B	A		B	A	D	C
D	C	B	A		B	A	D	C		C	D	A	B

Knowledge about sets of orthogonal Latin squares when n is not a prime power is rather sketchy. In no such case has a complete set been constructed, though methods are known for constructing a smaller number in certain cases (e.g. two orthogonal 12×12 squares). For n satisfying certain conditions it is known that the maximum number of squares in an orthogonal set is less than $n-1$ [12], while in the case $n = 6$, enumeration methods have been used to show that no orthogonal pair exists. This case was mentioned by Euler [18] but not finally settled until 1900 [35] (see also [20]). The existence of a complete orthogonal set of $n \times n$ Latin squares is equivalent to the existence of a finite projective plane geometry in which each line contains $n + 1$ points. Either of these systems can be constructed from a finite field of order n, so that a sufficient condition for their existence is that n be a prime power, but this condition is not known to be necessary. An open question at present is whether any set of two or more 10×10 orthogonal Latin squares exists.

Orthogonal squares which are not Latin squares can be useful in the construction of association schemes. The two following squares are obviously not Latin squares but they are orthogonal; that is, when they are superimposed every ordered pair of letters occurs exactly once.

R:

A	A	A	A
B	B	B	B
C	C	C	C
D	D	D	D

C:

A	B	C	D
A	B	C	D
A	B	C	D
A	B	C	D

An association scheme obtained by superimposing them on an array of the numbers 1, 2, . . . , 16 and taking numbers as first associates if they occur with the same letter in either square will be identical with the scheme L_2 in which associates are defined by rows and columns of the array. Moreover, any 4 x 4 Latin square is orthogonal to each of them, and any 4 x 4 square which is orthogonal to both must be a Latin square. The analogous statement for n x n squares is clearly true. Therefore a set of g-2 orthogonal Latin squares is equivalent to a set of g orthogonal squares of which two are R and C.[1] If the n^2 cells of each of g orthogonal squares are subjected to the same permutation, the resulting squares will still be orthogonal, though not necessarily Latin. Given any set of orthogonal squares, simultaneous permutation of the cells can be used to place any two of the squares in the form of R and C, still preserving orthogonality. The association scheme L_g may now be redefined by a set of g n x n orthogonal squares superimposed on an array of the numbers 1, 2, . . . , n^2, taking numbers as first associates if they occur with the same letter in any of the squares. Permutation of the numbers of the array together with the cells of the superimposed array will preserve all association relations, so that any such association scheme is equivalent to one in which two of the squares are R and C and any

---

1. The notion of non-Latin orthogonal squares is not new. In at least one recent publication [13] there is a description of the squares R and C and their relation to Latin squares.

remaining squares are necessarily Latin. In particular this shows that for a given n, all pairs of $n \times n$ orthogonal squares lead to L_2 schemes which are equivalent except for numbering of treatments. It may be noted that an L_1 scheme is a special case of a group divisible scheme.

It is convenient to use this definition of the L_g scheme to derive expressions for the parameters n_i and p^i_{jk} and to show that they satisfy the requirements for a PBIB design. This derivation will be illustrated with an example of an L_3 scheme for 16 treatments based on the squares R, C, and a Latin square of the orthogonal set. These squares are listed below for easy reference, along with the array of numbers with which they are to be superimposed. The orthogonal squares are numbered from 1 to 3 for identification in the discussion.

Array				Square 1				Square 2				Square 3			
1	2	3	4	A	A	A	A	A	B	C	D	A	B	C	D
5	6	7	8	B	B	B	B	A	B	C	D	B	A	D	C
9	10	11	12	C	C	C	C	A	B	C	D	C	D	A	B
13	14	15	16	D	D	D	D	A	B	C	D	D	C	B	A

Figure 5. Example of L_3 association scheme for 16 treatments.

It follows from the orthogonality of the squares that two cells occupied by the same letter in one square must be occupied by different letters in each other square. Thus the $n-1$ associates of a particular treatment in one square will be distinct from its associates in each other square, and the treatment will have as the number of its first associates

$$n_1 = g(n-1). \tag{2.6}$$

In the example, treatment 10 has as its first associates 9, 11, 12 in square 1; 2, 6, 14 in square 2; and 4, 7, 13 in square 3, for a total of 9 first associates.

Let two treatments θ and ϕ occur with letters a_k and b_k respectively in the k^{th} square, $k = 1, \ldots, g$. a_k and b_k will be distinct for all values of k if the two treatments are second associates, and equal for just one value of k if they are first associates. If θ and ϕ occur with distinct letters in the h^{th} and k^{th} squares, so that $a_h \neq b_h$ and $a_k \neq b_k$, and if these two squares are superimposed, the pair of letters a_h, b_k will occur in just one cell. The treatment in this position will be a common first associate of θ and ϕ. The total number of such treatments will be equal to the number of ordered pairs h, k such that $a_h \neq b_h$ and $a_k \neq b_k$. The number of pairs may be expressed $u(u-1)$, where u is the number of squares in which θ and ϕ occur with distinct letters. If θ and ϕ are second associates they occur with distinct letters in all g squares, $u = g$, and the number p^2_{11} of first associates the treatments have in common is

$$p^2_{11} = g(g-1) . \tag{2.7}$$

In the example, let $\theta = 5$ and $\phi = 10$. Then

$$a_1 = B, \qquad b_1 = C;$$
$$a_2 = A, \qquad b_2 = B;$$
$$a_3 = B, \qquad b_3 = D;$$

and since $a_k \neq b_k$ for all k, treatments 5 and 10 are second associates. The pairs of letters a_h, b_k obtained when squares are superimposed and the corresponding common first associates of the two treatments are as follows.

Pair of superimposed squares h, k	Ordered pair of letters a_h, b_k	Cell in which the pair of letters occurs
1, 2	B, B	6
2, 1	A, C	9
1, 3	B, D	7
3, 1	B, C	12
2, 3	A, D	13
3, 2	B, B	2

The six cells singled out represent the six common first associates of treatments 5 and 10.

If θ and ϕ are first associates, occurring with the same letter in one square, say the first, then $u = g-1$. In addition to the $(g-1)(g-2)$ common first associates found by superimposing pairs of distinct squares, θ and ϕ will have as common first associates the $n-2$ other treatments occurring with the same letter in the first square. No additional first associates are found by superimposing the first square with any of the others, for the pair a_h, b_k in this case is a_1, b_k, which is identical with b_1, b_k, and this pair of letters occurs in the position of ϕ itself rather than any distinct treatment. The number p_{11}^1 of common first associates of the two treatments is therefore

$$(2.8) \qquad p_{11}^{1} = (g-1)(g-2) + n-2 = g^2 - 3g + n.$$

In the example, let $\theta = 9$ and $\phi = 10$. Then

$$a_1 = C, \qquad b_1 = C;$$
$$a_2 = A, \qquad b_2 = B;$$
$$a_3 = C, \qquad b_3 = D;$$

and since $a_1 = b_1$, treatments 9 and 10 are first associates. They occur with distinct letters in squares 2 and 3. The pairs of letters a_h, b_k obtained when these squares are superimposed and the corresponding common first associates of the two treatments are as follows.

Pair of superimposed squares h, k	Ordered pair of letters a_h, b_k	Cell in which the pair of letters occurs
2, 3	A, D	13
3, 2	C, B	14

Treatments 9 and 10 occur with the same letter C in square 1, and the other $n-2 = 2$ cells of the square which also contain the letter C are 11 and 12. The four cells singled out represent the four common first associates of treatments 9 and 10.

The remaining parameters are quickly computed from n_1, p_{11}^{2} and p_{11}^{1} to give the following set.

$$(2.9) \qquad v = n^2 ,$$

$$n_1 = g(n-1) , \qquad P_1 = \begin{bmatrix} g^2 - 3g + n & (g-1)(n-g+1) \\ (g-1)(n-g+1) & (n-g)(n-g+1) \end{bmatrix} ,$$

$$n_2 = (n-g+1)(n-1) ,$$

$$P_2 = \begin{bmatrix} g(g-1) & g(n-g) \\ g(n-g) & (n-g)^2 + g-2 \end{bmatrix} .$$

The non-negative nature of n_2 implies the inequality

$$(2.10) \qquad g \leq n + 1 ,$$

proving the statement previously made that the maximum number of Latin squares in an orthogonal set is $n-1$. If a complete set of $n+1$ orthogonal squares is constructed and if g of them are used to define an L_g association scheme, then second associates are precisely those treatments which occur with the same letter in one of the $n-g+1$ remaining squares. It will be convenient later if the letter f is introduced to represent this number:

$$(2.11) \qquad f = n - g + 1 .$$

It is clear that if the designation of first and second associates is interchanged in the L_g association scheme, the result will be the L_f scheme based on the f remaining squares. A scheme with the parameter values and properties of L_f can be obtained in this way from any L_g scheme, whether or not the f orthogonal squares are actually constructed. Some examples will be given in Chapter IV of schemes of this kind for which the orthogonal squares can be shown not to exist. Since the schemes L_g and L_f are equivalent for any value of g , each L_g scheme is equivalent to one in which $g \leq \frac{n+1}{2}$, and in particular the L_n scheme is equivalent to L_1, which is simply a group divisible scheme. The dual roles of g and f are most clearly seen from the following expressions for the parameters.

$$(2.12) \qquad \begin{aligned} n &= (g+f-1)\,, \\ v &= n^2\,, \\ n_1 &= g(n-1)\,, \\ n_2 &= f(n-1)\,, \end{aligned} \qquad P_1 = \begin{bmatrix} (g-1)^2+f-2 & f(g-1) \\ f(g-1) & f(f-1) \end{bmatrix}, \quad P_2 = \begin{bmatrix} g(g-1) & g(f-1) \\ g(f-1) & (f-1)^2+g-2 \end{bmatrix}.$$

These expressions give the values of parameters of L_g schemes if g and f are positive; if group divisible schemes are to be excluded, g and f must be taken as ≥ 2.

It may be verified that certain negative values for g and f (and hence n) lead to values for the above parameters which are non-negative and different from those obtained with positive g and f. Conditions (2.2), (2.4) and (2.5) are algebraic identities in g and f and are satisfied in either case, so the new values represent the parameters of a possible new series of association schemes. Some connections of the new schemes with the ordinary L_g series will be discussed in Chapter III, and several of them will be constructed. They are found to fall outside the five known classes of association schemes. The name "negative Latin square" will be used for the series of schemes whose parameter values are given by (2.12) with g, f and n negative, and the symbol L_g^*, with g negative, will be used as an abbreviation. Parameter values of schemes in the L_g^* series are identified in Table II of the Appendix, and possible designs for the new schemes will be listed in Table IV. It should be mentioned that the ordinary Latin square, or L_g , series was defined to include only schemes in which first associates can be defined by means of a set of g orthogonal $n \times n$ squares. The term "scheme with L_g parameter values" will include the L_g schemes and any other schemes whose parameter values are given by (2.12) with positive values of g, f and n.

2.2 Enumeration of Association Schemes.

The parameters by which an association scheme is specified are v, n_i, p^i_{jk} . A procedure will be developed in this section for the construction of a table of all possible sets of values of these parameters for PBIB designs with two associate classes. For a given value of v , a unique group divisible (GD) scheme exists for each pair of integers $m \geq 2$, $n \geq 2$, such that $mn = v$; that is, there is one GD association scheme for each proper divisor of v. The construction of these association schemes is trivial and it is not considered necessary to list their parameters. The enumeration of possible association schemes of other types will be carried out for all values of $v \leq 100$.[1]

Theorem 2.0, due to Connor and Clatworthy [17], defines parameter values of one series of possible association schemes in terms of a parameter t . Theorem 2.1 uses two parameters s and t in the derivation of expressions for the parameters of all possible schemes not given by Theorem 2.0. Several necessary conditions on the parameters are also derived. Table Ia of the Appendix lists the sets of parameters given by Theorem 2.0. Table Ib also makes use of Theorem 2.5. Tables Ia and Ib give 101 possible sets of parameters for $v \leq 100$. These are listed in Table II. The necessary conditions applied in constructing these tables are by no means sufficient for the existence of an association scheme and

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1. An easy computation shows that there are 283 GD schemes for values of $v \leq 100$.

it is easy to derive additional necessary conditions applying to certain classes of the parameters. Theorems 2.2 to 2.8, which are of this nature, show the impossibility of four of the schemes in Table II and place restrictions on several others, as well as giving some general information on the structure of association schemes. The schemes which are proved impossible are indicated by the letter X in Table II, followed by a reference by number to the applicable theorem. Parameters of known schemes are indicated by the letter C, followed by a reference. The remaining 41 schemes have neither been constructed nor proved impossible. Further explanation of the tables precedes them in the Appendix.

Several necessary conditions satisfied by the parameters of a PBIB design are derived by Connor and Clatworthy [17] by using the matrix NN'. They show that this matrix has only three distinct characteristic roots, obtaining expressions for the roots and their multiplicities. The same results may be obtained rather easily by methods mentioned in Section 1.3. In their notation,

rk is a root with multiplicity 1,

$r - z_1$ is a root with multiplicity α_1,

$r - z_2$ is a root with multiplicity α_2.

The α_i depend only on the parameters of the association scheme, and are thus the same for all designs having a given association scheme. The z_i depend in addition on λ_1 and λ_2 and will not be needed in this section. Equations (5.9) and (5.10) of [17] give

$$(2.13) \qquad \alpha_1 = \frac{(v-1)(-\gamma + \sqrt{\Delta} + 1) - 2n_1}{2\sqrt{\Delta}} ,$$

$$(2.14) \qquad \alpha_2 = \frac{(v-1)(\gamma + \sqrt{\Delta} + 1) - 2n_2}{2\sqrt{\Delta}} ,$$

where

$$\text{(a)} \qquad \gamma = p_{12}^2 - p_{12}^1 ,$$

$$(2.15) \quad \text{(b)} \qquad \beta = p_{12}^1 + p_{12}^2 ,$$

$$\text{(c)} \qquad \Delta = \gamma^2 + 2\beta + 1 .$$

Useful necessary conditions on the parameters may be obtained from the fact that the multiplicities α_1 and α_2 must be non-negative integers. They are of course not independent; $\alpha_1 + \alpha_2 = v-1$. Connor and Clatworthy in their theorems 5.3 to 5.5 investigate the nature of Δ . One of their results will be stated as a theorem.

THEOREM 2.0. (Connor and Clatworthy) [17]. If Δ is not a square, it is necessary that

$$\text{(a)} \qquad p_{12}^1 = p_{12}^2 = t ,$$

$$(2.16) \quad \text{(b)} \qquad n_1 = n_2 = \alpha_1 = \alpha_2 = 2t ,$$

$$\text{(c)} \qquad v = \Delta = 4t + 1 ,$$

where t is a non-negative integer defined by (2.16) (c).

This series of possible association schemes is easily enumerated. The possible parameters are listed in Table Ia. In every other case

Δ is an integral square. This will be used to develop a method of systematic enumeration of other possible association schemes. From (2.15) (c),

$$\beta = \frac{\Delta - \gamma^2 - 1}{2} . \tag{2.17}$$

Solving (2.15) (a) and (b) for the p_{12}^i , then using (2.17),

$$p_{12}^1 = \frac{\beta - \gamma}{2} = \frac{\Delta - \gamma^2 - 2\gamma - 1}{4} = \frac{\Delta - (\gamma + 1)^2}{4} \tag{2.18}$$

$$= \frac{(\sqrt{\Delta} - \gamma - 1)}{2} \frac{(\sqrt{\Delta} + \gamma + 1)}{2} ,$$

$$p_{12}^2 = \frac{\beta + \gamma}{2} = \frac{\Delta - \gamma^2 + 2\gamma - 1}{4} = \frac{\Delta - (\gamma - 1)^2}{4} \tag{2.19}$$

$$= \frac{(\sqrt{\Delta} - \gamma + 1)}{2} \frac{(\sqrt{\Delta} + \gamma - 1)}{2} .$$

Statement (2.17) shows that the integers Δ and γ must be of opposite parity. Therefore $\sqrt{\Delta} \pm \gamma$ must be odd integers, $\sqrt{\Delta} \pm \gamma \pm 1$ must be even integers for all choices of signs, and s and t defined as follows will be integers.

$$(2.20) \qquad \begin{aligned} s &= \frac{\sqrt{\Delta} - \gamma - 1}{2} , \\ t &= \frac{\sqrt{\Delta} + \gamma - 1}{2} . \end{aligned}$$

Equations (2.18) and (2.19) may now be rewritten as follows.

$$p_{12}^1 = s(t + 1) , \tag{2.21}$$

$$p_{12}^2 = (s + 1)t . \tag{2.22}$$

Also,

$$(2.23) \qquad \sqrt{\Delta} = s + t + 1 .$$

A preliminary enumeration of possible pairs of values p^1_{12} , p^2_{12} now reduces to the listing of pairs of integers s, t and application of (2.21) and (2.22). Bose and Connor [7] show that a PBIB design with two associate classes is of GD type if and only if $p^i_{12} = 0$ for $i = 1$ or 2. This case will be excluded by requiring s and t to be <u>positive</u> integers.

For each pair p^1_{12} , p^2_{12} , it is next desired to enumerate possible sets of the remaining parameters, particularly n_1 and n_2 . It will be convenient to do this by finding values of p^1_{22} and p^2_{11} . Multiplying equations (2. 5), we obtain

$$n_1 \, n_2 \, p^1_{12} \, p^2_{12} = n_1 \, n_2 \, p^1_{22} \, p^2_{11} ,$$

$$(2.24) \qquad p^1_{12} \, p^2_{12} = p^1_{22} \, p^2_{11} .$$

Pairs of possible values of p^1_{22} and p^2_{11} may thus be obtained by expressing the product $p^1_{12} \, p^2_{12}$ in every possible way as the product of two positive integers p^1_{22} and p^2_{11} . Relations (2.4) then give values of the remaining parameters, including

$$n_2 = p^1_{12} + p^1_{22} ; \quad n_1 = p^2_{12} + p^2_{11} .$$

To avoid duplication, we make the restriction

$$(2.25) \qquad p^1_{12} \leq p^2_{12} ; \text{ if } p^1_{12} = p^2_{12} , \text{ then } n_1 \leq n_2 .$$

A design for which $p^1_{12} > p^2_{12}$ or $p^1_{12} = p^2_{12}$ and $n_1 > n_2$ may be reduced to one for which (2.25) holds by changing the designations of first and second associates.

The enumeration will be carried out only for values of $v \leq 100$. Since $v = n_1 + n_2 + 1 = p_{12}^2 + p_{11}^2 + p_{12}^1 + p_{22}^1 + 1$,

this means $p_{12}^2 + p_{11}^2 + p_{12}^1 + p_{22}^1 \leq 99$,

implying

$$\tfrac{1}{4}(p_{12}^1 + p_{12}^2 + p_{22}^1 + p_{11}^2) < 25 .$$

Since the geometric mean of any set of positive numbers is $\leq$ their arithmetic mean,

$$(p_{12}^1\, p_{12}^2\, p_{22}^1\, p_{11}^2)^{\frac{1}{4}} < 25 .$$

Using (2.24) we obtain

$$(2.26) \quad (p_{12}^1\, p_{12}^2)^{\frac{1}{2}} < 25 .$$

Only finitely many values of p_{12}^1 and p_{12}^2 satisfy (2.21), (2.22) and (2.26). A convenient form for listing them is a table of values of the function $\sigma(\sqrt{\Delta} - \sigma)$, where σ is an integer. The following portion of the table will illustrate its form.

$\sqrt{\Delta}$ \ σ	0	1	2	3	4
0	0	-1	-4	-9	-16
1	0	0	-2	-6	-12
2	0	1	0	-3	-8
3	0	2	2	0	-4
4	0	3	4	3	0
5	0	4	6	6	4

The table is most easily constructed by noting that the diagonal entries are 0's and that the entries in column σ form an arithmetic progression with difference σ. In the row of the table corresponding to a fixed value of $\sqrt{\Delta} = s + t + 1$, the consecutive entries for $\sigma = s$ and $\sigma = s + 1$ are precisely $s(t + 1) = p^1_{12}$ and $(s + 1)t = p^2_{12}$. For example, for $\sqrt{\Delta} = 4$, the possible pairs of values of p^1_{12}, p^2_{12} are 0, 3; 3, 4; 4, 3; 3, 0. All values of the p^i_{12} satisfying $0 < p^1_{12} \leq p^2_{12}$; $(p^1_{12}\, p^2_{12})^{\frac{1}{2}} < 25$ are given in the portion of the table in Figure 6.

Thus the sets of parameters to be listed in Table Ib include only 31 possible pairs of values of p^1_{12} and p^2_{12}. For a given pair, the number of values for the remaining parameters depends in part on the number of divisors of the product $p^1_{12}\, p^2_{12}$. The relation $n_2 = p^2_{22} + p^2_{12} + 1$ and the non-negative nature of p^2_{22} lead to

$$p^2_{12} \leq n_2 - 1 = p^1_{12} + p^1_{22} - 1,$$

$$p^1_{22} \geq p^2_{12} - p^1_{12} + 1.$$

$$(2.27) \qquad p^1_{22} \geq \gamma + 1.$$

This restriction on the value of p^1_{22} will be used to shorten the computation somewhat. A similar restriction on p^2_{11} turns out to be vacuous in view of (2.25). It will be shown in Theorem 2.5 that at least one of n_1 and n_2 must be an even number. If both are odd in a

$\sqrt{\Delta}$ \ σ	1	2	3	4	5
3	2	2			
4	3	4			
5	4	6	6		
6	5	8	9		
7	6	10	12	12	
8	7	12	15	16	
9	8	14	18	20	20
10	9	16	21	24	25
11	10	18	24		
12	11	20	27		
13	12	22			
14	13	24			
15	14	26			
16	15	28			
17	16	30			
18	17	32			
19	18	34			

Figure 6. Array giving possible values of p^1_{12} and p^2_{12} .

line of Table Ib, their values are omitted and the rest of the line is left blank, further shortening the computation.

The full strength of the positive integral condition on α_1 and α_2 has not yet been imposed. Using $v-1 = n_1 + n_2$, expression (2.13) may be written

$$(2.28) \quad \alpha_1 = \frac{n_2(\sqrt{\Delta} - \gamma + 1) + n_1(\sqrt{\Delta} - \gamma - 1)}{2\sqrt{\Delta}} = \frac{n_2(s+1) + n_1 s}{\sqrt{\Delta}}$$

The value of this quotient is readily computed for each set of values of p_{12}^1, p_{12}^2, n_1, n_2 in Table Ib. If it is not an integer, no association scheme exists and the letter f is entered in column α_1 of the table. If α_1 is an integer, it is entered and followed by the value of $v = n_1 + n_2 + 1$.

The results that have been obtained for the construction of Table Ib are collected for convenience and stated as the following theorem.

THEOREM 2.1. An association scheme for a PBIB design with two associate classes and not of group divisible type must have parameter values given either by the expressions stated in Theorem 2.0 or by the following conditions.

$\sqrt{\Delta}$ is a positive integer,

s and t are positive integers satisfying (2.23): $s + t + 1 = \sqrt{\Delta}$

$$(2.21) \quad p_{12}^1 = s(t+1),$$

$$(2.22) \quad p_{12}^2 = (s+1)t,$$

p^1_{22} is a divisor of $p^1_{12}\, p^2_{12}$ satisfying (2.27):

$$p^1_{22} \geq \gamma + 1 = p^2_{12} - p^1_{12} + 1 ,$$

$$p^2_{11} = \frac{p^1_{12} \;\; p^2_{12}}{p^1_{22}} \qquad \text{(obtained from (2.24))},$$

$$(2.4) \qquad \begin{aligned} n_1 &= p^2_{11} + p^2_{12} , \\ n_2 &= p^1_{22} + p^1_{12} , \end{aligned}$$

$$(2.28) \qquad \frac{(s+1)\, n_2 + s\, n_1}{\sqrt{\Delta}} = \alpha_1 \quad \text{must be an integer;}$$

moreover, if the requirements

$$(2.25) \qquad p^1_{12} \leq p^2_{12} \; ; \; \text{ if } \; p^1_{12} = p^2_{12} , \; \text{ then } \; n_1 \leq n_2$$

and

$$v \leq 100$$

are imposed, then p^1_{12} , p^2_{12} must be a pair of consecutive entries in row $\sqrt{\Delta}$ of the array in Figure 6.

The proof of Theorem 2.1 has already been completed. One additional necessary condition used in Table Ib will appear as Theorem 2.5. This condition is a special case of (2.28) but seems to be of sufficient interest to be stated separately. It may be remarked that taking negative integral values for s and t (which is equivalent to using the negative square root of Δ) leads to parameter values which are positive but no different from those already obtained.

The enumeration in Tables Ia and Ib gives 101 sets of possible parameter values for association schemes. These are reproduced in Table II, in order of increasing values of v, and numbered serially. These serial numbers are given for reference in Tables Ia and Ib. Table II gives values of v, n_i, p^i_{jk}, α_i and $\sqrt{\Delta}$. The parameter $\sqrt{\Delta}$ will be found convenient in locating particular sets of parameters in Table Ib, which is arranged in order of increasing values of $\sqrt{\Delta}$. Table II is standardized by listing only association schemes for which $n_1 \leq n_2$. In some cases this requires that designations of first and second associates be interchanged in the corresponding scheme of Table Ib. The same parameter values occur, but with the indices 1 and 2 interchanged wherever they appear.

Inspection of Table II suggests a number of remarks about the possible association schemes. Their abundance when v is a square is somewhat striking; so is their scarcity when v is a prime. n_1 and n_2 have a factor in common in every scheme in the list; α_1 and α_2 have a common factor in many cases but not all; there seems to be a high proportion of cases in which at least one of the α_i has a factor in common with v. The following theorems, some of which were suggested by this sort of observation, show that at least part of the apparent regularity is a result of general properties of association schemes. Proofs of impossibility of several association schemes are obtained as particular results of some of the theorems.

THEOREM 2.2. In a PBIB design with two associate classes, if the number of treatments v is a prime, then v must have the form $4t + 1$ and the parameters of the association scheme must satisfy (2.16).

PROOF: Except in the case specified in (2.16), the values of the association scheme parameters are given by Theorem 2.1. The following makes use of (2.2), (2.4), (2.21), (2.22) and (2.24).

$$v = n_1 + n_2 + 1 = p^1_{22} + p^1_{12} + p^2_{12} + p^2_{11} + 1$$

$$= p^1_{22} + s(t+1) + (s+1)t + 1 + \frac{st(s+1)(t+1)}{p^1_{22}}$$

$$= \frac{(p^1_{22})^2 + (2st + s + t + 1)\, p^1_{22} + st(s+1)(t+1)}{p^1_{22}} ,$$

$$v = \frac{\left[p^1_{22} + st\right]\left[p^1_{22} + (s+1)(t+1)\right]}{p^1_{22}} . \tag{2.29}$$

The greatest common divisor of two integers f and g will be denoted by the usual notation (f, g). Define c and d by

$$c = (p^1_{22}\ ,\ st)\ ,$$

$$cd = p^1_{22}\ .$$

$p^1_{22} + st$ is then divisible by c. Since d is a divisor of p^1_{22}, it must be prime to st. But $\dfrac{p^1_{12}\, p^2_{12}}{p^1_{22}} = \dfrac{st(s+1)(t+1)}{cd}$ is an integer p^2_{11}, so d must be a divisor of $(s+1)(t+1)$ and hence of $p_{22}^1 + (s+1)(t+1)$. We may therefore write

$$(2.30) \qquad v = \left[\frac{p_{22}^1 + st}{c}\right] \quad \left[\frac{p_{22}^1 + (s+1)(t+1)}{d}\right],$$

where both factors in the right member are integers. If s and t are both positive, (2.29) shows that both factors are greater than 1; v is then composite. If s or t is equal to 0, it has been shown that if the design exists, it must be of group divisible type, which is defined only for composite values of v. This completes the proof of Theorem 2.2.

THEOREM 2.3. If a PBIB design with two associate classes is not of group divisible type, then the number of treatments v cannot be of the form $p+1$ for any prime p.

PROOF: This theorem is a particular result of some general relations connecting the parameters n_1, n_2, p_{12}^1 and p_{12}^2, which will now be developed. Using (2.4),

$$p_{11}^2 + p_{12}^2 = n_1 ,$$

$$n_2\, p_{11}^2 + n_2\, p_{12}^2 = n_1\, n_2 .$$

Applying (2.5),

$$(2.31) \qquad n_1\, p_{12}^1 + n_2\, p_{12}^2 = n_1\, n_2 .$$

The following form of (2.31) was found useful as a check during the construction of Table Ib.

$$(2.32) \qquad \frac{p_{12}^1}{n_2} + \frac{p_{12}^2}{n_1} = 1 \,.$$ [1]

Next introduce the greatest common divisor (n_1 , n_2) of n_1 and n_2 and define a , m_1 , m_2 by

$$(2.33) \qquad \begin{aligned} (n_1 , n_2) &= a \,, \\ n_1 = a\, m_1 \,, \; n_2 &= a\, m_2 \,. \end{aligned}$$

Then $(m_1 , m_2) = 1$. (2.31) may now be written

$$a\, m_1\, p_{12}^1 + a\, m_2\, p_{12}^2 = a^2\, m_1\, m_2 \,,$$

$$(2.34) \qquad m_1\, p_{12}^1 + m_2\, p_{12}^2 = a\, m_1\, m_2 \,.$$

Equation (2.34) in integers, with m_1 and m_2 relatively prime, implies that p_{12}^1 is divisible by m_2 and p_{12}^2 is divisible by m_1 . Say

$$(2.35) \qquad \begin{aligned} p_{12}^1 &= u\, m_2 \,, \\ p_{12}^2 &= w\, m_1 \,. \end{aligned}$$

Substituting (2.35) in (2.34) and simplifying,

$$(2.36) \qquad u + w = a \,.$$

If the design is not to reduce to a balanced design, both n_1 and n_2

---

1. This admits a geometric interpretation if p_{12}^1 and p_{12}^2 are taken as rectangular coordinates of a point in a plane. Then the point (p_{12}^1 , p_{12}^2) must lie on the straight line with intercepts n_2 and n_1 .

must be non-zero, so that m_1 and m_2 are non-zero (tacitly assumed in some of the preceding statements). If the design is not of GD type, p^1_{12} and p^2_{12} must also be positive, so that each of u and w is ≥ 1. Then (2.36) shows that $a \geq 2$, so that n_1 and n_2 have a proper divisor a in common. Their sum $n_1 + n_2 = v - 1$ is also divisible by a, completing the proof of Theorem 2.3.

Relations (2.33), (2.35) and (2.36) can be used as the basis for an enumeration of possible sets of values n_1, n_2, p^1_{12}, p^2_{12}. It appears to be considerably less efficient than the method based on Theorem 2.1.

THEOREM 2.4. In a PBIB design with two associate classes, if $p^i_{ii} \geq 1$, then $p^i_{ij} \leq \frac{v}{3} - 1$, where i and j are equal to 1 and 2 in some order.

PROOF: The proof will be carried out for $i = 1$ and $j = 2$. The other case is similar. Let θ and ϕ be two treatments which are first associates. Since $p^1_{11} \geq 1$, there is at least one treatment which is a first associate of both. Denote one such treatment by π. Of the n_1 first associates of π, p^1_{11} are first associates of θ and p^1_{11} are first associates of ϕ. At most $2p^1_{11}$ of them are first associates of θ or ϕ or both. At least $n_1 - 2p^1_{11}$ are first associates of neither, and thus are second associates of both. But θ and ϕ are first associates and the number of treatments which are second associates of both is precisely p^1_{22}. This proves the inequality

$$n_1 - 2p^1_{11} \leq p^1_{22} .$$

Using the relations $p_{11}^1 = n_1 - p_{12}^1 - 1$ and $p_{22}^1 = n_2 - p_{12}^1$, from (2.4),

$$n_1 - 2n_1 + 2p_{12}^1 + 2 \leq n_2 - p_{12}^1 ,$$

$$3p_{12}^1 \leq n_1 + n_2 - 2 = v - 3 ,$$

$$p_{12}^1 \leq \frac{v}{3} - 1 . \tag{2.37}$$

The following sets of parameters from Table II violate Theorem 2.4 and are thus impossible.

#	v	p_{11}^1	p_{12}^1
36	50	4	16
40	56	3	18

The following theorems make use of the association matrices A_i. The details will be carried through for A_1, the incidence matrix of first associates. Let the numbering of the treatments be chosen so that the treatment 1 has treatments 2, 3, . . . , $n_1 + 1$ as its first associates. Treatment θ corresponds to row and column θ of A_1. The matrix may then be partitioned as follows.

$$(2.38)\qquad A_1 = \left[\begin{array}{c|c|c} 0 & 1 \; . \; . \; . \; 1 & 0 \; . \; . \; . \; 0 \\ \hline 1 & & \\ \vdots & R & S \\ 1 & & \\ \hline 0 & & \\ \vdots & S' & T \\ 0 & & \end{array}\right] \begin{array}{l} \\ \\ n_1 \text{ rows} \\ \\ \\ n_2 \text{ rows} \\ \\ \end{array}$$

R is a symmetric $n_1 \times n_1$ matrix; T is a symmetric $n_2 \times n_2$ matrix; both have 0's on the main diagonal. S' is the transpose of S. The rows and columns of A_1 will be taken as vectors. The inner product of row 1 with row θ is equal to the number of common first associates of treatments 1 and θ, and is equal to p^1_{11} or p^2_{11} according as treatments 1 and θ are first or second associates, respectively. This shows that each row of block R contains p^1_{11} 1's and each row of block S' contains p^2_{11} 1's. Each row of A_1 contains n_1 1's, and by subtraction the number of 1's in each row of S is equal to $n_1 - 1 - p^1_{11} = p^1_{12}$; the number in each row of T is equal to $n_1 - p^2_{11} = p^2_{12}$.

If the matrix A_2 is partitioned in an analogous way, blocks R and T are $n_2 \times n_2$ and $n_1 \times n_1$ matrices respectively, and the row totals of R, S, S' and T are p^2_{22}, p^2_{12}, p^1_{22} and p^1_{12} respectively.

THEOREM 2.5. In any PBIB design with two associate classes, the following statements are equivalent and true.

(a) The products $n_1\, p^1_{11}$, $n_1\, p^1_{12}$, $n_2\, p^2_{22}$, and $n_2\, p^2_{12}$ are all even.

(b) n_1 and n_2 cannot both be odd numbers.

PROOF: Each of the n_1 rows of submatrix R of A_1 contains p^1_{11} 1's , and R therefore contains $n_1\, p^1_{11}$ 1's. Since R is symmetric with 0's on the main diagonal, it contains 1's only in symmetrically located pairs and $n_1\, p^1_{11}$ must be even. Similar reasoning applied to T shows that $n_2\, p^2_{12}$ is even. The argument may be repeated for the matrix A_2 to show that $n_1\, p^1_{12}$ and $n_2\, p^2_{22}$ are even. (An equivalent argument using A_1 is based on the remark that $n_1\, p^1_{12}$ and $n_2\, p^2_{22}$ are equal to the numbers of off-diagonal 0's in R and T respectively.) This completes the proof of (a). Since both terms in the left member of (2.31) are even, $n_1\, n_2$ must be even, proving (b). It remains to show that (b) implies (a). Let (b) be true. If both n_1 and n_2 are even, (a) is true trivially. If one is odd, say n_2 , then n_1 is even, and $n_1\, p^1_{22} = n_2\, p^2_{12}$ is even, implying p^2_{12} is even. Therefore $n_2 - 1 - p^2_{12} = p^2_{22}$ is even and the products $n_2\, p^2_{12}$ and $n_2\, p^2_{22}$ are both even, as well as $n_1\, p^1_{11}$ and $n_1\, p^1_{12}$. A similar argument is used when n_1 is odd, completing the proof that (b) implies (a).

Statement (b) is used in the construction of Table Ib. It can be shown that it is weaker than (2.28), a condition which is also used, but it is used because it shortens the computation.

Additional information is now needed about the partitioned matrix A_1 . Squaring according to the rule for products of partitioned matrices, ([1], p. 24),

$$A_1^2 = \left[\begin{array}{c|ccc|ccc} n_1 & p^1_{11} & \cdots & p^1_{11} & p^2_{11} & \cdots & p^2_{11} \\ \hline p^1_{11} & & & & & & \\ \vdots & & U + R^2 + SS' & & & RS + ST & \\ p^1_{11} & & & & & & \\ \hline p^2_{11} & & & & & & \\ \vdots & & S'R + TS' & & & S'S + T^2 & \\ p^2_{11} & & & & & & \end{array}\right] \tag{2.39}$$

where U is a matrix all of whose elements are 1's. By (1.18), $A_1{}^2$ has diagonal entries n_1, entries p^1_{11} in the positions of 1's of A_1, and entries p^2_{11} in the positions of off-diagonal 0's of A_1. This proves

LEMMA 2.1: If R, S, S' and T are the submatrices of A_1 depicted in (2.38),

(a) $R^2 + SS'$ has diagonal entries $n_1 - 1$, entries $p^1_{11} - 1$ in any positions occupied by 1's in matrix R, and entries $p^2_{11} - 1$ elsewhere;

(b) $S'S + T^2$ has diagonal entries n_1, entries p^1_{11} in any positions occupied by 1's in T, and entries p^2_{11} elsewhere.

THEOREM 2.6. A necessary condition for the existence of an association scheme for a PBIB design with two associate classes and $p^i_{ii} = 0$, is the existence of a BIB design with parameters $v = n_i$, $r = n_i - 1$, $k = p^j_{ii}$, $b = n_j$, $\lambda = p^j_{ii} - 1$, where i and j are equal to 1 and 2 in some order. Moreover, given any block of the BIB design, there exist at least p^j_{ij} other blocks which have no treatments in common with the given block.

PROOF: The proof will be carried out for the case $i = 1$, $j = 2$, using the matrix A_1. A similar proof using A_2 applies in the other case.

When $p^1_{11} = 0$, the submatrix R contains no 1's and R^2 is a zero matrix. According to statement (a) of Lemma 2.1, SS' then has entries $n_1 - 1$ on the main diagonal and entries $p^2_{11} - 1$ elsewhere. S is thus an $n_1 \times n_2$ matrix with uniform row totals $p^1_{12} = n_1 - 1$, uniform column totals equal to the row totals p^2_{11} of S', and uniform row inner products $p^2_{11} - 1$, identifying it as the incidence matrix of the BIB design described in the Theorem. The number of treatments which a given block of the design has in common with another block is equal to the inner product of the two corresponding columns of S, which in turn is equal to an off-diagonal entry in the given row of $S'S$, and is $\leq$ the entry in the same position of $S'S + T^2$. The number of 1's in a row of T is equal to p^2_{12} and by statement (b) of Lemma 2.1, an equal number of entries in the same row of $S'S + T^2$ are equal to $p^1_{11} = 0$. Thus the given row of $S'S$ must have at least p^2_{12} entries equal to 0, proving the final statement of the theorem.

There are nine sets of parameters in Table II with $p^1_{11} = 0$, five of which belong to constructed association schemes. The others are schemes #15, 34, 39, 50. None of the balanced designs specified by the theorem are known to be impossible; the first three are known designs and the other has not been studied. Existence of the BIB design does not imply the existence of the association scheme, though it may give useful information about the structure of the scheme if it does exist. Whether or not the BIB design exists, the condition on blocks may be impossible, a fact which will now be used to show the impossibility of schemes 15 and 50.

Let $s_{\mu\nu}$ denote the number of treatments which blocks μ and ν of an incomplete block design have in common. Where N is the incidence matrix of the design, $s_{\mu\nu}$ will be the value of the element in the μ, ν position of the $b \times b$ matrix $N'N$. Let $f(n)$ denote the number of blocks of the design which have precisely n treatments in common with a chosen block, say the first. $f(n)$ may be interpreted as the number of indices ν for which $s_{1\nu} = n$, or as the number of occurrences of the entry n in the first row of $N'N$ (disregarding the entry in the 1,1 position). In the case of a BIB design, Hussain [24] proves the following identity in the integers x and y.

$$(2.40) \quad xy(b-1)-k(x+y-1)(r-1)+k(k-1)(\lambda-1) = \sum_{n=0}^{k} (x-n)(y-n)f(n).$$

Setting $x = y = 0$ we obtain

$$(2.41) \quad k(r-1)+k(k-1)(\lambda-1) = \sum_{n=0}^{k} n^2 f(n).$$

If we include the diagonal element of $N'N$,
[illegible]
[illegible]
$= k[r$ [illegible]

Setting $x = 0$, $y = 1$ we obtain

$$k(k-1)(\lambda - 1) = \sum_{n=0}^{k} (n^2 - n)\, f(n),$$

leading to

$$(2.42) \quad k(r-1) = \sum_{n=0}^{k} n\, f(n) .$$

Statements (2.41) and (2.42) give expressions for the sum and the sum of squares of the $b - 1$ off-diagonal entries of a row of $N'N$.

These results, valid for all BIB designs, will be applied to the particular designs introduced in Theorem 2.6. For these designs, at least p_{12}^2 of the $b - 1$ entries are equal to 0. The remaining $b - 1 - p_{12}^2$ entries are integers n satisfying $0 \leq n \leq k$, with sum and sum of squares still given by the left members of (2.41) and (2.42). In some cases it may be impossible to find such a set of integers. This will be demonstrated in the cases of schemes #15 and 50 by computing the variances of the proposed sets of integers. The pertinent parameter values are these.

#	$b = n_2$	$k = p_{11}^2$	$r = n_1 - 1$	$\lambda = p_{11}^2 - 1$	p_{12}^2
15	18	4	8	3	5
50	42	10	20	9	11

Using (2.41) and (2.42) these lead to the following values.

#	Number of integers	Sum	Sum of squares	Variance
15	12	28	52	$\frac{12 \cdot 52 - 28^2}{144} = \frac{-160}{144}$
50	30	190	910	$\frac{30 \cdot 910 - 190^2}{900} = \frac{-880}{900}$

Since negative values of variance are impossible, no such sets of integers can exist. This proves the impossibility of schemes 15 and 50.

THEOREM 2.7. A necessary condition for the existence of an association scheme for a PBIB design with two associate classes and $p^i_{ii} = 1$ is the existence of a PBIB design of GD type with parameters $v = n_i$, $r = n_i - 2$, $k = p^j_{ii}$, $b = n_j$, $\lambda_1 = 0$, $\lambda_2 = p^j_{ii} - 1$, based on an association scheme with parameters $n^*_1 = 1$, $n^*_2 = n_i - 2$, $P_1 = \begin{bmatrix} 0 & 0 \\ 0 & n_i - 2 \end{bmatrix}$,

$P_2 = \begin{bmatrix} 0 & 1 \\ 1 & n_i - 4 \end{bmatrix}$, where i, j are equal to 1 and 2 in some order and starred quantities refer to the GD design. Moreover, given any block of the GD design, there exist at least p^j_{ij} blocks which have at most one treatment in common with the given block.

PROOF: The proof will be carried out for $i = 1$ and $j = 2$, using the matrix A_1 . A similar proof using A_2 applies in the other case.

If $p_{11}^1 = 1$, then in (2.38) R has a single 1 in each row and column and is a symmetric permutation matrix. R^2 is equal to I , the identity matrix. It is easily verified that R has the necessary properties for the incidence matrix of first associates in the GD scheme specified in the theorem. By Lemma 2.1, SS' has diagonal entries equal to $n_1 - 2$, entries equal to $p_{11}^1 - 1 = 0$ in the off-diagonal positions that are occupied by 1's in block R , and entries equal to $p_{11}^2 - 1$ elsewhere. All the requirements are now satisfied for S to be the incidence matrix of the GD design specified in the theorem. The number of treatments which a given block of the design has in common with another block is equal to the inner product of the two corresponding columns of S, which in turn is equal to an off-diagonal entry in the given row of S'S, and is $\leq$ the entry in the same position of $S'S + T^2$. The number of 1's in a row of T is equal to p_{12}^2 and by statement (b) of Lemma 2.1, an equal number of entries in the same row of $S'S + T^2$ are equal to $p_{11}^1 = 1$. Thus the given row of S'S must have at least p_{12}^2 entries ≤ 1, proving the final statement of the theorem.

Seven schemes of Table II have $p_{11}^1 = 1$, including schemes #41, 45 and 90 which are unknown. The GD designs to which these lead do not seem to have been investigated and will not be taken up here. It is therefore not clear whether Theorem 2.7 can be used to prove the impossibility of any of these schemes.

A remark which will be used in the proof of the next theorem will now be stated as a lemma.

LEMMA 2.2. Sufficient conditions for a $v \times v$ matrix A_1 of 0's and 1's to be the incidence matrix for first associates in a PBIB design with two associate classes are

(a) A_1 is symmetric,

(b) the diagonal elements of A_1 are 0's,

(c) $A_1{}^2 = n_1 I + p^1_{11} A_1 + p^2_{11}(U-I-A_1)$, where I is the identity matrix, U is a matrix all of whose elements are 1's, and n_1, p^1_{11}, p^2_{11} are non-negative integers.

PROOF: Define $A_2 = U - I - A_1$. Then A_1 and A_2 are symmetric incidence matrices whose sum is the matrix with 0's on the main diagonal and 1's elsewhere. By Theorem 1.2, they are the association matrices of a PBIB design with two associate classes if the products A_1^2, A_1A_2, A_2A_1, A_2^2 have the form of (1.16), where the constant coefficients n_i and p^i_{jk} are non-negative integers. By hypothesis this is true for A_1^2. It is easy to compute the remaining products, but not necessary for this proof. The equality of the diagonal elements of A_1^2 implies that A_1 has equal row and column totals n_1, implying

$$A_1U = UA_1 = n_1U . \quad \text{Also } U^2 = vU .$$

Each of the products A_iA_j reduces to a linear combination of I, A_1, A_2, and $U = I + A_1 + A_2$, with constant coefficients. Since the elements in any product of incidence matrices must be non-negative integers, the coefficients are of this form and the proof is complete. The values of the coefficients are easily computed by (2.2) and (2.4).

THEOREM 2.8. The existence of an association scheme with two associate classes and parameters v, n_i, p^i_{jk} satisfying the condition given in (a) or (b) below is equivalent to the existence of the BIB design with v treatments described in (a) or (b) respectively. i and j are equal to 1 and 2 in some order.

(a) Condition: $p^i_{ii} = p^j_{ii}$.

BIB design: $v = b$; $r = k = n_i$; $\lambda = p^i_{ii} (= p^j_{ii})$; the incidence matrix N is symmetric with 0's on the main diagonal.

(b) Condition: $p^i_{ii} + 2 = p^j_{ii}$.

BIB design: $v = b$; $r = k = n_i + 1$; $\lambda = p^i_{ii} + 2 (= p^j_{ii})$; the incidence matrix N is symmetric with 1's on the main diagonal.

PROOF: The proof will be carried out for the case $i = 1$ and $j = 2$. The other case is similar.

Case (a) The treatments in block θ of the design will be taken as the first associates of treatment θ in the association scheme. Then $N = A_1$, and by (1.18)

$$NN' = A_1 A_1' = A_1^2 = n_1 I + p^1_{11} A_1 + p^2_{11} A_2 .$$

Defining r and λ as in (a),

$$NN' = rI + \lambda(A_1 + A_2) = rI + \lambda(U - I) .$$

Thus N is a $v \times v$ incidence matrix with uniform row and column totals r and uniform row inner products λ , identifying it as the incidence matrix of the BIB design described in (a). Conversely, let N be the incidence matrix of such a design. Defining $A_1 = N$, the conditions (a) and (b) of Lemma 2.2 are satisfied immediately and condition (c) follows from the expression for NN' which holds for all BIB designs.

$$NN' = r\,I + \lambda(U - I) = r\,I + \lambda N + \lambda(U - I - N) .$$

Case (b). The treatments in block θ of the design are taken as treatment θ and its first associates. Then $N = A_1 + I$ and

$$NN' = (A_1 + I)(A_1 + I)' = (A_1 + I)^2 = A_1^2 + 2A_1 + I$$

$$= (n_1 + 1)\,I + (p_{11}^1 + 2)\,A_1 + p_{11}^2\,A_2$$

Defining r and λ as in (b),

$$NN' = r\,I + \lambda(A_1 + A_2) = r\,I + \lambda(U - I) .$$

Thus N is the incidence matrix of the design described in (b). Conversely, let N be the incidence matrix of such a design and define $A_1 = N - I$. Again conditions (a) and (b) of Lemma 2.2 are satisfied and we have

$$A_1^2 = (N - I)^2 = N^2 - 2N + I = NN' - 2N + I = r\,I + \lambda(U - I) - 2N + I$$

$$= (r - 1)\,I + (\lambda - 2)(N - I) + \lambda(U - N) .$$

Therefore by Lemma 2.2, A_1 so defined leads to the required association scheme. This completes the proof of Theorem 2.8.

Parts (a) and (b) of Theorem 2.8 are not independent. If either of them applies to the matrix A_1 of an association scheme the other applies to A_2. The two BIB designs will be complementary, a given block of one containing exactly the treatments not occurring in the corresponding block of the other. The conditions $p^1_{11} = p^2_{11}$ and $p^2_{22} + 2 = p^1_{22}$ are easily shown to be equivalent, either by direct application of (2.4) or by the device of applying one part of Theorem 2.8, taking the complement of the resulting BIB design, then applying the converse of the other part of the theorem.

The conditions of Theorem 2.8 are satisfied by 18 of the sets of parameters listed in Table II, of which 11 belong to known association schemes. There remain schemes #22, 39, 84, 85, 92, 100, 101. These lead to 5 distinct BIB designs, all of which have $r > 10$ and have not been studied so far. Scheme #22 is equivalent to a design with $v = b = 36$, $r = k = 15$, $\lambda = 6$, N symmetric with 1's on the main diagonal. A design with these parameter values is constructible from the known scheme #23, but with an incidence matrix N having 0's on the main diagonal. These designs all fall within the class of "symmetric" BIB designs which have the property that $v = b$; symmetric BIB designs have been investigated more thoroughly than any others. However, none of the known necessary conditions exclude any of the designs in question. In particular, some deep conditions due to Shrikhande [30] are satisfied automatically whenever $r - \lambda$ is a perfect square, which it is for all of these designs.[1] Therefore Theorem 2.8 does not furnish conclusive information about any unknown association schemes.

1. We remark without proof that the value of $r - \lambda$ is a perfect square for all the BIB designs specified by Theorem 2.8. This is a fairly direct result of the conditions of the theorem and the expressions given in Theorem 2.1.

Theorems 2.2 to 2.8 prove the impossibility of four association schemes of Table II and may provide the basis for other such proofs. They do not represent an exhaustive list of theorems on the structure of association schemes, but they show that such theorems may be proved rather easily, and illustrate some methods of proof. Most of them make use of algebraic properties of the expressions for parameter values of the schemes, or of properties of the association incidence matrices. It is not illustrated here but deserves to be mentioned that empirical attempts to construct an association scheme may lead quickly to a constructed scheme or to a proof that the scheme is impossible. This method requires too much enumeration to be practicable for most schemes with more than 20 treatments, but there are exceptions. Some empirical proofs of impossibility of designs will be mentioned in Section 2.3 , and two association schemes are constructed in Section 3.3 by methods which are largely empirical.

2.3 Enumeration of Possible Designs for Particular Association Schemes.

If a balanced or partially balanced design is to be used in an experiment, the first parameters to be specified by the experimenter are likely to be v and k, which are determined by the number of treatments and the variability of the experimental material. From the designs available for the particular v and k, he will try to choose one for which the number of replications r is large enough to provide the precision desired but not too large to be economically feasible. This will determine the value of b and will leave little or no choice in the values of the other parameters.

A somewhat different procedure is used for our purpose of enumerating possible designs. It has been convenient to classify designs first by association scheme, so that the first parameters specified are v, n_i, p^i_{jk}, leaving the parameters b, r, k, λ_1 and λ_2. Since these five parameters must satisfy relations (2.1) and (2.3), at most three of them may be chosen independently. The requirement that all be non-negative integers is also a considerable restriction. The existence of any design implies the existence of an infinite class of other designs obtained by using each block r_1 times, $r_1 = 2, 3, \ldots$ The parameters v and k will be unchanged for designs obtained in this way, while the parameters r, b, λ_1 and λ_2 will be multiplied by r_1. Only a finite number of these will be useful to experimenters, since there are practical limits to the amount of experimental material that can be used. Fisher and Yates [21] enumerated only designs for which $r \leq 10$, and other writers have followed their example. Extremely large block sizes are likely to defeat the purpose of having homogeneous experimental conditions within blocks, and some limitation on k is also desirable. It is a property of balanced designs that $k \leq r$, so that no qualification was necessary for Fisher and Yates. k is somewhat larger than r in some PBIB designs, and Bose, Clatworthy and Shrikhande [6] enumerate only designs for which $k \leq 10$ also. The same restrictions will be adopted here, admitting only a finite number of designs for a given association scheme.

A fairly efficient enumeration of the possible designs for an association scheme may be begun by choosing a pair of values for λ_1 and λ_2, then computing the quantity $n_1 \lambda_1 + n_2 \lambda_2$. By (2.3), this is equal to $r(k-1)$ and by factoring it in every possible way as the product of two integers, possible pairs of values for r and k may be obtained. By (2.1), the fraction $^{vr}/_k$ is equal to b and must be integer valued. This, along with upper bounds on r and k, will eliminate some sets of values. Some additional restrictions depend on the characteristic roots of the matrix NN', which have been mentioned in Sections 1.3 and 2.2. In the notation of Connor and Clatworthy [17],

rk is a root with multiplicity 1,

$r - z_1$ is a root with multiplicity α_1,

$r - z_2$ is a root with multiplicity α_2,

where the α_i may be obtained from the parameters of the association scheme and the z_i depend in addition on λ_1 and λ_2. Since NN' is the product of a real matrix by its transpose, it is positive semi-definite, meaning that each of its roots must be non-negative. This gives the results

$$r - z_i \geq 0 \ , \ \text{or}$$

$$(2.43) \qquad r \geq z_i \ , \quad i = 1 \ , \ 2 \ .$$

If both of the multiple roots are positive, the $v \times v$ matrix NN' is non-singular and has rank v, meaning that the $v \times b$ matrix N has rank at least v, which is impossible if $b < v$. Therefore in this case $b \geq v$. This is identical with Fisher's inequality for balanced designs and is equivalent to the following statement.

$$(2.44) \qquad \text{If} \quad r > z_i \ , \quad i = 1 \ \text{and} \ 2 \ , \quad \text{then} \quad r \geq k \ .$$

If one of the multiple roots $r-z_i$ is equal to 0, the rank of NN' is $v-\alpha_i$, meaning that N has rank at least $v-\alpha_i$, which is impossible if $b < v-\alpha_i$. This leads to the following statement.

$$(2.45) \qquad \text{If} \quad r = z_i \ , \quad i = 1 \ \text{or} \ 2 \ , \quad \text{then} \quad b \geq v - \alpha_i \ .$$

The situation that both z_1 and z_2 are equal to r does not arise, since it can be shown that any design for which $z_1 = z_2$ will be a balanced design.

Attention will now be restricted to association schemes of the L_g and L_g^* series. The following expressions for parameter values, which use the notation of (2.12), apply to both series. For the L_g schemes, g, f and n are all positive integers; for the L_g^*, they are all negative integers.

$$
(2.46) \qquad \begin{aligned} n &= g + f - 1 \, , \\ n_1 &= g(n - 1) \, , \\ n_2 &= f(n - 1) \, . \end{aligned}
$$

For the L_g schemes, in the same notation,

$$
(2.47) \qquad \begin{aligned} \alpha_1 &= g(n - 1) \, , \\ \alpha_2 &= f(n - 1) \, , \\ z_1 &= (1 - f)\lambda_1 + f\lambda_2 \, , \\ z_2 &= g\lambda_1 + (1 - g)\lambda_2 \, . \end{aligned}
$$

The expressions of (2.46) apply to schemes of both series, but for reasons which will be stated in Section 3.1, the designation of the multiple roots is reversed for $L_g{*}$ schemes, giving the following expressions instead of (2.47).

$$
(2.48) \qquad \begin{aligned} \alpha_1 &= f(n - 1) \, , \\ \alpha_2 &= g(n - 1) \, , \\ z_1 &= g\lambda_1 + (1 - g)\lambda_2 \, , \\ z_2 &= (1 - f)\lambda_1 + f\lambda_2 \, . \end{aligned}
$$

It will be noticed that for either series, the multiplicities α_i of the characteristic roots of NN' are equal in some order to the numbers n_i of treatments in the associate classes. This relation holds only for certain classes of association schemes and will be discussed in Section 3.1.

The work of enumerating design parameters is shortened by some preliminary restrictions placed on λ_1 and λ_2, which will be described first for the L_g case. Taking $r \leq 10$ and $k \leq 10$ implies $r(k-1) \leq 90$. Using (2.43) and (2.47),

$$(1 - f)\lambda_1 + f\lambda_2 \leq 10 ,$$
$$g\lambda_1 + (1 - g)\lambda_2 \leq 10 ,$$

and from (2.3),

$$n_1\lambda_1 + n_2\lambda_2 \leq 90 .$$

Solution for λ_1 leads to the following inequalities, which define the quantities m, M and M' for L_g schemes.

$$\lambda_1 \geq \frac{f}{f-1}\lambda_2 - \frac{10}{f-1} = m , \tag{2.49}$$

$$\lambda_1 \leq \frac{10}{g} + \frac{g-1}{g}\lambda_2 = M , \tag{2.50}$$

$$\lambda_1 \leq \frac{90 - n_2\lambda_2}{n_1} = M' . \tag{2.51}$$

In the L_g^* case, g and f are negative, leading to the following inequalities and different definitions for m and M. Inequality (2.51) and the definition of M' hold without change.

$$\lambda_1 \geq \frac{10}{g} + \frac{g-1}{g}\lambda_2 = m , \tag{2.52}$$

$$\lambda_1 \leq \frac{f}{f-1}\lambda_2 - \frac{10}{f-1} = M . \tag{2.53}$$

For a particular set of association scheme parameters g, f, n_1 and

n_2 , the lower bound m and the upper bounds M and M' for λ_1 are quickly listed for each non-negative value of λ_2 .

The enumeration which has been outlined in this section is carried out in Table III of the appendix. The section of the table for each association scheme is preceded by a list of the values of g , f , n_1 and n_2 and the expressions for m , M , M' , z_1 , z_2 , and $r(k-1)$. In the table, values of λ_2 are listed, followed by the value of m if it is positive and the value of the smaller of M and M' . The possible values of λ_1 are then listed. The value $\lambda_1 = \lambda_2$ is omitted, since it leads only to balanced designs. Also if $n_1 = n_2$, values $\lambda_1 > \lambda_2$ are omitted since they lead to designs which can be obtained from designs with $\lambda_1 < \lambda_2$ by interchanging the designation of first and second associates. For each pair λ_1 , λ_2 , the quantities z_1 , z_2 and $r(k-1)$ are entered in the next columns of the table for use in computing values of r and k . Only values $r \leq 10$ and $k \leq 10$ consistent with (2.43) and (2.44) are listed. The value of b is then computed and entered in the table if it is integral. Finally, in case $r = z_i$, (2.45) is applied, eliminating a few more sets of parameters. Table III is intended as an illustration of the computations and is presented only for a representative sample of the association schemes.

Table IV is a list of those parameter values which satisfy all the conditions applied in Table III. The designs for each association scheme are listed together, preceded by a list of the scheme parameters for reference. Design parameters are identified by the numbers given to the scheme in Table II, and by a serial numbering of the designs for each scheme. All known schemes of the L_g and L_g^* series are included, and values of v, r, k, b, λ_1, λ_2, z_1 and z_2 are listed. Designs which are known to have been constructed or have been proved impossible are marked by the letter C or X respectively, followed by an explanatory remark or reference.

Several methods which are frequently of use in constructing designs will now be listed in the form of theorems. These are presented here for easy reference and no claim is made that they are new, although the author is not aware of any publications which include theorems 2.12 to 2.14.

THEOREM 2.9. A PBIB design with $k = 2$ treatments per block may be formed from any association scheme by taking as the blocks all pairs of i^{th} associates. The parameter values will be v, $r = n_i$, $k = 2$, $b = \frac{1}{2}vn_i$, $\lambda_i = 1$, $\lambda_j = 0$. i and j represent 1 and 2 in some order.

PROOF: Since each pair of i^{th} associates occurs together in a block exactly once and since no treatments which are not i^{th} associates occur together in any block, the design satisfied the requirements specified.

THEOREM 2.10. In a Latin square type association scheme with $v = n^2$ treatments and g constraints, a PBIB design with parameter values

$$v = n^2, \quad r = g, \quad k = n, \quad b = ng, \quad \lambda_1 = 1, \quad \lambda_2 = 0$$

may be formed by taking as blocks the sets of n treatments occurring in the rows of the g orthogonal squares. If there exists a set of $f = n - g + 1$ additional squares which may be adjoined to form a complete orthogonal set, a PBIB design with parameters

$$v = n^2 , \quad r = f , \quad k = n , \quad b = nf , \quad \lambda_1 = 0 , \quad \lambda_2 = 1$$

may be formed by taking as blocks the sets of n treatments occurring in the rows of the f additional orthogonal squares.

PROOF: These are square lattice designs, whose properties are well known. They are discussed, for example, in Chapter 10 of [14]. By the orthogonality property of the $n \times n$ squares, no pair of treatments occurs together more than once in a row of any of the squares. By definition of the L_g association scheme, the treatments occurring together are precisely those treatments which are first associates in the case of the first design described, or second associates in the second.

All designs of either of these types will be identified in Table IV of the appendix by the word "Lattice".

THEOREM 2.11. Let two PBIB designs based on the same association scheme have the same number k of treatments per block, so that their parameter values may be represented by

$$v , r^* , k , b^* , \lambda_1^* , \lambda_2^*$$

and

$$v , r^{**} , k , b^{**} , \lambda_1^{**} , \lambda_2^{**}$$

respectively. Then a design with parameter values

$$v\ ,\ r = r^{*}+r^{**},\ k\ ,\ b = b^{*}+b^{**},\ \lambda_1 = \lambda_1^{*}+\lambda_1^{**},\ \lambda_2 = \lambda_2^{*}+\lambda_2^{**}$$

may be formed by taking each block of the two original designs as a block of the new design.

PROOF: It is obvious that the set of blocks obtained in this way leads to the values specified for b and r and that the total number of occurrences within blocks of a given pair of treatments is equal to the sum of the numbers of occurrences in the two original designs. Since the two designs have the same association scheme, the number of occurrences of a pair of treatments is $\lambda_i^{*}+\lambda_i^{**}$ when they are i^{th} associates, $i = 1$ or 2.

It is an immediate extension of the theorem that three or more component designs with the same association scheme and the same value of k may be combined in the same way. The designs need not all be distinct. In Table IV, a design which may be formed in this way from other designs for the same association scheme will be identified by the letter R, followed by the serial numbers of the other designs.

THEOREM 2.12. Given any association scheme with 2 associate classes, a PBIB design with the parameter values

$$v = b\ ,\quad r = k = n_i\ ,\quad \lambda_1 = p^i_{ii}\ ,\quad \lambda_2 = p^j_{ii}\ ,$$

where i and j are equal to 1 and 2 in some order, may be formed by taking block θ as the set of i^{th} associates of treatment θ.

PROOF: If the design is formed in this way, its incidence matrix will be identical with the association matrix A_i of i^{th} associates, giving the result

$$NN' = A_iA_i' = A_i^2 = n_iI + p_{ii}^iA_i + p_{ii}^jA_j .$$

This shows that each treatment occurs in n_i blocks and each pair of treatments occurs together in p_{ii}^i or p_{ii}^j blocks, according as the two treatments are first or second associates. Each block contains n_i treatments, and all the requirements for a PBIB design are therefore satisfied.

In Table IV, a design which may be formed by applying this theorem is identified by the statement

$$N = A_i, \ (i = 1 \text{ or } 2).$$

THEOREM 2.13. Given any association scheme with 2 associate classes, a PBIB design with the parameter values

$$v = b, \quad r = k = n_i + 1, \quad \lambda_1 = p_{ii}^i + 2, \quad \lambda_2 = p_{ii}^j ,$$

where i and j are equal to 1 and 2 in some order, may be formed by taking block θ as the set of treatments consisting of treatment θ and its i^{th} associates.

PROOF: If the design is formed in this way, its incidence matrix N will have the form $A_i + I$, giving the result

$$NN' = (A_i + I)(A_i + I)' = (A_i + I)^2 = A_i^2 + 2A_i + I$$

$$= (n_i + 1)I + (p_{ii}^i + 2)A_i + p_{ii}^jA_j .$$

This shows that each treatment occurs in $n_i + 1$ blocks and each pair of treatments occurs together in $p^i_{ii} + 2$ or p^j_{ii} blocks, according as the two treatments are first or second associates. Each block contains $n_i + 1$ treatments, and all the requirements for a PBIB design are therefore satisfied.

In Table IV, a design which may be formed by applying this theorem is identified by the statement

$$N = A_i + I \;, \left(i = 1 \text{ or } 2\right).$$

THEOREM 2.14. In a Latin square type association scheme with $v = n^2$ treatments and g constraints if a balanced incomplete block design with parameter values

$$v^* = n \;, \quad r^* \;, \quad k^* \;, \quad b^* \;, \quad \lambda^* \;,$$

is constructed on each of the sets of n treatments in the rows of the g orthogonal squares, the result is a PBIB design with parameter values

$$v = nv^* = n^2 \;, \; r = gr^* \;, \; k = k^* \;, \; b = gnb^* \;, \; \lambda_1 = \lambda^*, \; \lambda_2 = 0 \;.$$

If there exists a set of $f = n - g + 1$ additional squares which may be adjoined to form a complete orthogonal set, and the same BIBD is constructed on each of the nf rows, the result is a PBIB design with parameter values

$$v = n^2 \;, \; r = fr^* \;, \; k = k^* \;, \; b = fnb^* \;, \; \lambda_1 = 0 \;, \; \lambda_2 = \lambda^* \;.$$

PROOF: The proof will be stated for the first case. The necessary changes in wording for the second case are inserted in parentheses. Since

b^* blocks are constructed from the treatments of each of the n rows of each of the g (or f) squares, the total number of blocks will be gnb^* (or fnb^*) and each block will contain k treatments. Since each treatment occurs in just one row of each square, it will occur r^* times in the design formed from each of the g (or f) squares, leading to the stated value for r. By definition of the association scheme, each pair of first (or second) associates occurs together in just one row of one of the g (or f) squares, so that the number of occurrences within blocks of the PBIB design of the pair of treatments is equal to the number λ^* of blocks of the BIB design in which two treatments occur together. Two treatments which are second (or first) associates do not occur together in any rows of the squares used and will not occur together in the PBIB design, which means that λ_2 (or λ_1) is equal to 0.

THEOREM 2.15. In a Latin square type association scheme with $v = n^2$ treatments and g constraints, if the rows of each of the g orthogonal squares are identified with the treatments of a BIB design with the parameter values

$$v^* = n\ ,\quad r^*\ ,\quad k^*\ ,\quad b^*\ ,\quad \lambda^*\ ,$$

then a PBIB design may be constructed with b^* blocks formed from each of the $n \times n$ squares by replacing the treatments in each block of the balanced design by the sets of n treatments in the corresponding rows of the square. The parameter values of the partially balanced design will be

$$v = nv^* = n^2 \ , \ r = gr^* \ , \ k = nk^* \ , \ b = gb^* \ ,$$
$$\lambda_1 = r^* + (g-1)\,\lambda^* \ , \ \lambda_2 = g\,\lambda^* \ .$$

If there exists a set of $f = n-g+1$ additional squares which may be adjoined to form a complete orthogonal set, and the rows of the f squares are used in the same way with the same BIB design, a PBIB design is obtained with the parameter values

$$v = n^2 \ , \quad r = fr^* \ , \quad k = nk^* \ , \quad b = fb^* \ ,$$
$$\lambda_1 = f\,\lambda^* \ , \quad \lambda_2 = r^* + (f-1)\,\lambda^* \ .$$

PROOF: The proof will be stated for the first case. The necessary changes in wording for the second case are inserted in parentheses. Since b^* blocks are formed from each of the $n \times n$ squares, the total number of blocks is gb^* (or fb^*) . Since each treatment of a block of the balanced design is replaced by n treatments of the partially balanced design, the block size is nk^* . Each row of an $n \times n$ square occurs in r^* blocks and each pair of rows occurs together in λ^* blocks. Since each treatment occurs in just one row of an $n \times n$ square, it occurs in r^* of the blocks formed from each square, for a total of gr^* (or fr^*) occurrences. If two treatments are in the same row of a square, they will occur together in r^* of the blocks formed from that square; if they occur in different rows of a square, they will occur together in λ^* of the blocks formed from that square. First associates occur in the same row of one square and in different rows of the remaining $g-1$ squares, while second associates occur in different rows of all g squares. (In the case of f squares, first associates occur in different rows of all f squares, while second associates occur in the same row of just one square.) The total numbers of occurrences of pairs of treatments are therefore equal to the values given for λ_1 and λ_2.

The method of construction outlined in Theorem 2.15 is a rather direct extension of a construction given by Bose and Connor [7] for group divisible designs, and of a generalization by Zelen [40]. There are other general methods of generating PBIB designs, but the ones just given furnish constructions for most of the known designs of Table IV, which is sufficient for the purpose of this section. Of the remaining known designs, some are tabulated by Bose, Clatworthy and Shrikhande and are identified in Table IV by a reference to [6]. Others that have been constructed by miscellaneous methods are listed in Section A.3 of the Appendix.

It is known for many incomplete block designs and is probably true for many of those listed in Table IV that two or more solutions exist which are distinct under permutation of treatments or blocks. This is certainly the case for those designs which can be constructed from either of two inequivalent association schemes. The question of uniqueness of designs based on the same association scheme will not be taken up in this dissertation.

Proofs of impossibility of designs, which are given for several particular designs in Section A.3 of the Appendix, may involve the question of uniqueness of association schemes. Design #7.3 furnishes a useful example. The design is in the L_2 series with $v = 16$, and when the association scheme is based on a pair of orthogonal 4×4 squares, is easily shown to be impossible. However, the design can be constructed by using a different association scheme with the same parameter values, which will be used as an example in Section 4.1. This shows that different association schemes with the same parameter values may have different properties and that any proof of impossibility of a PBIB design must cover all association schemes with the appropriate parameter values. It will be shown in Section 4.2 that for L_2 designs with $n \neq 4$, the association scheme defined by $n \times n$ squares is unique, so that the squares may be assumed in any discussion of these designs. This is a necessary step in the proof of impossibility of designs such as #20-2, #30-2, and #93-1. On the other hand, design #12-2 in the L_3 series may be shown impossible with an association scheme based on three 5×5 squares, but another example in Section 4.1 will show that the scheme is not unique and the existence of the design remains in doubt.

A singular incomplete block design is one for which the matrix NN' is singular, and for PBIB designs with two associate classes, this means a design for which one of the values z_1 and z_2 is equal to r. It is easy to verify that Lattice designs, designs constructed by the method of Theorem 2.15, and designs formed by replicating a design of either of these types, are singular. These designs all have the property that the blocks may be partitioned into subsets of n treatments which are the sets occurring in the rows of the orthogonal squares. It is conjectured by the author that every singular design based on an association scheme of the L_g series has this property and may be formed in one of the ways described. This would be an extension of results proved by Bose and Connor [7] on the structure of singular group divisible designs. If this conjecture were proved, a necessary condition for the existence of a Latin square type design with the parameter values stated in Theorem 2.15 would be the existence of the BIB design described in the theorem. This would prove the impossibility of designs #7-20 and #12-8 of Table IV, since the BIB designs involved would have fractional values for some of the parameters r , k and b and are obviously impossible.

Table II gives parameter values of 20 schemes in the Latin square series, of which 18 are known and are listed in Table IV. 167 sets of design parameters are listed for these schemes, designs are constructed or indicated for 125, and three are proved impossible by enumeration methods. There remain 39 unknown designs.

Table II gives parameter values of 10 schemes in the negative Latin square series, aside from schemes which are also in the L_g series. Five of these schemes will be constructed in Chapter III and are included in Table IV. 22 sets of design parameters are listed, and designs are constructed for nine. The remaining 13 designs are unknown. The constructed schemes and designs of the L_g^* series are believed to be new.

In all, Table IV gives parameter values of 189 designs, of which 134 are constructed, three are shown to be impossible, and 52 are unknown.

III. NEGATIVE LATIN SQUARE TYPE ASSOCIATION SCHEMES

3.1 Relationships between Latin square and negative Latin square association schemes.

It was pointed out in Section 2.1 that formulas (2.12), developed for Latin square type (L_g) association schemes, give parameter values of a possible new series of association schemes when the arguments n, g, f were given negative integral values. This new series of "negative Latin square" type (L_g^*) schemes will be the principal topic of this chapter. Five of the schemes will be constructed in Sections 3.2 and 3.3, and have already been included in the tables discussed in Chapter II. In the present section it will be shown that the family resemblance in the parameter values is not the only thing the new series has in common with L_g schemes. A property related to the characteristic roots of NN', where N is the incidence matrix of a design, is shown to be shared by both series of association schemes and to come close to characterizing them, holding for only one other class of schemes.

Formulas (2.13) and (2.14), due to Connor and Clatworthy [17], for the multiplicities α_i of the characteristic roots of NN' are easily used to find general expressions for the α_i for any family of designs for which general expressions for the other parameters are available. When the formulas are applied to Latin square designs it is found that the expressions for α_1 and α_2 are identical with those for the parameters

n_1 and n_2, given in (2.9). This is not true for group divisible or triangular designs except in special cases, showing that it does not hold in general. On the other hand, reference to Table II of the Appendix shows that about half of the non-group-divisible schemes with $v \leq 100$, including all L_g and L_g^* schemes, have one of the two following properties.

Property A: $\alpha_1 = n_1$.

Property B: $\alpha_1 = n_2$.

Since $\alpha_1 + \alpha_2 = n_1 + n_2 = v - 1$, property A or B implies that α_2 is equal to n_2 or n_1 respectively. In this section we determine the class of designs which have either property A or property B.

First it will be shown that the two properties are practically identical, and that basically the difference between them is one of notation. n_1 and n_2 denote the numbers of other treatments which are first and second associates respectively of a treatment in the design. The two classes of associates play dual roles in many respects and nothing more than a choice of notation is involved in designating one class as the first. Once the choice is made for a particular design, the values of n_1, n_2, λ_1, λ_2, and the p^i_{jk} are uniquely determined. The designation of α_1 and α_2, however, depends in addition on the designation of the two characteristic roots $r - z_1$ and $r - z_2$ of NN'. These are obtained as the two roots of a quadratic equation whose coefficients are functions of r, λ_1, λ_2, p^1_{12} and p^2_{12}. Solution of the equation leads to

$$(3.1) \qquad z = \frac{(\lambda_1 + \lambda_2) + (\lambda_2 - \lambda_1)(\Upsilon \pm \sqrt{\Delta})}{2},$$

where

$$\Upsilon = p^2_{12} - p^1_{12} \qquad \text{and} \qquad \Delta = (p^2_{12} - p^1_{12})^2 + 2(p^1_{12} + p^2_{12}) + 1$$

This result for the two z_i is given in [17]. The expressions for z_1 and z_2 differ only in the sign of the terms involving Δ, which is a symmetric function of p^1_{12} and p^2_{12} and is thus independent of the designation of associate classes. Connor and Clatworthy denote by z_1 the root obtained by taking the $+$ sign, giving the expressions

$$(3.2) \qquad z_1 = \tfrac{1}{2}\lambda_1(1 - \Upsilon - \sqrt{\Delta}) + \tfrac{1}{2}\lambda_2(1 + \Upsilon + \sqrt{\Delta}),$$

$$(3.3) \qquad z_2 = \tfrac{1}{2}\lambda_1(1 - \Upsilon + \sqrt{\Delta}) + \tfrac{1}{2}\lambda_2(1 + \Upsilon - \sqrt{\Delta}).$$

This amounts to designating the i^{th} characteristic root $r - z_i$ as the one in which the coefficient of λ_i is positive. It needs to be emphasized that this convention is arbitrary and does not identify z_i with the i^{th} associate class. An expression which involves a positive multiple of λ_1 and a negative multiple of λ_2 is not thereby more closely related to one than to the other. While it is convenient to be able to refer to $r - z_1$ and $r - z_2$ without ambiguity, this does not reveal any intrinsic connection between the designation of these two characteristic roots and the designation of the two associate classes, and none should be inferred. One choice for the designation of z_1 and z_2 seems to be as good as another, and it is sensible to stick to the choice already made by Connor and Clatworthy. The values of α_1 and α_2 are then uniquely determined. This is the notation used in Tables I to IV of the Appendix. If the other choice of notation were made for any scheme of Table II which has property B , it would have property A instead.

It is now possible to clear up a discrepancy in the notation which has been used in this dissertation for association schemes of the negative Latin square (L_g^*) series. It was stated at the beginning of this section that schemes of the ordinary Latin square (L_g) series have property A . It is stated in section 2.1 that the expressions for the parameter values of the L_g and L_g^* schemes are identical, which would imply that the L_g^* schemes also have property A . However, the schemes of this series listed in Tables II, III, and IV have property B . The parameter values of the L_g^* schemes are given by the expressions (2.12) used for the L_g schemes provided the parameters n , g , f of those expressions are taken as negative integers. For both classes, $\Delta = n^2$ and $\sqrt{\Delta} = n$; taking n as a negative integer means using the negative square root of Δ in the expressions for z_1 and z_2 . If this is done, these schemes have property A . But this is the opposite of the sign convention agreed on in the previous paragraph and used in the tables, explaining why they appear there with property B . This concludes the discussion of the nature of properties A and B and we now return to the problem of finding the class of designs which have property A or property B .

For group divisible designs the values of n_i and α_i , which are given, for example, in [17] , are as follows.

$$n_1 = n - 1 \ , \quad n_2 = n(m - 1) \ ,$$

$$\alpha_1 = m - 1 \ , \ \alpha_2 = m(n - 1) \ ,$$

where m and n are positive integers. It is easily verified that these designs have property B only if $m = 1$ or $n = 1$, in which cases the design reduces to a balanced design. They have property A only if $m = n$. A group divisible design with $m = n$ is the simplest case L_1 of a Latin square type design.

All partially balanced designs with two associate classes and not of group divisible type have association schemes whose parameter values may be determined by the conditions of one of Theorems 2.0 and 2.1. For all schemes of the class defined by Theorem 2.0, $n_1 = n_2 = \alpha_1 = \alpha_2$, so that both of properties A and B hold. These schemes are defined only for v of the form $v = 4t + 1$; a scheme of the class may be constructed for each such value of v which is a prime or prime power, for example by the method to be described in Section 3.2. No schemes of this class are known at present for other values of v.

We now turn to the schemes specified by Theorem 2.1, in which expressions for the parameter values are given terms of positive integers s and t. Some of these expressions are now repeated for reference.

(2.21) $\quad p^1_{12} = s(t + 1)$,

(2.22) $\quad p^2_{12} = (s + 1)t$,

(2.28) $\quad \alpha_1 = \dfrac{(s + 1)n_2 + sn_1}{\sqrt{\Delta}}$;

using (2.23) ,

$$\alpha_1 = \frac{(s+1)n_2 + sn_1}{s+t+1} .$$

First assume that property A holds. It may be stated

$$(3.4) \qquad \frac{(s+1)n_2 + sn_1}{s+t+1} = n_1 ,$$

leading to

$$n_1(t+1) = n_2(s+1) .$$

This is now multiplied by t , followed by application of (2.22) and (2.5) :

$$n_1 \, t(t+1) = n_2(s+1)t = n_2 \, p_{12}^2 = n_1 \, p_{22}^1 .$$

Therefore,

$$(3.5) \qquad p_{22}^1 = t(t+1) .$$

Using (2.24) ,

$$(3.6) \qquad p_{11}^2 = s(s+1) .$$

The remaining association scheme parameters are now easily determined. In particular,

$$(3.7) \qquad n_1 = p_{12}^2 + p_{11}^2 = (s+1)t + s(s+1) = (s+1)(s+t) ,$$

$$(3.8) \qquad n_2 = p_{12}^1 + p_{22}^1 = s(t+1) + t(t+1) = (t+1)(s+t) ,$$

$$(3.9) \qquad v = n_1 + n_2 + 1 = (s+t+1)^2 = \triangle .$$

The notation will now be changed by defining new parameters n, g, f as follows.

$$s+1=g \ , \quad s=g-1 \ ,$$

$$t+1=f \ , \quad t=f-1 \ ,$$

$$s+t+1=g+f-1=n \ .$$

In terms of these parameters we have, for example,

$$v=n^2 \ ,$$

$$n_1=g(n-1) \ ,$$

$$p^1_{12}=f(g-1) \ .$$

These and the other expressions in n, g and f are identical with those given in (2.12) for schemes in the Latin square (L_g) series. Therefore every scheme specified by Theorem 2.1 which has property A must have the parameter values of the L_g series.

Next assume that property B holds. It may be stated

$$\frac{(s+1)n_2+sn_1}{s+t+1}=n_2 \ , \tag{3.10}$$

leading to

$$n_1 s=n_2 t \ .$$

This is now multiplied by $(s+1)$, followed by application of (2.22) and (2.5) :

$$n_1\, s(s+1) = n_2\,(s+1)t = n_2\, p^2_{12} = n_1\, p^1_{22} \ .$$

Therefore,

$$(3.11) \qquad p^1_{22} = s(s+1) \ .$$

Computing other parameter values as in the case of Property A ,

$$(3.12) \qquad p^2_{11} = t(t+1) \ ,$$

$$(3.13) \qquad n_1 = p^2_{11} + p^2_{12} = t(t+1) + t(s+1) = t(s+t+2) \ ,$$

$$(3.14) \qquad n_2 = p^1_{22} + p^1_{12} = s(s+1) + s(t+1) = s(s+t+2) \ ,$$

$$(3.15) \qquad v = n_1 + n_2 + 1 = (s+t+1)^2 = \Delta \ .$$

In order to make use of (2.12) , the notation will now be based on negative integers n^* , g^* , f^* , defined as follows.

$$t = -g^* \ , \quad t+1 = -g^* + 1 \ ,$$

$$s = -f^* \ , \quad s+1 = -f^* + 1 \ ,$$

$$s + t + 1 = -g^* - f^* - 1 = -n^* \ .$$

In terms of these parameters we have, for example,

$$v = (-n^*)^2 = (n^*)^2 ,$$
$$n_1 = -g^*(-n^* + 1) = g^*(n^* - 1) ,$$
$$p_{12}^1 = -f^*(-g^* + 1) = f^*(g^* - 1) .$$

These and the other expressions in n^* , g^* , f^* are of the form of (2.12), identifying the present series of schemes as the negative Latin square series. Therefore every scheme specified by Theorem 2.1 which has property B must be in the Lg^* series.

In some work with Lg^* schemes it is convenient to have expressions for the parameters as functions of positive integers. The letters n , g , f will still be used, but with the following relation to the parameters of Theorem 2.1.

$$g = t , \quad f = s , \qquad (3.16)$$
$$n = s + t + 1 .$$

In this notation the expressions for the parameter values are the following.

$$(3.17) \quad \begin{aligned} n &= g + f + 1 , \\ v &= n^2 , \\ n_1 &= g(n + 1) , \\ n_2 &= f(n + 1) , \end{aligned} \qquad P_1 = \begin{bmatrix} (g+1)^2 - f + 2 & f(g+1) \\ f(g+1) & f(f+1) \end{bmatrix} , \quad P_2 = \begin{bmatrix} g(g+1) & g(f+1) \\ g(f+1) & (f+1)^2 - g + 2 \end{bmatrix}$$

The classes of association schemes which have been characterized by properties A and B are not disjoint. When n is odd, say $n = 2a + 1$, the design with parameter values

$$(3.18) \quad \begin{aligned} v &= n^2 = 4a^2 + 4a + 1 , \\ n_1 &= n_2 = 2a^2 + 2a , \end{aligned} \qquad P_1 = \begin{bmatrix} a^2 + a-1 & a^2 + a \\ a^2 + a & a^2 + a \end{bmatrix} , \quad P_2 = \begin{bmatrix} a^2 + a & a^2 + a \\ a^2 + a & a^2 + a-1 \end{bmatrix}$$

is in the class defined by Theorem 2.0, with $t = a^2 + a$. It is also an Lg scheme with $g = f = a + 1$, and an Lg^* scheme with $g = f = a$. There are no other duplications. There are clearly no other Lg or Lg^* schemes which have the property $n_1 = n_2$ of Theorem 2.0 , and for a scheme with $v = n^2$ treatments to be simultaneously an Lg and an Lg^* scheme, it is necessary that n_1 be simultaneously a multiple of $n-1$ and $n + 1$. The only possible value less than $n^2 - 1$ is $\frac{1}{2}(n^2-1)$, with n odd.

The results that have been proved in this section will now be stated as a theorem.

THEOREM 3.1. Let N be the incidence matrix of a partially balanced incomplete block design with two associate classes. In order for the multiplicities α_1 and α_2 of the multiple characteristic roots of NN' to be equal in some order to the numbers n_1 and n_2 of treatments in the associate classes, it is necessary and sufficient that the design be in one of the following classes.

(i) The class specified by Theorem 2.0 ;

(ii) The Lg series, Latin square type designs with g constraints, $g \geq 1$, or other schemes with the same parameter values;

(iii) The Lg^* series, negative Latin square type designs, introduced in Section 2.1, with parameter values given by (3.17).

For v an odd square there is one possible association scheme with $n_1 = n_2$ which falls in all three of these classes; otherwise they have no schemes in common.

The specification of Lg^* schemes in terms of the negative integers n^*, g^*, f^*, is not very helpful in suggesting possible ways of constructing the schemes. The parameters g and n in the Lg series are related to a set of g orthogonal $n \times n$ squares, and there seems to be no analog to this for negative integers g^* and n^*. Expressions (3.17) in terms of positive arguments are a little more promising, at least in any case in which a complete set of orthogonal squares exists. This is more easily described in terms of the finite Euclidean plane geometry which may be constructed from such a set of squares. This geometry has n^2 points, any two of which determine a line; there are n points on each line and $n + 1$ lines on each point. For an association scheme the $v = n^2$ treatments are identified with the points of the geometry. If the scheme is of Lg type, each treatment has $n_1 = g(n-1)$ first associates, which for a given point may be taken as the $n-1$ remaining points on each of g suitably chosen lines through the point. This is discussed in further detail in the following section. If the scheme is of Lg^* type, the value $g(n+1)$ for the number n_1 of first associates suggests that the first associates of a particular point might be g suitably chosen points on each of the $n + 1$ lines through the given point. It appears that it would be a difficult combinatorial problem to select these points in a way that would satisfy all the requirements of partial balance, although two schemes constructed by another method in the following section have precisely this geometrical interpretation.[1]

It should be remarked that not all L_g schemes are associates with finite geometries. They may be constructed from sets of g orthogonal squares which cannot be extended to a complete set of $n+1$, and there are examples of association schemes which have the parameter values of the L_g series but which correspond to no set of g orthogonal squares. Some examples of this kind will be given in Section 4.1, while an example appeared in Section 2.1 of a 4×4 Latin square not belonging to a complete orthogonal set, which is equivalent to a set of $g = 3$ orthogonal squares which cannot be extended to a set of $n+1$. Such squares are known for many values of n and presumably exist for all values of $n > 3$. By analogy with this, there is no reason to expect all schemes of the L_g^* series to be related to complete sets of orthogonal squares or to finite geometries. On the other hand, there is at least the possibility of such a relation for each of the five schemes of the series which are known at present. The four which are constructed in the next section are all based on finite fields of order n^2, and in every case where such a field exists, the geometry and set of squares also exists. The one constructed in Section 3.3 is for $v = 100$ treatments, and while no field of this order exists, it has never been proved that the geometry and orthogonal squares do not exist.

1. These are schemes #6 and 51, for 16 and 64 treatments respectively.

3.2 Construction of Negative Latin Square Type Association Schemes by a Method Based on Finite Fields.

A method is developed in this section for the construction of an infinite class of association schemes with two or more associate classes. The method is applied to the construction of four schemes of the negative Latin square, or Lg^* , series.

The method is applicable when the number of treatments is equal to a power of a prime, $v = p^q$, so that there exists a finite field with v elements, denoted by the standard notation $GF(p^q)$. The treatments will be identified with the field elements or marks in any convenient order. It is well known that the multiplicative group of non-zero marks of the field is cyclic; denote a generator of this group by z . The marks of the field may be represented by 0 , 1 , z , z^2 , ... , z^{p^q-2} , where $z^{p^q-1} = 1$. Each non-zero mark x may be represented uniquely in the form $x = z^k$, $0 \leq k \leq p^q-2$. The integer k so defined is usually called the index of x relative to the base z and will be denoted by the symbol $\text{ind } x$, but the term "exponent" will be used in discussion, in order to reserve the term "index" for a different use.

Express the order of the multiplicative group as the product of two integers c and d ,

$$p^q - 1 = cd ,$$

and define a field mark e by $e = z^c$. Then e is the generator of a subgroup of order d , with elements

$$e^0 = 1 \ , \ e \ , \ e^2 \ , \ldots , \ e^{d-1} \ .$$

Since the group is cyclic, the subgroup of order d is unique. This subgroup and its cosets provide a partition of the non-zero field marks into c sets, each containing d marks. The j^{th} set contains the marks

$$z^j \ , \ ez^j \ , \ e^2z^j \ , \ldots , \ e^{d-1}z^j \ ,$$

and j may have the values $0 \ , \ 1 \ , \ldots , \ c-1$. It will be necessary to impose the condition that a coset contain with each element its additive inverse. It is easily seen that an equivalent condition is that the subgroup have this property, and that this reduces to the requirement that this (multiplicative) subgroup contain the additive inverse of the element 1 , denoted by -1 . If the prime p is equal to 2 , 1 is self-inverse (as is every other mark) and the condition is satisfied for every subgroup. If p is odd, 1 and -1 are the two solutions of the equation $x^2-1 = 0$. The corresponding exponents are the solutions of

$$2 \text{ ind } x \equiv 0 \bmod p^g-1 \ .$$

The solution corresponding to -1 is $\text{ind}(-1) = \frac{p^q-1}{2} = \frac{cd}{2}$, meaning

$$-1 = z^{cd/2} \ .$$

A necessary and sufficient condition that this be in the subgroup is that it is an integral power of the generator $e = z^c$, or equivalently, that d is an even integer. Accordingly it will be required that the chosen subgroup be of even order if the order of the field is odd.

We now define an association relation by saying that a mark θ_2 is a j^{th} associate of θ_1 if their difference $\theta_2 - \theta_1$ is a mark of the j^{th} coset. In this case, $\theta_1 - \theta_2$ is in the same coset by the condition just imposed, so that the association relation is symmetric. In order to show that it satisfies the definition of the association relation in a PBIB design it remains to show that condition iii(c) of the definition is satisfied. Let θ_1 and θ_2 be i^{th} associates, so that for some t_o, $\theta_2-\theta_1 = e^{t_o}z^i$. The j^{th} associates of θ_1 are of the form $\theta_1 + e^{t_1}z^j$, $t_1 = 0, 1, \ldots, d-1$. The k^{th} associates of θ_2 are of the form $\theta_2 + e^{t_2}z^k$, $t_2 = 0, 1, \ldots, d-1$. It is necessary to show that the number p^i_{jk} of marks in the intersection of these two sets is independent of the particular pair of i^{th} associates chosen as θ_1 and θ_2. This number is equal to the number of pairs t_1, t_2 for which

$$\theta_1 + e^{t_1}z^j = \theta_2 + e^{t_2}z^k .$$

This reduces to

$$\theta_2 - \theta_1 = e^{t_1}z^k - e^{t_2}z^j ,$$

$$(3.19) \qquad e^{t_o}z^i = e^{t_1}z^k - e^{t_2}z^j ,$$

$$z^i = e^{t_1-t_o}z^j - e^{t_2-t_o}z^k .$$

Now as t_1 (or t_2), runs over all the values $0, 1, \ldots, d-1$ modulo d, t_1-t_o (or t_2-t_o) runs over the same values, so the number of solutions p^i_{jk} is the same as the number of solutions of the equation

$$z^i = e^{t_1} z^j - e^{t_2} z^k .$$

This number p^i_{jk} is independent of the pair of i^{th} associates chosen as θ_1 and θ_2. Moreover, e^{t_2} and $-e^{t_2}$ run over the same set of values, so we may replace the previous equation by

$$(3.20) \qquad z^i = e^{t_1} z^j + e^{t_2} z^k .$$

Since this equation is symmetric in j and k, we have $p^i_{jk} = p^i_{kj}$. This is the last condition necessary for the association relation for a PBIB design with c associate classes. The classes are of equal size, $n_i = d$, $i = 0, 1, \ldots, c-1$.

The standard relations (1.7) hold and reduce in this case to

$$(3.21) \qquad p^i_{jk} = p^j_{ik} = p^k_{ij} , \quad i, j, k = 0, 1, \ldots, c-1 .$$

Multiplying (3.20) by z gives the following equivalent equation, which must have the same number of solutions t_1, t_2,

$$z^{i+1} = e^{t_1} z^{j+1} \quad e^{t_2} z^{k+1} .$$

This proves the relations

$$(3.22) \qquad p^i_{jk} = p^{i+1}_{j+1,k+1} , \quad i, j, k = 0, 1, \ldots, c-1 .$$

In applying (3.22), the indices may be reduced modulo c if necessary.

The method used for the construction of these schemes has led to a notation in which the c associate classes are numbered from 0 to $c-1$, rather than in the usual way from 1 to c. The matrices P_i will be numbered in the same way, and in particular their rows and columns will be indexed from 0 to c-1 . A change of notation would be easy enough, but would necessitate another definition and more symbols; instead, the reader is asked to bear with minor inconveniences such as referring to the 0^{th} row of a matrix. These association schemes will be used only in the present section, and only for the purpose of constructing schemes with two associate classes, for which the usual notation will be resumed.

The addition table of the field will serve as a convenient form for the association scheme if the first row is arranged with 0 as the leading mark and the marks of each coset in adjacent positions. The associates of any mark θ are read from the row of the table containing θ in the first column. The j^{th} associates are the marks appearing in the columns corresponding to the j^{th} coset.

If θ and ϕ are any two i^{th} associates, then p^i_{jk} is by definition equal to the number of treatments which are j^{th} associates of θ and k^{th} associates of ϕ . In determining the value of p^i_{jk} there is no loss of generality in taking $\phi = 0$ and θ any mark of the i^{th} coset; the k^{th} associates of ϕ are then the marks of the k^{th} coset. p^i_{jk} is then equal to the number of marks of the k^{th} coset in the set obtained by adding θ to each of the marks in the j^{th} coset. With a fixed value of θ , using only one row of the addition table, and assigning all of the values 0 , 1 , ... , c-1 to j and k , all the elements of the matrix P_i may be determined for a particular i . The elements of P_0 may be determined by using only the row of the table corresponding to

$\theta = z^0 = 1$. When the values p^0_{jk} are obtained, the remaining p^i_{jk} values are easily obtained from (3.22) without further use of the addition table. Equations (3.21) and the symmetry relation $p^i_{jk} = p^i_{kj}$ may be used to shorten the work and check the values.

The derivation of this association scheme has made use of a particular primitive mark z and the subgroup of order d generated by the mark $e = z^c$. Since this subgroup is unique, any other primitive mark y will lead to the same subgroup and hence to the same c cosets. The use of y leads to a notation in which the j^{th} coset is the one containing y^j , rather than z^j . The numbering of the cosets, other than the 0^{th} , which is the subgroup itself, may thus be different. This means that the c classes of associates may be numbered differently for different choices of the primitive mark, but are otherwise identical. That is, the association scheme is unique except for numbering of the associate classes.

The results that have been obtained in this section will now be stated as a theorem.

THEOREM 3.2: For any number v of treatments of the form $v = p^q$ for p a prime and q a positive integer, identify the treatments with the marks of the finite field of order p^q . Let a divisor d of $p^q - 1$ be chosen subject to the requirement that d be even if p is odd, and define $c = \frac{p^q - 1}{d}$. Let a subgroup of order d of the multiplicative group of the field, and the cosets of the subgroup, be used to partition the $p^q - 1$ group marks into c disjoint sets, each containing d marks.

For any primitive field mark z refer to the coset containing z^i as the i^{th} coset, $i = 0, 1, \ldots, c-1$. Let the set obtained by adding the field mark θ to each of the marks of the i^{th} coset be defined as the set of i^{th} associates of θ. This defines an association scheme with c associate classes, with $n_i = d$, $i = 0, 1, \ldots, c-1$, and satisfying all the conditions of partial balance. The value p^i_{jk} is equal to the number of marks of the k^{th} coset in the set obtained by adding a fixed mark of the i^{th} coset to each of the d marks of the j^{th} coset. The following special relations hold for the p^i_{jk}, $i, j, k = 0, 1, \ldots, c-1$.

(3.21) $$p^i_{jk} = p^j_{ik} = p^k_{ij} ,$$

(3.22) $$p^i_{jk} = p^{i+1}_{j+1,\, k+1} .$$

For particular values of p^q and d, the scheme is unique except for numbering of the associate classes. Only one scheme by this method. This doesn't rule out other schemes with the same param!

An example will now be given to illustrate the procedure just described. An association scheme which will be useful later is based on the field of order $v = p^q = 2^4 = 16$, with $d = 5$, $c = 3$. The field marks will be represented by the integers $0, 1, \ldots, 15$. Rules for forming sums and products in this field are easily stated but it will suffice here to give the addition and multiplication tables.

ADDITION TABLE

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

MULTIPLICATION TABLE

	1	3	5	15	2	6	10	13	4	12	7	9	8	11	14
1	1	3	5	15	2	6	10	13	4	12	7	9	8	11	14
3	3	5	15	2	6	10	13	4	12	7	9	8	11	14	1
5	5	15	2	6	10	13	4	12	7	9	8	11	14	1	3
15	15	2	6	10	13	4	12	7	9	8	11	14	1	3	5
2	2	6	10	13	4	12	7	9	8	11	14	1	3	5	15
6	6	10	13	4	12	7	9	8	11	14	1	3	5	15	2
10	10	13	4	12	7	9	8	11	14	1	3	5	15	2	6
13	13	4	12	7	9	8	11	14	1	3	5	15	2	6	10
4	4	12	7	9	8	11	14	1	3	5	15	2	6	10	13
12	12	7	9	8	11	14	1	3	5	15	2	6	10	13	4
7	7	9	8	11	14	1	3	5	15	2	6	10	13	4	12
9	9	8	11	14	1	3	5	15	2	6	10	13	4	12	7
8	8	11	14	1	3	5	15	2	6	10	13	4	12	7	9
11	11	14	1	3	5	15	2	6	10	13	4	12	7	9	8
14	14	1	3	5	15	2	6	10	13	4	12	7	9	8	11

The multiplication table is in a cyclic form based on the primitive element 3 as a generator. In the notation of this section,

$$z = 3 \;,$$

$$z^c = 3^3 = 15 \;,$$

and the subgroup of order $d = 5$ consists of the marks 1 , 15 , 10 , 12 , 8 . Note: Since the order of the field is even, it is not necessary that d be even. The subgroup and its cosets are displayed in the horizontal rows of the following array, which is formed by filling in one column at a time with the entries of the first row of the multiplication table in order.

Subgroup (0-coset)	1	15	10	12	8
1-coset	3	2	13	7	11
2-coset	5	6	4	9	14

We can now define an association scheme with three associate classes by saying that two field marks are i^{th} associates, $i = 0 , 1 , 2$, if their difference is a mark of the i-coset. The following is the addition table rearranged to serve as a table of the association scheme.

	0-coset					1-coset					2-coset				
0	1	15	10	12	8	3	2	13	7	11	5	6	4	9	14
1	0	14	11	13	9	2	3	12	6	10	4	7	5	8	15
15	14	0	5	3	7	12	13	2	8	4	10	9	11	6	1
10	11	5	0	6	2	9	8	7	13	1	15	12	14	3	4
12	13	3	6	0	4	15	14	1	11	7	9	10	8	5	2
8	9	7	2	4	0	11	10	5	15	3	13	14	12	1	6
3	2	12	9	15	11	0	1	14	4	8	6	5	7	10	13
2	3	13	8	14	10	1	0	15	5	9	7	4	6	11	12
13	12	2	7	1	5	14	15	0	10	6	8	11	9	4	3
7	6	8	13	11	15	4	5	10	0	12	2	1	3	14	9
11	10	4	1	7	3	8	9	6	12	0	14	13	15	2	5
5	4	10	15	9	13	6	7	8	2	14	0	3	1	12	11
6	7	9	12	10	14	5	4	11	1	13	3	0	2	15	8
4	5	11	14	8	12	7	6	9	3	15	1	2	0	13	10
9	8	6	3	5	1	10	1	4	14	2	12	15	13	0	7
14	15	1	4	2	6	13	12	3	9	5	11	8	10	7	0

For use in obtaining the values p^0_{jk} the coset designation of each of the j^{th} associates of treatment 1 will be noted, first for $j = 0$.

0^{th} associates of 1 :	0	14	11	13	9 ,
coset designation:	-	2	1	1	2 .

The frequencies of marks of the 0^{th} , first and second cosets are 0 , 2 and 2 respectively, giving the values $p^0_{00} = 0$, $p^0_{01} = 2$, $p^0_{02} = 2$. The remaining p^0_{jk} are obtained similarly.

1^{st} associates of 1 :	2	3	12	6	10 ,
coset designation:	1	1	0	2	0 ,

giving $p^0_{10} = 2$, $p^0_{11} = 2$, $p^0_{12} = 1$.

2^{nd} associates of 1 :	4	7	5	8	15 ,
coset designation:	2	1	2	0	0 ,

giving $p^0_{20} = 2$, $p^0_{21} = 1$, $p^0_{22} = 2$. (3.22) gives such results as

$$p^0_{01} = p^1_{12} = p^2_{20}$$

and the following set of matrices $P_i = (p^i_{jk})$ is obtained.

$$P_0 = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} , \quad P_1 = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} , \quad P_2 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} .$$

The set of field marks consisting of 0 and the multiplicative subgroup 1 , e , e^2 , ... , e^{d-1} has some of the properties of an additive group. Addition is simply field addition and is commutative and associative, and the condition that the element -1 is included insures that the set contains with each element its additive inverse. The set is not necessarily closed under addition. This is the only remaining requirement for the set to be an additive abelian group, and will be satisfied if and only if the set is a finite field. It follows from general properties of finite fields that this must be a subfield of the original field of p^q elements, containing p^s elements, where s is a divisor of q. This case has been studied rather extensively in connection with a variety of combinatorial problems; a recent application to incomplete block designs will be mentioned at the end of this section. In 1938 Singer [32] showed that it may be used to generate finite projective geometries, in particular projective planes. In the present setting we make use instead of the finite Euclidean plane which may be obtained from the projective plane by designating one line as the line at infinity and deleting it with the points on it. The number of remaining points is a square, say n^2, where n is the number of points on a line and in every known case is a prime power, say $n = p^s$. Two particular association schemes will now be discussed for $v = n^2 = p^{2s}$ treatments. The first scheme will be shown to be equivalent to this geometry and leads to association schemes of the L_g series. The second scheme is used in constructing the new schemes of the L_g^* series. In the finite field with $p^{2s} = n^2$ marks, the order of the multiplicative group is $n^2 - 1 = (n-1)(n+1)$. The first association scheme, leading to the finite geometry, uses $d = n-1$ and $c = n+1$; in the other scheme the same values are used in the opposite order.

An arbitrary treatment θ in either scheme will now be considered. An arbitrary pair of distinct i^{th} associates of θ may be represented by

$$\theta + e^{t_1}z^i \ , \quad \theta + e^{t_2}z^i \ , \quad \text{where } e^{t_1} \text{ and } e^{t_2} \text{ are distinct .}$$

We investigate whether these two treatments are i^{th} associates. This will be the case if and only if the difference between the field marks is a mark of the i^{th} coset.

$$\theta + e^{t_1}z^i - (\theta + e^{t_2}z^i) = z^i(e^{t_1} - e^{t_2}) \ .$$

This expression represents a mark of the i^{th} coset if and only if $e^{t_1} - e^{t_2}$ is an element of the multiplicative subgroup generated by e. Since e^{t_1} and e^{t_2} are distinct, their difference is not 0. The additive inverse of e^{t_2} is an element of the subgroup, say e^{t_3}, giving

$$e^{t_1} - e^{t_2} = e^{t_1} + e^{t_3} \ .$$

This sum will be an element of the subgroup for all choices of $e^{t_1} \neq e^{t_2}$ if and only if the set consisting of 0 and the subgroup is closed under addition, or equivalently if and only if the set is a field. Therefore the i^{th} associates of an arbitrary treatment θ are pairwise i^{th} associates if and only if the subgroup used in defining the association scheme is the multiplicative group of a subfield. This remark is used in the discussion of both association schemes for n^2 treatments.

While properties of the Euclidean geometry are well known and closely similar constructions of it have been published [5], [2], enough of the derivation of it will be presented here to be used in describing the association scheme. This will be useful for comparison with the second scheme. The marks of the finite field will be identified with the treatments of the design as for all of the association schemes treated in this section; for the present scheme they will also be identified with the points of the geometrical system. The set consisting of a treatment and its i^{th} associates will be identified with the n—points on a line. In order for this line to be well-defined it is necessary to show that the treatments in such a set are pairwise i^{th} associates. The subgroup of order $d = n-1$ may be taken as the multiplicative group of the subfield of order $n = p^s$. Therefore, by the previous paragraph, the treatments are pairwise i^{th} associates. Since each of them has the same number $n-1$ of i^{th} associates, each of them determines the same set of n marks, and the line is well-defined. Every mark of the field must lie in such a set of n marks related as i^{th} associates and defined as a line. This implies that the n^2 points of the system may be divided into n disjoint sets, each containing the n points of a line determined by the i^{th} association relation. Since the lines have no points in common, they will be called parallel lines and will be described for convenience as lines in the i^{th} direction. Corresponding to the $n+1$ associate classes there are $n+1$ systems of parallel lines in as many different directions, each system containing n lines of n points each and exhausting the set of n^2 points.

Since any two distinct treatments are i^{th} associates for some i, any two distinct points of the geometrical system determine a unique line.

This implies that the number of points common to two distinct lines cannot be as large as 2, and must be either 0 or 1. The n points of a line in direction i must be distributed over the n lines of the set of lines in direction j in such a way that not more than one point falls on each line. Since the number of points is equal to the number of lines, this means that a line of direction i intersects each line of direction j in just one point, $i \neq j$. This completes the proof of the relevant properties of the geometrical system, showing that it is indeed a finite Euclidean geometry, and furnishing a convenient way of computing the parameter values of the association scheme.

The parameter n_i has the geometrical interpretation of the number of additional points on the line through an initial point in the i^{th} direction, and it is clear from the geometry or general properties of the association schemes under discussion that $n_i = n - 1$ for all i. The parameter p^i_{jk} may be defined by means of any two points θ and ϕ joined by a line of the i^{th} direction. p^i_{jk} is equal to the number of points other than possibly θ and ϕ themselves, common to the line through θ in direction j and the line through ϕ in direction k. It is clear from geometrical reasoning that

$$
\begin{aligned}
& p^i_{ii} = n - 2 , \\
(3.23) \qquad & p^i_{jj} = p^j_{ij} = p^j_{ji} = 0 \quad \text{when} \quad i \neq j , \\
& p^i_{jk} = 1 \quad \text{when} \quad i, \ j, \ k \quad \text{are all distinct.}
\end{aligned}
$$

In view of the known relations among the p^i_{jk}, particularly for the class of association schemes of the present section, the three preceding statements are far from independent. Straightforward use of (1.6) and (3.21) shows that each of the first two implies the other, while the third implies both of the first two. Finally, the first two may be shown to im imply the geometric structure and hence the third statement.

In the second scheme a subgroup of order $d = n + 1 = p^s + 1$ of the field multiplicative group is used. This subgroup and its cosets determine $c = n-1$ associate classes. The zero element of the field and the subgroup form a set of $n + 2 = p^s + 2$ marks which will not be a subfield, and the set of treatments consisting of an element and its i^{th} associates will accordingly not be pairwise i^{th} associates. It is therefore not possible to use association relations in this case to define lines in a plane geometry, and there is no obvious way to compute the p^i_{jk} values. However, direct computation gives the values fairly easily in a particular case. The example already given for $n^2 = 16$ is an association scheme of this class and illustrates the computation involved. The p^i_{jk} values for several other cases will be given later.

The association schemes constructed by Theorem 3.2 have in general more than two associate classes. In most cases where schemes with two classes are derived, it will be by the device of combining classes, that is, by forming a set C_1 of one or more of the associate classes of the original scheme, and defining two treatments to be first associates in the new scheme if and only if they are associates of one of the classes of the set C_1. An association scheme formed in this way does not necessarily satisfy the conditions of partial balance. Conditions that it will do so, in a more general setting, are derived in the next theorem. It will then be easy to show that the schemes related to the Euclidean plane lead to a wide class of schemes of the L_g series. The second family of schemes is more difficult to deal with but will be used to construct several schemes of the L_g^* series.

THEOREM 3.3. Let an association scheme with m classes of associates be formed from a scheme with a larger number of classes by partitioning the classes of the original scheme into m disjoint sets C_1, ..., C_m, with two treatments defined as α^{th} associates in the new scheme if in the original scheme they are associates of one of the classes of set C_α. The notation C_α will be used interchangeably for the set of associate classes and for the set of indices by which they are identified. Parameter values will be denoted by n_i, p^i_{jk} in the original scheme and by n_α, $p^\alpha_{\beta\gamma}$ in the new scheme. Then

$$(3.24) \qquad n_\alpha = \sum_{i \in C_\alpha} n_i$$

and a necessary and sufficient condition that the new scheme satisfy the conditions of partial balance is

$$(3.25) \qquad p^\alpha_{\beta\gamma} = \sum_{j \in C_\beta} \sum_{k \in C_\gamma} p^i_{jk} \quad \text{for all } \alpha, \beta, \gamma, \text{ and uniformly for each } i \in C_\alpha .$$

PROOF: The association matrices of the original scheme will be denoted by B_i. It may be recalled that the B_i are symmetric matrices of 0's and 1's whose sum is the matrix with 0's on the main diagonal and 1's elsewhere, and that they satisfy relation (1.16),

$$B_j B_k = \delta_{jk} n_j I + \sum_i p^i_{jk} B_i \qquad (= B_k B_j) \quad .$$

For fixed i, the matrix with p^i_{jk} in the j, k position will be denoted as usual by P_i. The definition of the new scheme implies that its association matrices, denoted by A_α, $\alpha = 1, \ldots, m$ must have the form

$$(3.26) \qquad A_\alpha = \sum_{i \in C_\alpha} B_i \quad .$$

It follows from Theorems 1.1 and 1.2 that in order for the new scheme to satisfy the conditions of partial balance, it is necessary and sufficient that the association matrices A_α satisfy (1.16), that is, that there exist constants n_α and $p^\alpha_{\beta\gamma}$ such that

$$(3.27) \qquad A_\beta A_\gamma = A_\gamma A_\beta = \delta_{\beta\gamma} n_\beta I + \sum_{\alpha=1}^{m} p^\alpha_{\beta\gamma} A_\alpha \quad .$$

Since n_α has the interpretation of the uniform row total of A_α and n_i is the uniform row total of B_i, it is clear from (3.26) that (3.24) is satisfied. Using (3.26), the product $A_\beta A_\gamma$ may be written

$$(3.28) \qquad A_\beta A_\gamma = (\sum_{j \in C_\beta} B_j)(\sum_{k \in C_\gamma} B_k) = \sum_{j \in C_\beta} \sum_{k \in C_\gamma} B_j B_k \quad .$$

Since $B_k B_j = B_j B_k$, it is clear at this stage that $A_\beta A_\gamma = A_\gamma A_\beta$, and only $A_\beta A_\gamma$ will be discussed. Using (1.16), (3.28) may be written

$$(3.29) \qquad A_\beta A_\gamma = \sum_{j \in C_\beta} \sum_{k \in C_\gamma} (\delta_{jk} n_j I + \sum_i p^i_{jk} B_i) \quad .$$

Since the sets C_α are disjoint, j and k can be equal in this summation only if $\beta = \gamma$, in which case the first term in the parentheses leads to $\sum_{j \in C_\beta} n_j I$. By (3.24), this is equal to $n_\beta I$ and (3.29) may be written

$$(3.30) \qquad A_\beta A_\gamma = \delta_{\beta\gamma} n_\beta I + \sum_{j \in C_\beta} \sum_{k \in C_\gamma} \sum_{i} p^i_{jk} B_i \, .$$

This expression for the product $A_\beta A_\gamma$ will now be compared with the following, obtained from (3.27) when A_α is written in the form given by (3.26) .

$$(3.31) \qquad A_\beta A_\gamma = \delta_{\beta\gamma} n_\beta I + \sum_{\alpha=1}^{m} p^\alpha_{\beta\gamma} \sum_{i \in C_\alpha} B_i \, .$$

Since the B_i and the identity matrix I are linearly independent, (3.30) reduces to the form of (3.31) if and only if the coefficients of B_i in the two expressions are equal for all i . The coefficient of B_i in (3.31) reduces to the single term $p^\alpha_{\beta\gamma}$, and the necessary and sufficient condition is identical with (3.25) , completing the proof of Theorem 3.3.

Thus in order for the matrices A_α defined by (3.26) to multiply in accordance with (1.16), it is necessary and sufficient that each $p^\alpha_{\beta\gamma}$ value be equal to the sum of the elements in the submatrix of P_i determined by the row indices j belonging to the set C_β and the column indices k belonging to the set C_γ , where i is a member of set C_α . The crucial condition is that the same total be obtained from every P_i matrix for which $i \in C_\alpha$. Because of the relations (1.6) and (1.7) satisfied by the parameter values of every PBIB design, it is not necessary to verify (3.25) for all of the $p^\alpha_{\beta\gamma}$. In the special case of a scheme with two associate classes, the values p^1_{11} and p^2_{11} , with n_1 and n_2 , are sufficient to determine the remaining p^i_{jk} values, and the following corollary results.

COROLLARY 3.3. If $m = 2$ in Theorem 3.3, condition (3.25) may be replaced by the simpler condition

$$(3.32) \qquad p_{11}^{\alpha} = \sum_{j \in C_1} \sum_{k \in C_1} p_{jk}^{i} \quad \text{for } \alpha = 1 \text{ or } 2 \text{ and uniformly for each } i \in C_{\alpha} .$$

If the new scheme has $m = 2$ classes, set C_2 is the complement of set C_1, and the association relation is most simply defined by saying that two treatments are first associates in the new scheme if in the original scheme they are associates of one of the classes of set C_1, and the two treatments are second associates otherwise. In the application of the corollary, the same symmetric submatrix, determined by the rows and columns whose indices are in set C_1, is used in each of the original P_i matrices. The necessary and sufficient condition that the new scheme satisfy the conditions of partial balance is that the sum of all the elements of the submatrix be the same for all P_i with $i \in C_1$, the common value being taken as p_{11}^{1} for the new scheme, and the same for all P_i with $i \notin C_1$, the common value being taken as p_{11}^{2} for the new scheme.

Equations (3.22) show that for the schemes with $n + 1$ classes, obtained from the Euclidean plane, each P_i matrix may be obtained by a cyclic permutation of rows and columns of the matrix P_0, which is an $(n + 1) \times (n + 1)$ matrix with the following form.

$$P_0 = \begin{bmatrix} n-2 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & . & 1 & 1 \\ 0 & 1 & 0 & . & . & . & 1 & 1 \\ . & . & . & . & . & . & . & . \\ 0 & 1 & 1 & . & . & . & 0 & 1 \\ 0 & 1 & 1 & . & . & . & 1 & 0 \end{bmatrix}$$

In the matrix P_i , the diagonal element p_{ii}^i is equal to $n-2$, the remaining entries in row i and column i are 0's , and the remaining diagonal entries are 0's . Application of Corollary 3.3 to find the values of p_{11}^1 and p_{11}^2 requires finding the sum of the elements of the symmetric submatrix of each P_i determined by the rows and columns whose indices are in class C_1 . Suppose that g classes of the original scheme are to be combined into set C_1 ; then the symmetric submatrix will be of order $g \times g$. If the diagonal element $n-2$ is not in the submatrix, the $g(g-1)$ off-diagonal elements will all be 1's ; if the diagonal element $n-2$ is in the submatrix, then $(g-1)(g-2)$ off-diagonal elements will be 1's and other elements will be 0's . This means that the sum is $n-2 + (g-1)(g-2) = g^2-3g+n$ whenever the index i of the matrix P_i is in the class C_1 which determines the submatrix, and the sum is $g(g-1)$ for all P_i with $i \notin C_1$. But this is precisely the requirement of Corollary 3.3, proving that the association scheme defined by taking two treatments as first associates if they are associates of one of the g classes of set C_1 of the original, Euclidean geometry, scheme, satisfies the conditions of partial balance with two associate classes. There was no restriction on the value of g. The expressions obtained for n_1, p^1_{11} and p^2_{11} are identical with those derived in Section 2.1 for Latin square type schemes with g constraints.

Some of the special features of the association schemes constructed by Theorem 3.2 may be used to simplify the application of Theorem 3.3 and Corollary 3.3. This will be discussed in the case of Corollary 3.3 for the present purpose of constructing schemes with two associate classes. In the schemes of Theorem 3.2, the c associate classes all have equal numbers of treatments $n_i = d$, so that if first associates are defined in the new scheme by a set C_1 of g of the original associate classes, the number of treatments in the first associate class of the new scheme is $n_1 = gd$, regardless of the particular set of g classes chosen for set C_1. Suppose that a set C_1 is known to satisfy conditions (3.32) of Corollary 3.3; define a new set C_1' by adding 1 to each index in the set C_1: that is,

$$i+1 \in C_1' \text{ if and only if } i \in C_1 .$$

$i+1$ is reduced modulo c if necessary. The following equalities are obtained by successive use of (3.22), of the definition of set C_1', and of a change of notation in the indices of summation.

$$\sum_{j\in C_1}\sum_{k\in C_1} p^i_{jk} = \sum_{j\in C_1}\sum_{k\in C_1} p^{i+1}_{j+1,k+1} = \sum_{j+1\in C_1'}\sum_{k+1\in C_1'} p^{i+1}_{j+1,k+1} = \sum_{j\in C_1'}\sum_{k\in C_1'} p^i_{jk} .$$

Since the first sum is equal to the same value p^1_{11} uniformly for $i \in C_1$ and equal to p^2_{11} uniformly for $i \notin C_1$, the last sum will be equal to p^1_{11} uniformly for $i \in C_1'$ and equal to p^2_{11} uniformly for $i \notin C_1'$.

Thus the set C_1' satisfies conditions (3.32) of Corollary 3.3, giving the same values for the parameters n_1, p_{11}^1 and p_{11}^2, and hence for the remaining association scheme parameters as well, as are given by the set C_1. The operation of increasing by unity the index of each associate class in C_1 may evidently be repeated as often as desired, giving in each instance a scheme equivalent to that obtained with C_1. A sufficient number of repetitions will result in a set which contains the 0^{th} class of associates.

Therefore, in application of Corollary 3.3 to scheme constructed by Theorem 3.2, it may be assumed without loss of generality that the set C_1 of classes which are combined to form the class of first associates in the new scheme contains the 0^{th} class of associates in the original scheme. The submatrix determined by set C_1 will then contain the leading diagonal element of each P_i matrix. This fact may be used to reduce the amount of empirical search necessary to find a suitable set C_1, although it was not needed in discussion of the schemes related to the finite geometry. The search may also be simplified if the parameter values of the possible new scheme are known. In the second family of schemes discussed for $v = n^2$ treatments, each associate class has $d = n + 1$ treatments, and the only schemes which can be formed by combining classes are those in which n_1 is a multiple of $n + 1$. Inspection of Table II of the Appendix shows that most of the schemes with appropriate values of v and n_1 are in the L_g^* series, and that in this case the number of classes to be combined in set C_1, which is equal to the order of the symmetric submatrices of the P_i matrices specified in Corollary 3.3, is given by the numerical value of the subscript g. For a particular possible scheme, the values of p_{11}^1 and p_{11}^2 may also be obtained from Table II. The sum of the elements of the submatrix of P_0 must be equal to p_{11}^1, a condition which is easily checked for any submatrix and may eliminate many of the possible submatrices.

Corollary 3.3 and the remarks which have just been made will now be used to attempt to construct schemes with two associate classes from the second family of schemes constructed by Theorem 3.2 for $v = n^2 = p^{2s}$ treatments. For $v = 9$, the scheme of this family has only two classes of associates and there is no need to combine classes. The scheme is listed in Table II as #2, and is also in the L_g series. For $v = 16$, the scheme has three associate classes of 5 treatments each and has been given as an example. The P_i matrices are repeated here for reference.

$$P_o = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} .$$

The only scheme of Table II in which n_1 is a multiple of 5 is scheme #6, in which $n_1 = 5$. If this scheme is to be formed by combining associate classes, the set C_1 must consist of a single associate class and will be taken as the 0^{th} class. The submatrix in this case is the single element p_{00}^i, and is equal to 0 for $i = 0$, and 2 for $i = 1$ or 2. These are the required values for p_{11}^1 and p_{11}^2 in scheme #6, proving that the scheme can be constructed by this method.

Table II includes no association schemes for $v = 25$ treatments which cannot be constructed by other methods. Scheme #22 is an L_{-2}^{*} scheme but the number of treatments is 36, which is not a prime power, and the present method is not applicable.

The scheme constructed by Theorem 3.2 with $v = 49$ treatments has six associate classes, each containing 8 treatments. The schemes of Table II with $v = 49$ and n_1 a multiple of 8 are schemes #31 and #33. The latter is a known L_3 scheme. Scheme #31 is an L_{-2}^{*} scheme with $p_{11}^{1} = 3$. The set C_1 must therefore determine a symmetric 2 x 2 submatrix which may be assumed to contain the leading diagonal element of each P_i matrix, and the sum of the four elements of the submatrix of P_0 must be 3. Matrix P_0, computed by the methods already illustrated in the example with $v = 16$, is as follows.

$$P_0 = \begin{bmatrix} 0 & 0 & 2 & 2 & 1 & 2 \\ 0 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 2 & 2 & 0 \\ 1 & 0 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 & 0 \end{bmatrix} .$$

If the elements of a symmetric matrix are integers and their sum is odd, it is clear that a diagonal element must be odd. The only odd diagonal element of P_0 is $p_{22}^{0} = 1$, which must be in the 2 x 2 submatrix. This determines a submatrix with the form $\begin{matrix} 0 & 2 \\ 2 & 1 \end{matrix}$, which has sum 5 instead of the required 3. It is therefore impossible to choose a set C_1 of associate classes which satisfies all the conditions of Corollary 3.3, and scheme #31 cannot be constructed by the method of this section.

The next scheme constructed by Theorem 3.2 has $v = 64$ treatments, with seven associate classes, each containing 9 treatments. The schemes of Table II with $v = 64$ and n_1 a multiple of 9 are schemes #48 and #51.

Scheme #48 is an L_{-2}^* scheme with $p_{11}^1 = 2$. The set C_1 therefore must determine a symmetric 2×2 submatrix containing the leading diagonal element of each P_i matrix, and the sum of the four elements of the submatrix of P_0 must be 2. The matrices P_i are listed below and it is easily verified that no submatrix of P_0 satisfies these requirements. Therefore scheme #48 cannot be constructed by the method of this section.

Scheme #51 is an L_{-3}^* scheme with $p_{11}^1 = 10$ and $p_{11}^2 = 12$. The set C_1 must contain three associate classes whose indices determine a 3×3 submatrix of each of the seven P_i matrices. The sum of the elements of the submatrix must be 10 or 12, according as the index i of the matrix is or is not the index of a class in the set C_1. The seven matrices will now be listed.

$$P_0 = \begin{bmatrix} 2&0&0&2&0&2&2 \\ 0&2&2&0&2&1&2 \\ 0&2&2&1&2&2&0 \\ 2&0&1&0&2&2&2 \\ 0&2&2&2&2&0&1 \\ 2&1&2&2&0&0&2 \\ 2&2&0&2&1&2&0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0&2&2&0&2&1&2 \\ 2&2&0&0&2&0&2 \\ 2&0&2&2&0&2&1 \\ 0&0&2&2&1&2&2 \\ 2&2&0&1&0&2&2 \\ 1&0&2&2&2&2&0 \\ 2&2&1&2&2&0&0 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0&2&2&1&2&2&0 \\ 2&0&2&2&0&2&1 \\ 2&2&2&0&0&2&0 \\ 1&2&0&2&2&0&2 \\ 2&0&0&2&2&1&2 \\ 2&2&2&0&1&0&2 \\ 0&1&0&2&2&2&2 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 2&0&1&0&2&2&2 \\ 0&0&2&2&1&2&2 \\ 1&2&0&2&2&0&2 \\ 0&2&2&2&0&0&2 \\ 2&1&2&0&2&2&0 \\ 2&2&0&0&2&2&1 \\ 2&2&2&2&0&1&0 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 0&2&2&2&2&0&1 \\ 2&2&0&1&0&2&2 \\ 2&0&0&2&2&1&2 \\ 2&1&2&0&2&2&0 \\ 2&0&2&2&2&0&0 \\ 0&2&1&2&0&2&2 \\ 1&2&2&0&0&2&2 \end{bmatrix}, \quad P_5 = \begin{bmatrix} 2&1&2&2&0&0&2 \\ 1&0&2&2&2&2&0 \\ 2&2&2&0&1&0&2 \\ 2&2&0&0&2&2&1 \\ 0&2&1&2&0&2&2 \\ 0&2&0&2&2&2&0 \\ 2&0&2&1&2&0&2 \end{bmatrix},$$

$$P_6 = \begin{bmatrix} 2&2&0&2&1&2&0 \\ 2&2&1&2&2&0&0 \\ 0&1&0&2&2&2&2 \\ 2&2&2&2&0&1&0 \\ 1&2&2&0&0&2&2 \\ 2&0&2&1&2&0&2 \\ 0&0&2&0&2&2&2 \end{bmatrix}.$$

It is not difficult to verify that the set C_1 containing classes 0, 1, 5 satisfies the requirements and leads to a construction of scheme #51. The seven 3×3 submatrices are the following. The association scheme is given in section A.4 of the Appendix.

$$i = 0 : \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} ; \qquad i = 1 : \begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} ;$$

$$i = 2 : \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} ; \qquad i = 3 : \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 2 \\ 2 & 2 & 2 \end{bmatrix} ;$$

$$i = 4 : \begin{bmatrix} 0 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} ; \qquad i = 5 : \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix} ;$$

$$i = 6 : \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} .$$

The next scheme constructed by Theorem 3.2 has $v = 81$ treatments, with eight associate classes, each containing 10 treatments. The schemes of Table II with $v = 81$ and n_1 a multiple of 10 are schemes #68 and #70 in the L_g^* series, #73, a known scheme of the L_g series which is also in the L_g^* series, and #72, in neither series.

Scheme #68 is an L_{-2}^* scheme with $p_{11}^1 = 1$ and $p_{11}^2 = 6$. The set C_1 must contain two of the eight associate classes and determine a 2×2 submatrix of each of the eight P_i matrices; the sum of the elements of the submatrix must be equal to 1 if i is the index of either class in set C_1 , and equal to 6 if i is any of the six other indices. The usual assumption that C_1 contains the 0^{th} associate class means that the submatrix of each P_i matrix includes the leading diagonal element p_{00}^i , and the set $C_1 = (0 , 4)$ is quickly determined. The 2×2 submatrices have the required totals, showing that the construction of the scheme is possible. The eight 8×8 P_i matrices appear below, with the 2×2 submatrices. The association scheme is given in Section A.4 of the Appendix.

Scheme #70 is an L_{-3}^{*} scheme with $p_{11}^{1} = 9$ and $p_{11}^{2} = 12$. The set C_1 must contain three of the eight associate classes and determine a 3×3 submatrix of each of the eight P_i matrices; the sum of the elements of the submatrix must be equal to 9 if i is the index of any of the three classes in set C_1, and equal to 12 if i is any of the five other indices. The set $C_1 = (0, 1, 6)$ is found to be satisfactory. The 3×3 submatrices appear below and the association scheme appears in Section A.4 of the Appendix.

No construction has been found for scheme #72.

The P_i matrices for the scheme with 81 treatments and eight associate classes are the following.

$$P_0 = \begin{bmatrix} 1&0&2&0&0&2&2&2 \\ 0&2&2&0&2&2&0&2 \\ 2&2&2&0&1&2&1&0 \\ 0&0&0&2&2&2&2&2 \\ 0&2&1&2&0&2&1&2 \\ 2&2&2&2&2&0&0&0 \\ 2&0&1&2&1&0&2&2 \\ 2&2&0&2&2&0&2&0 \end{bmatrix}; \quad P_1 = \begin{bmatrix} 0&2&2&0&2&2&0&2 \\ 2&1&0&2&0&0&2&2 \\ 2&0&2&2&0&2&2&0 \\ 0&2&2&2&0&1&2&1 \\ 2&0&0&0&2&2&2&2 \\ 2&0&2&1&2&0&2&1 \\ 0&2&2&2&2&2&0&0 \\ 2&2&0&1&2&1&0&2 \end{bmatrix};$$

$$P_2 = \begin{bmatrix} 2&2&2&0&1&2&1&0 \\ 2&0&2&2&0&2&2&0 \\ 2&2&1&0&2&0&0&2 \\ 0&2&0&2&2&0&2&2 \\ 1&0&2&2&2&0&1&2 \\ 2&2&0&0&0&2&2&2 \\ 1&2&0&2&1&2&0&2 \\ 0&0&2&2&2&2&2&0 \end{bmatrix}; \quad P_3 = \begin{bmatrix} 0&0&0&2&2&2&2&2 \\ 0&2&2&2&0&1&2&1 \\ 0&2&0&2&2&0&2&2 \\ 2&2&2&1&0&2&0&0 \\ 2&0&2&0&2&2&0&2 \\ 2&1&0&2&2&2&0&1 \\ 2&2&2&0&0&0&2&2 \\ 2&1&2&0&2&1&2&0 \end{bmatrix};$$

$$P_4 = \begin{bmatrix} 0&2&1&2&0&2&1&2 \\ 2&0&0&0&2&2&2&2 \\ 1&0&2&2&2&0&1&2 \\ 2&0&2&0&2&2&0&2 \\ 0&2&2&2&1&0&2&0 \\ 2&2&0&2&0&2&2&0 \\ 1&2&1&0&2&2&2&0 \\ 2&2&2&2&0&0&0&2 \end{bmatrix}; \quad P_5 = \begin{bmatrix} 2&2&2&2&2&0&0&0 \\ 2&0&2&1&2&0&2&1 \\ 2&2&0&0&0&2&2&2 \\ 2&1&0&2&2&2&0&1 \\ 2&2&0&2&0&2&2&0 \\ 0&0&2&2&2&1&0&2 \\ 0&2&2&0&2&0&2&2 \\ 0&1&2&1&0&2&0&2 \end{bmatrix};$$

$$P_6 = \begin{bmatrix} 2&0&1&2&1&0&2&2 \\ 0&2&2&2&2&2&0&0 \\ 1&2&0&2&1&2&0&2 \\ 2&2&2&0&0&0&2&2 \\ 1&2&1&0&2&2&2&0 \\ 0&2&2&0&2&0&2&2 \\ 2&0&0&2&2&2&1&0 \\ 2&0&2&2&0&2&0&2 \end{bmatrix}; \quad P_7 = \begin{bmatrix} 2&2&0&2&2&0&2&0 \\ 2&2&0&1&2&1&0&2 \\ 0&0&2&2&2&2&2&0 \\ 2&1&2&0&2&1&2&0 \\ 2&2&2&2&0&0&0&2 \\ 0&1&2&1&0&2&2&2 \\ 2&0&2&2&0&2&0&2 \\ 0&2&0&0&2&2&2&1 \end{bmatrix}.$$

The 2×2 submatrices used in the construction of scheme #68 are the following.

$$i = 0 : \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \qquad i = 1 : \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix};$$

$$i = 2 : \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \qquad i = 3 : \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix};$$

$$i = 4 : \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \qquad i = 5 : \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix};$$

$$i = 6 : \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \qquad i = 7 : \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}.$$

The 3×3 submatrices used in the construction of scheme #70 are the following.

$$i = 0 : \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}; \qquad i = 1 : \begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix};$$

$$i = 2 : \begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}; \qquad i = 3 : \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix};$$

$$i = 4 : \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}; \qquad i = 5 : \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix};$$

$$i = 6 : \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}; \qquad i = 7 : \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Theorem 3.2 is not applicable to any schemes with $v = 100$, since there is no finite field of order 100 .

The schemes for which a construction by the method of this section has been discussed thus far have all been in the L_g and L_g^* series. It has been shown that no more schemes of these series with $v \leq 100$ can be constructed in this way. Most of the author's attempts to construct new schemes for which v is a prime power but which are not in these series have been inconclusive. When the number of treatments $v = p^q$ is of the form $4t + 1$, Theorem 3.2 may be used directly with $d = 2t$ and $c = 2$ to form an association scheme with two classes. When $q = 1$ this is the known scheme of the cyclic series in which the first associates of 0 are the quadratic residues of p; when q is even, the scheme is in the L_g series. For odd $q \geq 3$ the scheme is not of cyclic or Latin square type, but the first example is for $v = 125$. This example illustrates that the methods of Theorems 3.2 and 3.3 are not limited to the cyclic, Latin square and negative Latin square association schemes, but it does not seem likely that they will provide solutions to any more schemes within the range of Table II.

It was pointed out in Section 3.1 that if the Euclidean plane geometry with n^2 points and the L_g^* association scheme with n^2 treatments both exist, the number n_1 of first associates of a treatment of the scheme is equal to $|g|(n + 1)$, where $(n + 1)$ is equal to the number of lines through a point of the geometry and the numerical value of the negative integer g is used. It could therefore be conjectured that for some correspondence between the treatments of the scheme and the points of the geometry, the first associates of a given treatment would correspond to $|g|$ points on each of the $(n+1)$ lines through the given point. The treatments of the four L_g^* schemes which have been constructed in this section are already identified with the elements of a finite field, which in turn are identified with the points of the Euclidean plane geometry, giving a very natural one-to-one correspondence between treatments and points. It will be shown that with this correspondence, schemes #6 and #51 have the geometric property described.

It will be convenient to discuss both the scheme and the geometry in terms of the finite field. For any element θ, the first associates of θ in the L_g^* scheme are obtained by adding θ to each of the first associates of the additive identity 0. The points of the line through θ in direction i are obtained by adding θ to each of the points of the line in direction i through 0. There is therefore a one-to-one correspondence between the first associates of 0 which lie on the line through 0 in direction i, and the first associates of θ which lie on the line through θ in direction i. The distribution of the first associates of any treatment θ in the L_g^* scheme over the $n+1$ lines through θ in the plane geometry is therefore the same for any element θ as it is for the element 0 and it is sufficient to consider the element 0. The first associates of 0 and the remaining points on any line through 0 all correspond to non-zero field elements and will be replaced for the rest of the discussion by their indices or exponents with respect to a primitive element of the field.

The remaining points on the line through 0 in direction i have exponents which are congruent to i modulo $n+1$. The first associates of 0 in the L_g^* scheme have the $|g|(n+1)$ exponents obtained by combining the sets of $(n+1)$ exponents which are congruent to j modulo $n-1$, for $|g|$ suitably chosen values of j. A typical set of this kind will be considered and may be written

$$j + u(n-1) \ , \quad u = 0 \ , 1 \ , \ldots , \ n \ .$$

Suppose that two exponents in this set are congruent modulo $n+1$.

$$j + u_1(n-1) \equiv j + u_2(n-1) \quad \text{mod } (n+1) \ ,$$

$$u_1(n-1) \equiv u_2(n-1) \quad \text{mod}(n+1) \ .$$

Only the case in which n is even will be considered. In this case, $n-1$ is prime to $n+1$ and may be cancelled; since u_1 and u_2 are both between 0 and n this gives the result $u_1 = u_2$, showing that no two of the exponents of the set fall into the same residue class modulo $n+1$, and that the $n+1$ points corresponding to the set lie one each on the $n+1$ lines through 0. Since the same is true for each of the $|g|$ sets of first associates of 0, exactly $|g|$ of the first associates of 0 must lie on each of the $n+1$ lines through 0. Finally, this shows that if an L_g^* association scheme with n^2 treatments is obtained by the method of this section, and if n is even (meaning that n is a power of 2), then the $|g|(n+1)$ first associates of any treatment θ correspond to $|g|$ points on each of the $n+1$ lines through the point corresponding to θ in the finite Euclidean plane geometry with n^2 points.

In the case of scheme #6, $n = 4$ and $g = -1$, and the 5 first associates of any treatment θ lie one each on the five lines through the point θ. In the case of scheme #51, $n = 8$ and $g = -3$, and the 27 first associates of any treatment θ lie three each on the nine lines through θ.

If n is odd, it is still true that the distribution of the first associates of a treatment θ over the $n + 1$ lines through the corresponding point is the same for all choices of θ, but the distribution is not necessarily uniform. It proves not to be uniform for the schemes constructed for 81 treatments.

The results of Theorem 3.2 have also been found by D. A. Sprott and were published in two papers [33][34]. The second of these, which is the only one dealing with partially balanced designs, appeared in 1955, after the present work had been completed. The first article appeared in 1954 but did not come to this author's attention until after the second had been published. Both articles are on the construction of incomplete block designs from finite fields and make use of sets of field elements equivalent to the subgroup used in Theorem 3.2. The designs described in sections 4 and 5 of the second paper have association schemes which are identical with those constructed in Theorem 3.2. Theorem 3.2 was motivated by the desire for a class of association schemes for $v = n^2$ treatments in which the numbers of treatments in the associate classes are multiples of $n + 1$, and the method of proof was originally suggested by some work published by Bose in 1942 [5]. The author is indebted to

Sprott's papers, however, for the realization that the final statement of Theorem 3.2 was necessary in the proof that schemes such as #31 and #48 cannot be constructed by the present methods. Sprott's work is different from that appearing here in many details. The present discussion is limited to association schemes, while Sprott constructs actual designs, including some classes of them whose association schemes are not related to Theorem 3.2. He treats a field as an instance of a module and bases his construction on a general theorem of Bose and Nair [8] on the construction of partially balanced designs from a module. The proof of Theorem 3.2, dealing directly with properties of the finite field, is self-contained, may be simpler in some respects, and is certainly different in its arrangement. Sprott does not consider combining associate classes to form designs with fewer classes, and the only designs of the L_g or L_g^* series obtained are those with $v = p^q = 4t + 1$, $d = 2t$ and $c = 2$. In particular, the new schemes #6, #51, #68 and #70 are not obtained.

3.3 Construction of a Negative Latin Square Type Scheme with 100 Treatments by Enumeration.

It is possible to solve some combinatorial problems by making systematic trials of possible solutions until a solution is found or all possibilities are shown to fail. This method entails too much computation to be usable in the construction of most incomplete block designs or association schemes, but it will be used in this section to construct the L_g^* scheme which appears in Table II as #94. The parameter values of the scheme are

$$v = 100, \quad n_1 = 22, \quad n_2 = 77, \qquad P_1 = \begin{bmatrix} 0 & 21 \\ 21 & 56 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 6 & 16 \\ 16 & 60 \end{bmatrix}.$$

Since no Galois field of order 100 exists, the method of Section 3.2 cannot be applied here. The scheme would seem to have some special interest because of its possible connection with the unsolved question of the existence of orthogonal 10 x 10 Latin squares. The reason this problem is amenable to empirical study is the parameter value $p_{11}^1 = 0$, which permits use of Theorem 2.6.

In this section the symbol $U_{c,d}$ will be used to denote a $c \times d$ matrix all of whose elements are 1's. The subscripts will sometimes be omitted when the order is clear from the context. The orders of matrices and matrix products occurring in certain equations will be indicated in parenthetic statements which appear to the right of the equations.

If scheme #94 exists, let an arbitrary treatment be designated as treatment 0, and its 22 first associates numbered from 1 to 22. Then the matrix A_1 of first associates may be partitioned in the form of (2.38), with submatrix R a zero matrix.

$$
(3.33) \qquad A_1 = \left[\begin{array}{c|ccc|ccc}
0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \hline
1 & & & & & & \\
\vdots & & 0 & & & S & \\
1 & & & & & & \\ \hline
0 & & & & & & \\
\vdots & & S' & & & T & \\
0 & & & & & &
\end{array}\right]
\begin{array}{l}
\\ \\ \\ 22 \text{ rows}, \\ \\ \\ 77 \text{ rows}. \\
\end{array}
$$

Theorem 2.6 may be applied to show that submatrix S is the incidence matrix of a BIB design with parameter values

$$
(3.34) \qquad v = 22 , \quad r = 21 , \quad k = 6 , \quad b = 77 , \quad \lambda = 5 .
$$

Moreover, each row of the 77×77 matrix $S'S$ must have at least
P.60 P.59
$p^2_{12} = 16$ off-diagonal elements equal to 0. By (2.42) and (2.41) the 60 remaining off-diagonal elements of each row of $S'S$ must have sum $k(r-1) = 6(20) = 120$ and sum of squares $k(r-1) + k(k-1)(\lambda-1) = 6(20) + 6(5)(4) = 240$. The variance of this set of 60 numbers is then $\frac{240}{60} - (\frac{120}{60})^2 = 0$, showing that they must all be equal to their mean value 2. Thus each row of $S'S$ contains a 6 on the main diagonal, 16 0's and 60 2's. This means that each block in the balanced design has no treatments in common with 16 of the other blocks, and exactly 2

treatments in common with each of the remaining 60 blocks. The existence of such a BIB design is therefore a necessary condition for the existence of scheme #94.[1]

As pointed out in the discussion accompanying (2.38), each row of submatrix T contains 1's in $p_{12}^2 = 16$ off-diagonal positions. By statement (b) of Lemma 2.1, $S'S + T^2$ must have entries $p_{11}^1 = 0$ in these positions and entries $p_{11}^2 = 6$ in the other 60 off-diagonal positions of each row. Since T^2 has non-negative elements, the 60 2's in each row of S'S must fall in these same 60 positions. By difference, the element of T^2 in each of these positions must be a 4 . This determines the following structure for T^2 .

$$T^2 = 16\,I + 0 \cdot T + 4(U-I-T) , \qquad (77 \times 77 \text{ matrices}) .$$

Lemma 2.2 may now be applied to show that T is the matrix A_1 of first associates in an association scheme with two classes of associates and the parameter values

$$v = 77 , \quad n_1 = 16 , \quad n_2 = 60 , \quad p_{11}^1 = 0 , \quad p_{11}^2 = 4 .$$

This is scheme #64 of Table II . Thus the existence of scheme #64 is another necessary condition for the existence of scheme #94.[1]

Either of submatrices S or T would presumably be easier to investigate than the 100 x 100 matrix, and it will be shown below that the

---

1. In other cases in Table II where Theorem 2.6 applies, either matrix T^2 is not determined or T is not an association matrix of any scheme with two classes.

balanced design corresponding to S can actually be constructed. It is therefore important to show that the existence of S is sufficient as well as necessary for the existence of scheme #94. This will now be done. Let S be the incidence matrix of a BIB design with parameter values (3.34). This implies that SS' has the form

$$(3.35) \qquad SS' = 16\,I + 5U \;, \qquad (22 \times 22 \text{ matrices}) \;.$$

Also, let S have the property that each column has inner product 0 with each of 16 other columns. This was shown to imply that it has inner 2 with each of the 60 remaining columns. Then $S'S$ has the form

$$(3.36) \qquad S'S = 6\,I + 0 \cdot B_1 + 2B_2 \;, \qquad (77 \times 77 \text{ matrices}) \;,$$

where B_2 is a symmetric matrix with 0's on the main diagonal, 60 1's in each row, and 0's elsewhere, and B_1 is defined by

$$(3.37) \qquad B_1 = U - I - B_2 \;, \qquad (77 \times 77 \text{ matrices}) \;.$$

The following useful equations are easily derived from the fact that S has uniform row totals 21 and uniform column totals 6 .

$$(3.38) \qquad SU_{77,d} = 21\,U_{22,d} \;.$$

$$(3.39) \qquad U_{c,22}S = 6\,U_{c,77} \;.$$

$(S'S)^2$ will now be computed in two ways. From (3.36) ,

$$(3.40) \qquad (S'S)^2 = 36\,I + 24B_2 + 4B_2^{\,2} \;.$$

The next chain of equalities uses (3.35), (3.36), (3.39) and (3.37) in the order stated.

$$\begin{aligned}(S'S)^2 &= S'(SS')S \\ &= S'(16\ I + 5U)S = 16S'S + 5S'US \\ &= 16(6\ I + 2B_2) + 180\ U \\ &= 96\ I + 32B_2 + 180(I + B_1 + B_2)\quad ,\end{aligned}$$

$$(3.41)\qquad (S'S)^2 = 276\ I + 180B_1 + 212B_2\quad .$$

Solution of (3.40) and (3.41) for $B_2^{\ 2}$ gives

$$B_2^{\ 2} = 60\ I + 45B_1 + 47B_2\quad .$$

Application of Lemma 2.2, with the designations of first and second associates interchanged, shows that B_2 is the incidence matrix of second associates in scheme #64. An incidental result is that S' is the incidence matrix of a PBIB design with this association scheme. This is the dual of the BIB design represented by S and is obtained by interchanging the roles of treatments and blocks.

So far, it has been shown that the existence of S with the given properties implies the existence of a matrix with the properties required for T, defined by $T = B_1$. It remains to show that if S and T so defined are used as the submatrices in (3.33), then A_1 will meet the requirements for the association matrix of scheme #94. Using A_1 in this form, and squaring according to the rule for partitioned matrices,

$$(3.42)\qquad A_1^{\,2} = \left[\begin{array}{c|c|c} 22 & 0 \; . \; . \; . \; 0 & 6 \; . \; . \; . \; 6 \\ \hline 0 & & \\ \cdot & U + SS' & SB_1 \\ \cdot & & \\ \cdot & & \\ 0 & & \\ \hline 6 & & \\ \cdot & & \\ \cdot & B_1 S' & S'S + B_1^{\,2} \\ \cdot & & \\ 6 & & \end{array}\right] \begin{array}{l} \\ \\ 22 \text{ rows} , \\ \\ \\ \\ \\ \\ 77 \text{ rows} . \\ \\ \\ \end{array}$$

The forms of the submatrices of $A_1^{\,2}$ will be computed separately.

$$(3.43)\qquad U + SS' = U + 16\,I + 5U = 22\,I + 6(U - I) , \qquad (22 \times 22) .$$

The value of SB_1 may be obtained by solving (3.36) and (3.37) for B_1, then multiplying by S.

$$S'S = 6\,I + 2(U - I - B_1) .$$

$$B_1 = 2\,I + U - \tfrac{1}{2}S'S , \qquad (77 \times 77 \text{ matrices}) .$$

$$\begin{aligned} SB_1 &= 2S + SU - \tfrac{1}{2}SS'S \qquad (22 \times 77 \text{ matrices.}) \\ &= 2S + 21U - \tfrac{1}{2}(16\,I + 5U)S \\ &= 2S + 21U - 8S - 15U \end{aligned}$$

$$(3.44)\qquad SB_1 = 6(U - S) .$$

The value of $B_1^{\,2}$ is easily obtained from (1.16), recalling that B_1 is the incidence matrix of first associates in scheme #64. Then

$$S'S + B_1^{\,2} = (6\,I + 2B_2) + (16I + 4B_2) = 22\,I + 6B_2 ,$$

$$(3.45)\qquad S'S + B_1^{\,2} = 22\,I + 6(U - I - B_1) , \qquad (77 \times 77 \text{ matrices}) .$$

Using equations (3.43), (3.44), (3.45), equation (3.42) may be written

$$A_1^2 = 22\ I + 6(U - I - A_1) = 22\ I + 6A_2\ , \quad (100 \times 100)\ ,$$

and by Lemma 2.2, A_1 and A_2 are the association matrices of scheme #94. This completes the proof that the existence of the BIB design implies the existence of scheme #94. It will also furnish an easy way of constructing the scheme from the design.

The existence of scheme #94 is therefore equivalent to the existence of the specified BIB design. This does not mean that the number of distinct designs is equal to the number of schemes. For any matrix A_1 in scheme #94, there are 100 possible choices of the treatment to be designated as treatment 0, each leading to a different set of rows and columns in the submatrix S. These determine BIB designs which have the same parameter values, but which are not necessarily all equivalent under permutation of treatments and blocks. However, any given matrix S determines the rest of matrix A_1 uniquely, showing that the number of solutions of A_1 is at most equal to the number of solutions of S. It will appear below that there are at most 4 solutions for S, so that if the solution for scheme #94 is not unique, there are certainly no more than 4 solutions distinct under permutations of treatments. The question of uniqueness might be of interest if a connection is found between scheme #94 and sets of orthogonal 10×10 Latin squares.

The structure of S will now be taken up. S is the incidence matrix of a BIB design with 77 blocks, each containing 6 of treatments 1 , 2 , ... , 22 . Each treatment occurs 21 times and each pair of treatments occurs together in the same block 5 times. The design satisfies the additional requirement that for any choice of an initial block, there are 16 blocks which have no treatments in common with the initial block; this was shown to imply that each of the 60 remaining blocks has exactly 2 treatments in common with the initial block. There is no loss of generality in assigning numbers 1 , 2 , 3 , 4 , 5 , 6 to the treatments in the initial block, then considering separately the set of 60 blocks each of which contains 2 of the treatments 1-6 and the set of 16 blocks which contain only treatments 7-22. Denote these by Set I and Set II respectively. In what follows the number of special cases will be reduced greatly by showing that certain cases are equivalent under changes of notation, that is, under permutation of treatments and/or permutation of blocks. The reader may verify that the only treatments or blocks involved in any of these permutations are those which play symmetrical roles in the part of the design which has been previously specified. At the present stage this permits any permutation of treatments 1-6 among themselves and any permutation of treatments 7-22 , but no interchange of treatments not in the same set. Similarly, blocks may be renumbered within Set I or Set II but the sets of blocks will be left intact.

Repeated use will be made of the fact that the number of treatments common to any two blocks of this design must be either 2 or 0 .

Each of the 15 pairs of treatments 1-6 occurs once in the initial block, not at all in the 16 blocks of Set II, and $\lambda = 5$ times in all, so that it must occur in exactly four of the 60 blocks of Set I. Since none of the 60 blocks can contain more than 2 of treatments 1-6, the blocks must fall into 15 sets of 4, blocks in different sets containing different pairs of treatments 1-6, and the 4 blocks of each set containing the same pair. No two blocks in the same set of 4 can have more than 2 treatments in common, meaning that no two of them can contain the same treatment of set 7-22. Therefore each of the 16 treatments in this set must occur just once in each of the 15 sets of 4 blocks. It will be convenient to identify the sets of 4 blocks by the pair of numbers of set 1-6 which they have in common. The symbol /i,j/ will be used to denote the set of 4 blocks containing the treatments i and j of set 1-6. Since each treatment must occur 21 times in the BIB design and 15 occurrences of each of treatments 7-22 have been accounted for, each must occur just 6 times in the 16 blocks of Set II.

Let M denote the submatrix of S whose rows are determined by treatments 7-22 and whose columns correspond to the blocks of Set II. M is a 16 x 16 incidence matrix whose row totals are all 6 by the final sentence of the preceding paragraph, whose column totals are all equal to 6, the number of treatments in a block, and whose column inner products are at most equal to 2. This means that the symmetric matrix M'M has diagonal entries equal to 6 and 15 off-diagonal entries in each row which are at most equal to 2. It is not difficult to show that (2.42) holds for any incidence matrix with equal row totals r and equal column totals k, whether it is the matrix of a BIB design or not. This shows in the present example that the sum of the 15 off-diagonal entries of each row of $M'M$ is 30, proving that each of these elements is equal to 2. This proves that M' is the matrix of a BIB design with parameters $v = b = 16$, $r = k = 6$, $\lambda = 2$. Since this is a symmetric design, a well-known result originally obtained by Fisher shows that the column inner products of M' are also all equal to 2, the same as the row inner products. This is the same as saying that M is the matrix of an equivalent BIB design. This is useful in the construction of Set II of 16 blocks. Also, since each pair of treatments 7-22 occurs 2 times in Set II and must occur 5 times in all, each pair must occur exactly 3 times in the 60 blocks of Set I, a fact which is helpful in the construction of Set I. The fact is not essential, and rather than digress to prove that (2.42) can be applied, the construction of Set I will be based on the fact that no two blocks of the set can have more than 2 treatments in common. The fact that Set II determines a BIB design then follows without any appeal to (2.42).

The numbering of treatments 7-22 will be chosen so that $\underline{/1,2/}$ has the form

$$\begin{array}{cccccc} 1 & 2 & 7 & 8 & 9 & 10 \\ 1 & 2 & 11 & 12 & 13 & 14 \\ 1 & 2 & 15 & 16 & 17 & 18 \\ 1 & 2 & 19 & 20 & 21 & 22 \end{array}$$

Next consider the sets $\underline{/3,4/}$, $\underline{/3,5/}$, $\underline{/4,5/}$. Each contains the treatments 7-22 once each, and the rows of each must have either 2 treatments or no treatments in common with the blocks of $\underline{/1,2/}$. This means that the last 4 numbers of any row of /1,2/ must occur in pairs in two of the rows of each of /3,4/ , /3,5/ , /4,5/ . Rearrange rows of each of these sets if necessary so that the first blocks of each set contain treatments 7 , 8 , 9 , 10 , with 7 in the first row. The blocks 3 4 7 _ _ _ , 3 5 7 _ _ _ , 4 5 7 _ _ _ already have the maximum number of treatments in common, so they must contain the treatments 8 , 9 , 10 in some order. Renumber these treatments if necessary so that they occur in the order given. This determines the following portions of sets /3,4/ , /3,5/ , /4,5/ .

3	4	7	8	_	_	3	5	7	9	_	_	4	5	7	10	_	_
3	4	9	10	_	_	3	5	8	10	_	_	4	5	8	9	_	_
3	4	_	_	_	_	3	5	_	_	_	_	4	5	_	_	_	_
3	4	_	_	_	_	3	5	_	_	_	_	4	5	_	_	_	_

Renumber treatments 11-22 if necessary so that the remaining treatments in block 3 4 7 8 _ _ are 11 and 12 . Then a repetition of the reasoning used for treatments 7 , 8 , 9 , 10 shows that the pairs of treatments 11 13 and 11 14 must occur in sets /3,5/ and /4,5/ ; renumber treatments 13 and 14 if necessary so that 11 13 occurs in /3,5/ . Since no block can have more than 2 treatments in common with the block 3 4 7 8 11 12 , no other block can have an 11 or 12 along with a 3 or 4 and a 7 or 8 . This determines the following.

3	4	7	8	11	12	3	5	7	9	_	_	4	5	7	10	_	_
3	4	9	10	_	_	3	5	8	10	_	_	4	5	8	9	_	_
3	4	_	_	_	_	3	5	11	13	_	_	4	5	11	14	_	_
3	4	_	_	_	_	3	5	12	14	_	_	4	5	12	13	_	_

There is now a choice of placing the pair 13 14 in the second block of set /3,4/ or in another block, say the third. The latter case will be investigated first. Renumbering treatments of the sets 15 , 16 , 17 , 18 and 19 , 20 , 21 , 22 if necessary and remembering that not more than two treatments of either of these sets can occur in the same block of /3,4/ , the following is obtained.

3	4	7	8	11	12
3	4	9	10	15	16
3	4	13	14	19	20
3	4	17	18	21	22

Since neither 15 nor 16 can now fall in a block of /3,5/ which contains a 9 or 10 , the set 15 , 16 , 17 , 18 must fall in the last two blocks of /3,5/ , in the following arrangement after renumbering them if necessary.

3	5	7	9	–	–
3	5	8	10		
3	5	11	13	15	17
3	5	12	14	16	18

The pair 15 18 must now occur in some block of /4,5/, but it is easy to verify that any such block would then have 3 treatments in common with some block of /3,4/ or /3,5/ , showing that this case is impossible, and that treatments 13 14 must occur in the second block of /3,4/ . Treatments 7-14 , which occur in blocks 1 2 7 8 9 10 and 1 2 11 12 13 14 of /1,2/ , have now been assigned to the blocks of /3,4/ . Notation was chosen so that treatments 7 , 8 , 11 and 12 all occurred in the same block. This was found to imply that treatments 9 , 10 , 13 and 14 all occur in the same block. By symmetry, this shows that if treatments from any two blocks of set /1,2/ occur in the same block of any of sets /3,4/ , /3,5/ , /4,5/ , then all eight treatments (other than 1 and 2) of those two blocks must occur in the same two blocks of the set. This fact is useful in completing the blocks of /3,4/ , /3,5/ and /4,5/ .

Treatments 15-22 may be renumbered if necessary so that number 15 is given to one of the remaining treatments in block 3 5 7 9 _ _ . Then after possible further changes of notation, the following is quickly obtained.

3	4	7	8	11	12		3	5	7	9	15	17		4	5	7	10	19	22	or	20	21
3	4	9	10	13	14		3	5	8	10	16	18		4	5	8	9	20	21	or	19	22
3	4	15	16	19	20		3	5	11	13	19	21		4	5	11	14	15	18	or	16	17
3	4	17	18	21	22		3	5	12	14	20	22		4	5	12	13	16	17	or	15	18

The blocks of /4,5/ may be completed in any of four ways. There seems to be no immediate way of reducing this number of cases by choice of notation, and from this point on only the first case will be considered (for each block, the first pair of the two possible pairs listed). It may be verified that the other three cases give similar results.

No - two of these fail. At most two solutions. 2-19-64

The next blocks to be constructed are those in sets /3,6/ , /4,6/ , /5,6/ . The block containing treatments 3 6 7 must contain another treatment of set 7 , 8 , 9 , 10 ; comparison with blocks 3 4 7 8 11 12 and 3 5 7 9 15 17 shows that the treatment must be 10 . Further comparison with the blocks already constructed shows that the block must also contain treatments 20 and 21 . This sort of argument quickly determines that the three sets of blocks are as follows.

3	6	7	10	20	21	4	6	7	9	16	18	5	6	7	8	13	14
3	6	8	9	19	22	4	6	8	10	15	17	5	6	9	10	11	12
3	6	11	14	16	17	4	6	11	13	20	22	5	6	15	16	21	22
3	6	12	13	15	18	4	6	12	14	19	21	5	6	17	18	19	20

Each block of sets /1,3/ , /1,4/ , /1,5/ , /1,6/ , /2,3/ , /2,4/ , /2,5/ , /2,6/ already has treatment 1 or 2 in common with each block of set /1,2/ , and must contain just one treatment of the set 7 , 8 , 9 , 10, one treatment of set 11 , 12 , 13 , 14 and so on. Comparison with either block 3 4 7 8 11 12 or 5 6 7 8 13 14 shows that treatments 13 and 14 , in some order, must occur in blocks 1 3 7 _ _ _ and 2 3 7 _ _ _. Treatments 1 and 2 occupy symmetrical positions in the part of the design which has been specified so far, and they may be interchanged if necessary to give the blocks

1 3 7 13 _ _ and 2 3 7 14 _ _ .

Further comparison with blocks containing treatments 3 , 7 and 13 determines the following.

1	3	7	13	16	22	2	3	7	14	18	19
1	3	8	14	__	__	2	3	8	13	__	__
1	3	9	__	18	__	2	3	9	__	16	__
1	3	10	__	__	19	2	3	10	__	__	22

Comparison with blocks already containing the pairs 3 14 , 3 18 or 3 19 then determines the remaining treatments. Entirely similar considerations determine the blocks of sets /1,4/ , /1,5/ , /1,6/ , /2,4/ , /2,5/ , /2,6/ , completing the construction of all the blocks of Set I.

Each pair of treatments must occur together in exactly $\lambda = 5$ blocks. Enumeration shows that each of the pairs of treatments 7 to 22 has occurred together 3 times in the blocks of Set I, meaning that each pair must occur twice in the 16 blocks of Set II. It has already been noted that each of treatments 7 to 22 occurs 6 times in the set. This verifies that the blocks of Set II form a BIB design with parameter values

$$v = b = 16 , \quad r = k = 6 , \quad \lambda = 2 .$$

Consider the two blocks of Set II which contain treatments 7 and 8 ,

7 8 _ _ _ _
7 8 _ _ _ _ .

Comparison with blocks of Set I which contains treatments 7 and 8 shows that the remaining eight treatments in these two blocks must be treatments 15 to 22 in some order. Comparison with block 4 5 11 14 15 18 shows that the pair 15 18 must occur in the same one of these two blocks. Comparison with block 3 4 15 16 19 20 shows that block 7 8 15 18 _ _ must contain either 19 or 20 ; comparison with block 2 3 7 14 18 19 shows that it cannot contain 19 ; comparison with block 3 6 7 10 20 21 , for instance , then shows that the remaining treatment in the block must be 22 , determining the structure

7 8 15 18 20 22
7 8 16 17 19 21 .

A similar procedure determines the remaining blocks of Set II. This completes the construction of the 77 blocks of the BIB design desired. The blocks are listed on the following page.

The following are the blocks of the balanced incomplete block design just constructed, with parameter values $v = 22$, $r = 21$, $k = 6$, $b = 77$, $\lambda = 5$.

Initial block: 1 2 3 4 5 6

Set I:

1	2	7	8	9	10
1	2	11	12	13	14
1	2	15	16	17	18
1	2	19	20	21	22

3	4	7	8	11	12	3	5	7	9	15	17	4	5	7	10	19	22
3	4	9	10	13	14	3	5	8	10	16	18	4	5	8	9	20	21
3	4	15	16	19	20	3	5	11	13	19	21	4	5	11	14	15	18
3	4	17	18	21	22	3	5	12	14	20	22	4	5	12	13	16	17

3	6	7	10	20	21	4	6	7	9	16	18	5	6	7	8	13	14
3	6	8	9	19	22	4	6	8	10	15	17	5	6	9	10	11	12
3	6	11	14	16	17	4	6	11	13	20	22	5	6	15	16	21	22
3	6	12	13	15	18	4	6	12	14	19	21	5	6	17	18	19	20

1	3	7	13	16	22	2	3	7	14	18	19
1	3	8	14	15	21	2	3	8	13	17	20
1	3	9	11	18	20	2	3	9	12	16	21
1	3	10	12	17	19	2	3	10	11	15	22

1	4	7	14	17	20	2	4	7	13	15	21
1	4	8	13	18	19	2	4	8	14	16	22
1	4	9	12	15	22	2	4	9	11	17	19
1	4	10	11	16	21	2	4	10	12	18	20

1	5	7	12	18	21	2	5	7	11	16	20
1	5	8	11	17	22	2	5	8	12	15	19
1	5	9	14	16	19	2	5	9	13	18	22
1	5	10	13	15	20	2	5	10	14	17	21

1	6	7	11	15	19	2	6	7	12	17	22
1	6	8	12	16	20	2	6	8	11	18	21
1	6	9	13	17	21	2	6	9	14	15	20
1	6	10	14	18	22	2	6	10	13	16	19

Set II:

7	8	15	18	20	22	8	10	11	14	19	20
7	8	16	17	19	21	8	10	12	13	21	22
7	9	11	14	21	22	9	10	15	18	19	21
7	9	12	13	19	20	9	10	16	17	20	22
7	10	11	13	17	18	11	12	15	17	20	21
7	10	12	14	15	16	11	12	16	18	19	22
8	9	11	13	15	16	13	14	15	17	19	22
8	9	12	14	17	18	13	14	16	18	20	21

This design was required to have the property that each block be disjoint from 16 other blocks. It may be verified that this is satisfied. Therefore matrix S exists and may be taken as the incidence matrix of this design. The columns of matrix S fall in columns 23 to 99 of the association matrix A_1 , and for discussion of scheme #94 it is convenient to number the blocks of the BIB design from 23 to 99 . Then for any θ from 23 to 99 the six treatments in block θ correspond to six of the first associates of θ . The remaining 16 first associates correspond to the 16 blocks of the design which have no treatments in common with block θ . In this way the first associates of all of treatments 23-99 are specified. The first associates of treatment 0 are treatments 1 to 22 . The first associates of any treatment ϕ from 1 to 22 are 0 and the 21 treatments corresponding to the blocks of the design which contain ϕ . Since it was shown that the properties of matrix S implied that A_1 had the properties required for scheme #94, further examination of the 100 x 100 matrix is unnecessary.

This completes the construction of scheme #94. There are at most four solutions to the association scheme, corresponding to the four choices for the structure of the blocks of set /4,5/ . It is not known whether any of the four solutions are equivalent under some permutation of treatments.

For the construction of scheme #64, the blocks of the BIB design may be numbered from 1 to 77 . The first associates of treatment θ then correspond to the 16 blocks which have no treatment in common with block θ .

Because of the ease with which they may be constructed from the BIB design, schemes #94 and #64 will not be listed explicitly.

If there exists a finite Euclidean plane with 10 points on a line and 11 lines on each point, it is conceivable that scheme #94 has a geometrical interpretation similar to that discussed in Section 3.2 for schemes #6 and #51. If so, the 22 first associates of a point θ would be two suitably chosen points on each of the 11 lines through θ. The choice of the two points on each line might be a difficult problem. Of considerable interest if true, but presumably even more difficult to prove or disprove, are the conjectures that the existence of the geometry is a necessary condition for the existence of the association scheme, or that the geometry can be constructed from the scheme. It thus appears that there is a possibility, but only that, that scheme #94 will shed some light on the unsolved problem of constructing orthogonal 10×10 squares. Several by-products of the scheme will now be mentioned. It has been pointed out that any of the 100 possible choices of an initial treatment in the association scheme leads to a different submatrix T which is an association matrix for scheme #64 and a different submatrix S which is a solution of the balanced design with 77 blocks. The 77 blocks correspond to the second associates of the initial treatment. Any choice of an initial column of S to be taken as an initial block leads to a different submatrix M which was shown to give a solution of the BIB design with $v = 16$ and $r = 6$. A distinct submatrix M of A_1 is determined by every choice of a pair of second associates, and there are 3850 pairs of second associates. There are an equal number of sets of 60 blocks with the properties of Set I. The blocks of Set I may be partitioned into the 15 sets denoted by /i,j/ . In discussing these blocks the treatments will be numbered as in the constructed example. The 16 treatments from 7 to 22 fall by fours into the 4 blocks of each set /i,j/ . If the cells of a square 4 x 4 array are numbered from 7 to 22 , an array of the letters A , B , C , D may be formed from each set /i,j/ by assigning the same letter to cells which correspond to treatments in the same block. There are five sets /i,j/ for any fixed value of i , for example /1,3/ , /2,3/ , /3,4/ , /3,5/ , /3,6/ . If the arrays formed from /i,j/ and /i,k/ , where $j \neq k$, are superimposed, the number of cells in which a particular ordered pair of letters occurs is equal to the number of treatments of the set 7-22 which the corresponding blocks of /i,j/ and /i,k/ have in common. The two blocks are not disjoint, having treatment i in common, so must have exactly 2 treatments in common, including one treatment of the set 7-22 . Each pair of letters therefore occurs in exactly one cell when the two squares are superimposed, meaning that the squares are orthogonal. The five sets /i,j/ for a fixed value of i form a complete orthogonal set of 4 x 4 squares. The fifteen squares defined by the 60 blocks of Set I include six complete orthogonal sets, each square occurring in two of the orthogonal sets. To the best of the author's knowledge, this configuration is new, and it is included in Section A.5 of the appendix.

IV. THE STRUCTURE OF LATIN SQUARE TYPE ASSOCIATION SCHEMES

4.1 Preliminary Discussion of Uniqueness, with Counter-Examples.

By definition, the terms "Latin square type" and "L_g" apply only to association schemes in which first associates may be defined by the rows of a set of g orthogonal squares. An association scheme with the parameter values of an L_g scheme but constructed by some other method does not necessarily satisfy this requirement. Some of the properties of association schemes of the Latin square series which have been treated in the previous chapters depend on the existence of the set of orthogonal squares; other properties hold as a consequence of the numerical values of the parameters of the schemes. For example, a scheme constructed by any method for which the values of v, n_i and p^i_{12} are the same as for an L_g scheme will have the property $\alpha_i = n_i$ which was discussed in Section 3.1. On the other hand, if an association scheme with these parameter values is constructed by any method other than the actual orthogonal squares, the compact representation of the scheme by means of the square array of numbers may not be available. If there are different constructions of the same scheme which are not equivalent under some permutation of treatments, there may be designs which are impossible with one scheme but can be constructed with another. Design #7-3 is an example. Important results on the existence of some types of incomplete block designs have been obtained by use of the Hasse-Minkowski invariants of the symmetric matrix NN' [30], [15]. In order to compute these invariants it is necessary to compute certain minor determinants of the matrix, and designs corresponding to inequivalent association schemes would have to be treated as separate special cases. (The Hasse-Minkowski invariant is not taken up in this thesis.) For these reasons it may be important to know whether association schemes are unique, and in particular, whether the existence of an association scheme with the parameter values of the L_g series implies the existence of a set of g orthogonal squares by which first associates in the scheme may be defined. In other words, does the set of L_g association schemes with a particular set of parameter values exhaust the set of all association schemes with the same parameter values? If this is the case, the L_g scheme will be said in this chapter to be unique.

The analogous question for group divisible designs was answered in the affirmative by Bose and Connor [7]. The definition of a group divisible scheme for $v = mn$ treatments uses an arrangement of the treatments into m groups of n treatments each and leads to a certain set of parameter values; it is shown in [7] that the existence of a scheme with these parameter values implies the partition of the treatments into the m groups. This includes the special case L_1 of the present question. It will appear in this chapter that the question is more complicated for the L_g schemes in general. Some counter-examples to be presented in this section will show that the analog of the Bose-Connor theorem cannot be true in full generality. It will be proved in Section 4.2 that with a single exception a scheme with the parameter values of an L_2 scheme implies the existence of the two orthogonal squares. In section 4.3 the result is extended to Latin square type schemes with 3 or more constraints, with a larger number of possible exceptions. The theorem for $g \geq 4$ makes the existence of the scheme equivalent to the existence of a set of two or more orthogonal Latin squares, a connection which could be useful in the study of such sets. Some of the results of the chapter apply to association schemes of any type or to more general incidence matrices.

A proof of the uniqueness of an L_g association scheme in that first associates can be defined only by the rows of some set of orthogonal squares has no bearing on the question of uniqueness of the set of orthogonal squares. It was shown in Section 2.1 that all pairs of $n \times n$ orthogonal squares are equivalent except for numbering of treatments, settling the question in the case of L_2. For $g \geq 3$, the number of sets of g mutually orthogonal squares which can be used to construct L_g schemes depends on the enumeration of Latin squares and sets of orthogonal Latin squares, and will not be considered here. Also omitted from any direct consideration will be any differences in the properties of solutions of the same association scheme based on distinct sets of orthogonal squares.

The interpretation of an association scheme or its incidence matrix in terms of a linear graph was mentioned in Section 1.3. Any symmetric incidence matrix with 0's on the main diagonal may be used to define a graph by identifying points with rows and columns, then joining points μ and ν of the graph if and only if the elements in the μ, ν and ν, μ positions of the matrix are 1's. In the case of the incidence matrix of first associates of an association scheme, the points of the graph are identified with the treatments. A pair of points which are joined by a line is identified with a pair of first associates, either pair being indicated in the matrix by a pair of 1's symmetrically located with respect to the main diagonal. In this chapter, terms such as point and line will be used interchangeably with the corresponding terms for association schemes. A set of k treatments which are pairwise first associates will be identified with a set of k points each pair of which is joined by a line. This configuration in the graph will be termed a complete configuration on k points, a complete k-point, or simply a k-point. Many of the properties of the association scheme correspond to analogous properties of the graph, as has been mentioned in Section 1.3. Therefore some theorems proved in this dissertation for partially balanced designs have applications to linear graphs. Many of the known theorems of graph theory are potentially useful in the study of designs, but no applications of them will be made in this chapter. The graphs encountered here are highly special, owing to the properties of partial balance in the designs, and do not seem to have received much attention.

Examples will now be given of association schemes which have the parameter values of L_2, L_3 and L_4 schemes but in which it is not possible to define first associates by any set of orthogonal squares. The construction of these schemes is based on the following remark, already made in Section 2.1.

Remark 1: If the designation of first and second associates is interchanged in a Latin square type scheme with g constraints on $v = n^2$

treatments, the resulting association scheme has the parameter values of a Latin square type scheme with $f = n-g+1$ constraints. The association matrix A_1 in the L_f scheme may be taken as the matrix A_2 of the L_g scheme.

The demonstration that first associates in the schemes constructed cannot be defined by orthogonal squares makes use of the following remark.

Remark 2: If first associates in an association scheme are those treatments which occur with the same letter in one of a set of $n \times n$ orthogonal squares, then every pair of first associates is contained in a complete n-point.

Example 1 . Let an L_3 scheme for $v = 16$ treatments be defined by rows, columns and letters of the following 4×4 Latin square, which was used as an example in Section 2.1. The array of numbers is also given for reference.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array} \qquad \begin{array}{cccc} A & B & C & D \\ B & C & D & A \\ C & D & A & B \\ D & A & B & C \end{array}$$

Dualize to form a scheme in which first associates are the same as the second associates of the original scheme, namely those treatments not occurring in the same row, the same column, or with the same letter of the Latin square. Thus in the dual scheme the first associates of treatment 1 are 6 , 7 , 10 , 12 , 15 , 16 and the first associates of treatment

6 are 1 , 4 , 11 , 12 , 13 , 15 . By remark 1 , this scheme has the parameter values of an L_2 scheme. Treatments 1 and 6 are first associates and if the scheme corresponds to any set of two orthogonal squares, then by Remark 2 , treatments 1 and 6 must be contained in a set of $n = 4$ treatments which are pairwise first associates. The remaining two treatments in such a set would have to be the two treatments which are the common first associates of treatments 1 and 6 . These are treatments 12 and 15 . But treatments 12 and 15 are not first associates in the dual scheme, hence the first associates 1 and 6 are not contained in any set of 4 pairwise first associates and by remark 2 the L_2 scheme cannot correspond to any set of two orthogonal squares.

Example 2. Let an L_3 scheme for $v = 25$ treatments be defined by rows, columns, and letters of the following 5 x 5 Latin square. The array of numbers is also given for reference.

1	2	3	4	5	A	B	C	D	E
6	7	8	9	10	B	A	D	E	C
11	12	13	14	15	C	D	E	A	B
16	17	18	19	20	D	E	B	C	A
21	22	23	24	25	E	C	A	B	D

Form the dual scheme by interchanging the designation of first and second associates. The new scheme will have the parameter values of an L_3 scheme, by Remark 1 . First associates in the new scheme are those treatments <u>not</u> occurring in the same row or column or with the same letter of the Latin square. Thus treatments 1 and 8 are first associates in the new scheme. If first associates in this scheme can be defined by a set of three 5 x 5 orthogonal squares, then by Remark 2 treatments 1 and 8 must be included in a complete 5-point. The remaining three treatments in the 5-point must be first associates of both of treatments 1 and 8 . The common first associates of treatments 1 and 8 are treatments 15 , 17 , 19 , 22 , 24 . In order for three of these to form with 1 and 8 a complete 5-point, it is necessary and sufficient that they be pairwise first associates. The submatrix of the incidence matrix of first associates corresponding to the five treatments is the following.

$$\begin{array}{c c} & \begin{array}{ccccc} 15 & 17 & 19 & 22 & 24 \end{array} \\ \begin{array}{c} 15 \\ 17 \\ 19 \\ 22 \\ 24 \end{array} & \left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

It is easily verified that there is no set of three pairwise first associates among the five treatments. Therefore remark 2 is violated and there exists no set of three orthogonal 5 x 5 squares by which first associates in the scheme may be defined.

Example 3. Let an L_3 scheme for $v = 36$ treatments be defined by rows, columns and letters of any 6 x 6 Latin square. Now dualize by interchanging the designation of first and second associates, obtaining a scheme which by Remark 1 has the parameter values of an L_4 scheme for 36 treatments. If this scheme corresponded to any set of four 6 x 6 orthogonal squares, it would imply the existence of two orthogonal 6 x 6

Latin squares, which has been proved impossible [35] [20]. Therefore the scheme cannot correspond to any set of orthogonal squares.

4.2 On the Uniqueness of L_2 Association Schemes

The proof of uniqueness of certain L_g schemes is begun in this section and will be completed in the case of L_2 schemes. Lemmas 4.1 and 4.2 and Theorem 4.2 are proved for schemes having the parameter values of Latin square schemes with any number of constraints, showing what the existence of one or more complete n-points in the scheme implies for the rest of the scheme. Lemma 4.3, applying to the incidence matrix of a scheme of any type with two associate classes, brings out a useful fact about the structure of a certain submatrix. In Theorems 4.2 and 4.3 these results are specialized to the case of Latin square type schemes with two constraints and with the exception already noted in the preceding section they are shown to be unique in the sense being used in this chapter. It is shown that additional methods must be used in the case of three or more constraints.

The parameter values of L_g schemes, given in (2.9), are repeated here for easy reference.

$$v = n^2 , \qquad P_1 = \begin{bmatrix} g^2 - 3g + n & (g-1)(n-g+1) \\ (g-1)(n-g+1) & (n-g)(n-g+1) \end{bmatrix},$$

$$n_1 = g(n-1) ,$$

$$n_2 = (n-g+1)(n-1) , \qquad P_2 = \begin{bmatrix} g(g-1) & g(n-g) \\ g(n-g) & (n-g)^2 + g-2 \end{bmatrix}.$$

LEMMA 4.1. In any association scheme for $v = n^2$ treatments which has the parameter values of a Latin square type scheme with g constraints, if there is a set of n treatments which form a complete n-point, then each of the remaining n^2-n treatments is a first associate of exactly

$g-1$ treatments of the set of n .

PROOF: The set of n treatments specified in the theorem will be called a complete n-point. Treatments may be numbered so that the treatments in the n-point correspond to the first n rows and n columns of the incidence matrix A_1 of first associates; then the leading $n \times n$ principal minor, denoted by B , will have 1's in all off-diagonal positions. Submatrices C , C' , and D are defined by the following diagram of A_1 in partitioned form.

$$A_1 = \left[\begin{array}{c:c} B & C \\ \hdashline C & D \end{array}\right] \begin{array}{l} n \text{ rows }, \\ \\ n^2 - n \text{ rows }. \end{array}$$

The n rows of C correspond to the n treatments in the n-point. The n^2-n columns of C correspond to the remaining treatments.

The row totals of A_1 are equal to $n_1 = g(n-1)$, while the inner product of a pair of rows corresponding to first associates is equal to $p_{11}^1 = g^2-3g + n$. B has row totals $n-1$ and row inner products $n-2$. By difference, C has row totals $(g-1)(n-1)$ and row inner products $g^2-3g + n - (n-2) = g^2-3g + 2$. The total number of 1's in all n rows of C is $n(g-1)(n-1) = (n^2-n)(g-1)$. The total of the u^{th} column of C will be denoted by k_u and the mean column total by $\bar{k}$. Clearly $\bar{k} = g-1$. The sum of the inner products of all pairs of distinct rows of C will now be computed in two ways. The number of such pairs of rows is $\binom{n}{2}$ and each inner product has the value $g^2-3g + 2$, giving the total $\binom{n}{2}(g^2-3g + 2)$. The contribution of the elements of a single column to the total is equal to the number $\binom{k_u}{2}$ of pairs of 1's in the column and the total may be obtained by summing over all columns. Equating the two expressions for the total,

$$\sum_u \binom{k_u}{2} = \binom{n}{2}(g^2-3g+2) \quad ,$$

$$\sum_u (k_u^{\,2} - k_u) = (n^2-n)(g^2-3g+2) \quad ,$$

$$\frac{1}{n^2-n} \sum_u k_u^{\,2} - \bar{k} = g^2-3g+2 \quad ,$$

$$\frac{1}{n^2-n} \sum_u k_u^{\,2} = g^2-3g+2+(g-1) = (g-1)^2 \ .$$

The variance of the k_u will now be computed.

$$\begin{aligned} \mathrm{Var}(k_u) &= \frac{1}{n^2-n} \sum_u k_u^{\,2} - (\bar{k})^2 \\ &= (g-1)^2 - (g-1)^2 = 0 \ . \end{aligned}$$

Therefore the column totals k_u of C must all be equal to their mean value $g-1$. This has the interpretation that each of the n^2-n treatments corresponding to the columns of C is the first associate of exactly $g-1$ of the n treatments corresponding to the rows of C. This completes the proof of Lemma 4.1.

COROLLARY 4.1. No association scheme with L_g parameter values, $g \leq n$, contains a complete configuration with more than n treatments.

It is natural to attempt to prove comparable results from the hypothesis of a complete configuration with fewer than n points, in particular

$n-1$. The author has found that the method used in the proof of the lemma is much weaker in this case, and has been unable to demonstrate any regularity in the column totals of submatrix C on this hypothesis.

LEMMA 4.2. If an association scheme with the parameter values of the Latin square series with g constraints contains a complete n-point and a complete h-point which is not a subgraph of the n-point but has at least two points in common with it, then $h \leq (g-1)^2$.

PROOF: By Lemma 4.1, each of the n^2-n treatments not in the n-point is a first associate of just $g-1$ treatments in the n-point. No set of pairwise first associates which contains treatments outside of the n-point can contain more than $g-1$ treatments of the n-point.

Consider two treatments which are in both the n-point and the h-point. Each other treatment of the h-point must be a common first associate of these two. In all the two treatments have $p_{11}^1 = g^2-3g+n$ common first associates. Of these, $n-2$ are in the n-point, leaving g^2-3g+2 which are outside of the n-point. Therefore at most g^2-3g+2 treatments of the h-point are outside of the n-point.

The largest possible number of treatments which the h-point can contain, in the n-point and outside of it, is therefore
$(g-1) + (g^2-3g+2) = (g-1)^2$, completing the proof.

This lemma shows that any set of $h > (g-1)^2$ treatments which form a complete configuration cannot have more than one treatment in common with any complete n-point unless all h treatments are contained in the n-point.

THEOREM 4.1. If an association scheme has the parameter values of the L_g series, if each pair of treatments is contained in a set of n treatments which are pairwise first associates, and if $n > (g-1)^2$, then there exists a set of g orthogonal $n \times n$ squares which may be used to define first associates in the scheme.

PROOF: The language of linear graphs will be used. Treatments will be referred to as points, pairs of first associates as lines, and a set of k treatments which are pairwise first associates will be termed a "complete k-point", or briefly a "k-point".

By hypothesis, each line on an arbitrary initial point of the graph is contained in a complete n-point. Each such n-point contains exactly $n-1$ lines through the initial point. By Lemma 4.2, since $n > (g-1)^2$, no line through the initial point can be in more than one n-point, and by Corollary 4.1, no complete configuration can contain more than $n-1$ lines through the initial point. Therefore the set of $n_1 = g(n-1)$ first associates of the initial point must fall into disjoint sets of $n-1$, each forming a complete n-point with the initial point. Therefore there are just g n-points containing the initial point; it was an arbitrary point and the same remark applies to each of the n^2 points. From this or from the remark that there is just one n-point on each line, it follows that there are exactly ng n-points in the entire graph.

Denote an arbitrary initial n-point by A. Each point of A lies on $g-1$ additional n-points, for a total of $n(g-1)$ n-points intersecting A. If any of these were counted twice, for different points of A, it would have more than one point in common with A, which is impossible by Lemma 4.2. Therefore A intersects exactly $n(g-1)$ of the remaining $ng-1$ n-points, leaving $n-1$ of them with which it has no points in common.

Take any n-point B disjoint from A. By Lemma 4.1, a particular point of B is joined by lines to $g-1$ points of A. Each of these lines lies on an n-point. If two of them lie on the same n-point, then that n-point would have more than one point in common with A, which is impossible, so the $g-1$ lines determine the remaining $g-1$ n-points through the point of B, for a total of $n(g-1)$ n-points intersecting A and B and distinct from both. If any of these were counted twice, for different points of B, it would have two points in common with B, which is impossible. Therefore the $n(g-1)$ n-points are all distinct, and must be the entire set of n-points which intersect B. Therefore the $n-1$ n-points disjoint from B must be A and the other $n-2$ in the set of n-points disjoint from A. B was any one of the $n-1$ n-points disjoint from A, so each of these n-points is disjoint from all the others, and the whole set of n are mutually disjoint, exhausting the n^2 points.

The argument carried out for A applies to any of the g n-points through an arbitrary point of the graph, showing that there are g systems of n "parallel" n-points. n-points in the same system are disjoint; any two in different systems have just one point in common. Let the n^2 points be identified in a 1-to-1 manner with the cells of an $n \times n$ square array. Identify the n "parallel" n-points of one of the

g systems with n distinct letters, and form a square array of these n letters by assigning each letter to those cells of the array corresponding to the points of the n-point. The g squares which may be formed in this way satisfy the requirements of a set of g orthogonal $n \times n$ squares, and may be used to define first associates in the association scheme.

Some terminology to be used in the next lemma and in some of the theorems of this chapter will now be introduced. In any association scheme with two associate classes, choose notation so that treatments 1 and 2 are first associates and number the remaining treatments so that the treatments in each of the following sets have consecutive numbers and the four sets are numbered in the order listed.

Set 1: p_{11}^1 common first associates of treatments 1 and 2 ,
Set 2: p_{12}^1 treatments which are first associates of treatment 1 and second associates of treatment 2 ,
Set 3: p_{12}^1 treatments which are first associates of treatment 2 and second associates of treatment 1 ,
Set 4: p_{22}^1 common second associates of treatments 1 and 2 .

In the incidence matrix A_1 of first associates, the treatments of each set correspond to a set of consecutive rows and columns, and the four sets of treatments determine sixteen submatrices, which are indicated in the diagram below. Orders of the submatrices are shown in the margins of the diagram. The notation $A_{\mu\nu}$ will be used for the submatrix with rows corresponding to Set μ and columns corresponding to Set ν .

$$(4.1)\qquad A_1 = \left[\begin{array}{cc|c|c|c|c}
0 & 1 & 1 \cdot\cdot 1 & 1 \cdot\cdot 1 & 0 \cdot\cdot 0 & 0 \cdot\cdot 0 \\
1 & 0 & 1 \cdot\cdot 1 & 0 \cdot\cdot 0 & 1 \cdot\cdot 1 & 0 \cdot\cdot 0 \\
\hline
1 & 1 & & & & \\
\cdot & \cdot & A_{11} & A_{12} & A_{13} & A_{14} \\
\cdot & \cdot & & & & \\
1 & 1 & & & & \\
\hline
1 & 0 & & & & \\
\cdot & \cdot & A_{21} & A_{22} & A_{23} & A_{24} \\
\cdot & \cdot & & & & \\
1 & 0 & & & & \\
\hline
0 & 1 & & & & \\
\cdot & \cdot & A_{31} & A_{32} & A_{33} & A_{34} \\
\cdot & \cdot & & & & \\
0 & 1 & & & & \\
\hline
0 & 0 & & & & \\
\cdot & \cdot & A_{41} & A_{42} & A_{43} & A_{44} \\
\cdot & \cdot & & & & \\
0 & 0 & & & &
\end{array}\right]
\begin{array}{l}
\\ \\ \hline \\ \text{Set } 1\,, \\ p^1_{11}\ \text{rows.} \\ \\ \hline \\ \text{Set } 2\,, \\ p^1_{12}\ \text{rows.} \\ \\ \hline \\ \text{Set } 3\,, \\ p^1_{12}\ \text{rows.} \\ \\ \hline \\ \text{Set } 4\,, \\ p^1_{22}\ \text{rows.} \\ \\
\end{array}$$

The number of 1's in a row of $A_{\mu\nu}$ will be denoted by $t_{\mu\nu}$; the row totals of a submatrix are not necessarily equal but the symbol will be used only in statements which are true for all rows. The symbols $T_{\mu\nu}$ and $Z_{\mu\nu}$ will be used, respectively, for the total number of 1's in $A_{\mu\nu}$ and the total number of 0's in $A_{\mu\nu}$ which are not on the main diagonal of A_1 .

LEMMA 4.3. If the incidence matrix A_1 of first associates in an association scheme with two associate classes is partitioned in the form of (4.1) , then the number Z_{11} of off-diagonal 0's in submatrix A_{11} satisfies the following inequality.

$$(4.2)\qquad Z_{11} \leq p^1_{12}(p^2_{11} - 1) \quad .$$

PROOF: Since A_{11} is a $p^1_{11} \times p^1_{11}$ matrix whose diagonal elements are 0's of the main diagonal of A_1 ,

$$(4.3) \qquad T_{11} + Z_{11} = p^1_{11}(p^1_{11} - 1) \ .$$

Considering inner products of row 2 with rows of Set 1 ,

$$t_{11} + t_{13} = p^1_{11} - 1 \ .$$

Summing over the p^1_{11} rows of Set 1 ,

$$(4.4) \qquad T_{11} + T_{13} = p^1_{11}(p^1_{11} - 1) \ .$$

From (4.3) and (4.4) ,

$$(4.5) \qquad Z_{11} = T_{13} \ .$$

Considering inner products of row 1 with rows of Set 3 ,

$$t_{31} + t_{32} = p^2_{11} - 1 \ .$$

Summing over the p^1_{12} rows of Set 3 ,

$$(4.6) \qquad T_{31} + T_{32} = p^1_{12}(p^2_{11} - 1) \ .$$

By symmetry of A_1 ,

$$(4.7) \qquad T_{13} = T_{31} \ .$$

Combining (4.5) , (4.6) and (4.7) ,

$$Z_{11} = T_{31} = p^1_{12}(p^2_{11} - 1) - T_{32} \ ,$$

$$Z_{11} \leq p^1_{12}(p^2_{11} - 1) \ .$$

This completes the proof of the lemma.

THEOREM 4.2. If an association scheme has two associate classes and $v = n^2$ treatments, $n \neq 4$, then necessary and sufficient conditions that it be a Latin square type scheme with 2 constraints are

$$(4.8) \qquad n_1 = 2(n - 1) ,$$

$$(4.9) \qquad p^1_{11} = n - 2 .$$

If $n = 4$, the condition is necessary but not sufficient.

PROOF: Necessity is proved by the general expressions (2.9) for the parameter values of Latin square type association schemes with g constraints, which reduce in the case of $g = 2$ to

$$n_1 = 2(n-1) , \quad n_2 = (n-1)^2 , \qquad P_1 = \begin{bmatrix} n-2 & n-1 \\ n-1 & (n-1)(n-2) \end{bmatrix} ,$$

$$P_2 = \begin{bmatrix} 2 & 2n-4 \\ 2n-4 & (n-2)^2 \end{bmatrix} .$$

Also, the parameters specified in (4.8) and (4.9) determine the remaining values, so that any of them may be assumed in the sufficiency proof.

The sufficiency proof will make use of the incidence matrix A_1, partitioned in the form of (4.1) . An important step will be to show that any pair of first associates and their $p^1_{11} = n-2$ common first associates form a set of n treatments which are pairwise first associates. In the notation of (4.1), this amounts to showing that submatrix A_{11} has 1's in all off-diagonal positions or equivalently, that $Z_{11} = 0$. Lemma 4.3 provides an upper bound for Z_{11} which reduces in the present case to

$$(4.10) \qquad Z_{11} \leq n - 1 .$$

Suppose that $Z_{11} > 0$, meaning that among the $n-2$ treatments of Set 1 there is at least one pair of second associates. For convenience, number treatments so that numbers 3 and 4 are second associates. Then the entries in the 3,4 and 4,3 positions of A_1 will be 0's. Since treatments 3 and 4 are second associates, the inner product of rows 3 and 4 of A_1 must be equal to $p_{11}^2 = 2$, meaning that the submatrix consisting of these two rows must contain exactly 2 columns with 1's in both positions. But columns 1 and 2 are of this form, meaning that each of columns $5, 6, \ldots, n^2$ of this submatrix must contain at least one 0.

Since $n-4$ of these columns are in submatrix A_{11}, it must contain at least $n-4$ 0's in rows 3 and 4 in addition to the two 0's originally assumed. By symmetry of the matrix there are also at least $n-4$ additional 0's in columns 3 and 4, for a total of at least $2n-6$ off-diagonal 0's in A_{11}. Therefore,

(4.11) if $Z_{11} > 0$, then $Z_{11} \geq 2n - 6$.

The latter inequality contradicts (4.10) for $n \geq 6$, proving that for $n \geq 6$, $Z_{11} = 0$. For $n = 3$, A_{11} is a 1×1 matrix which trivially has no off-diagonal 0's. For $n = 5$, it will be proved below that $Z_{11} = 0$. Therefore for all $n \neq 4$, the $n \times n$ submatrix of A_1 whose rows and columns are determined by a pair of first associates and their $n-2$ common first associates has all off-diagonal elements equal to 0, which means that the n treatments of this set are pairwise first associates.

This completes a proof that for $n \neq 4$, every pair of first associates is in a set of n treatments which are pairwise first associates. By Theorem 4.1, there exists a set of $g = 2$ orthogonal squares which may be used to define first associates in the scheme, which means precisely that it is a Latin square type scheme with two constraints.

In the special case $n = 4$, Counter-example 1 of Section 4.1 shows the existence of a scheme whose parameter values satisfy conditions (4.8) and (4.9) but in which it is not possible to define first associates by a set of two orthogonal squares. Therefore the conditions are not sufficient in this case.

It remains to prove that $Z_{11} = 0$ when $n = 5$. (4.10) and (4.11) show that if $Z_{11} > 0$, then $Z_{11} = 4$. Assume that for some choice of two first associates as treatments 1 and 2, $Z_{11} = 4$. That is, the 3×3 submatrix A_{11} determined by the set of $p^1_{11} = 3$ common first associates of treatments 1 and 2 has 4 off-diagonal elements equal to 0. After assigning the numbers 3, 4, 5 in a suitable order to these three treatments, the leading 5×5 principal minor of A_1 will have the form

$$\begin{matrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{matrix} \quad .$$

Since treatments 1 and 3 are first associates, they must have $p^1_{11} = 3$ first associates in common, of which two are treatments 2 and 4.

Number the remaining one as treatment 6 and adjoin row and column 6 to the submatrix, remembering that no further treatments can be common first associates of treatments 1 and 2 .

$$\begin{matrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & x \\ 1 & 1 & 0 & 0 & 0 & y \\ 1 & 0 & 1 & x & y & 0 \end{matrix}$$

Treatments 2 and 6 are second associates and the inner product of rows 2 and 6 cannot exceed $p_{11}^{2} = 2$. Therefore the letters x and y must represent 0's . Treatments 1 and 4 are first associates and must have three first associates in common, of which two are treatments 2 and 3 . Number the remaining one as treatment 7 and adjoin row and column 7 to the submatrix, remembering that no further treatments can be common first associates of treatments 1 and 3 .

$$\begin{matrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & z \\ 1 & 0 & 1 & 0 & 0 & 0 & w \\ 1 & 0 & 0 & 1 & z & w & 0 \end{matrix}$$

Treatment 7 is a second associate of treatments 2 and 3 and the inner product of row 7 with either of rows 2 and 3 cannot exceed 2 . Therefore the letters z and w must represent 0's . Treatment 1 has $n_1 = 8$ first associates, of which the remaining two may be numbered 8 and 9 . Then the next two elements of the row 1 of A_1 will be 1's and the remaining elements will be 0's . Treatments 5 , 6 and 7 are first associates of 1 and rows 5 , 6 and 7 must have inner product 3 with row 1 , meaning that each of these rows must have 1's in the next two positions. Then the inner product of any two of rows 5 , 6 and 7 will be 3 , which is impossible since these treatments are pairwise second associates. This contradiction disproves the assumption that $Z_{11} > 0$ for some pair of first associates and proves that every pair of first associates is contained in a set of $n = 5$ treatments which are pairwise first associates. This completes the proof of Theorem 4.2.

The principal object of the remainder of this chapter is to prove as much as possible of the statement that if an association scheme has the parameter values of a Latin square type scheme with g constraints, $g \geq 3$, there must exist a set of g mutually orthogonal squares which may be used to define first associates in the scheme. The counter-examples of Section 4.1 show that this statement is not true without exception, but it will be shown in Section 4.3 that for any g , the statement is true except for a finite number of values of n . When it is attempted to prove this by the methods used in the proof of Theorem 4.2, difficulties are encountered which will be described in the case $g = 3$. For $g \geq 4$, the difficulties are of the same kind but more severe.

The proof of Theorem 4.2 hinged on showing that an arbitrary pair of first associates, corresponding to an arbitrary line of the graph, was contained in a complete n-point. This was equivalent to showing that A_{11} , an $(n-2) \times (n-2)$ submatrix, contained no off-diagonal 0's , and was accomplished by showing that if any such 0's were present, the restrictions on the inner product of rows corresponding to second associates implied the existence of enough additional 0's in A_{11} to violate inequality (4.2) . In the case of $g = 3$ constraints, A_{11} is an $n \times n$ submatrix, and rather than prove that it has no off-diagonal 0's , it is desired to show that it has an $(n-2) \times (n-2)$ principal minor which is of this form. The symbol s_1 , to be used in the next section, will be borrowed for the sake of brevity. In this section, s_1 will denote the maximum order for a principal minor of A_{11} which has no off-diagonal 0's . If any contradiction to inequality (4.2) is to be obtained, it must be on the assumption that $s_1 \leq n-3$. Inequality (4.2) is weaker in the case $g = 3$, reducing to

$$(4.12) \qquad Z_{11} \leq 10n - 20 \text{ .}$$

This is consistent with forms of A_{11} such as the following, in which the leading $(n-5) \times (n-5)$ principal minor has 1's in all off-diagonal positions and the other submatrices have 0's in all positions.

$$A_{11} = \left|\begin{array}{ccccccc|ccccc}
0 & 1 & 1 & . & . & . & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & . & . & . & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & . & . & . & 1 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & & & & . & . & . & . & . & . \\
. & . & . & & & & . & . & . & . & . & . \\
. & . & . & & & & . & . & . & . & . & . \\
1 & 1 & 1 & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right|
\begin{array}{l}
\\ \\ \\ n-5 \text{ rows} \\ \\ \\ \\ \hline \\ \\ 5 \text{ rows} \\ \\ \\ \\
\end{array}$$

The number of off-diagonal 0's is $10n-30$, satisfying (4.12), and

$s_1 = n-5$. Moreover, it appears that s_1 can be still smaller without giving any easy proof that Z_{11} is large enough to contradict (4.12). The principal tool used in showing that Z_{11} is large is the restriction on the inner product of rows of A_{11} corresponding to second associates, and this restriction is also weaker than in the case of Theorem 4.2. With two constraints, the inner product of such rows was necessarily 0 ; with three constraints, the maximum value for the inner product is $p^2_{11}-2 = 4$. It may be possible with the methods used in Theorem 4.2 to prove the existence of a complete k-point on each line of order $s_1 + 2 = n-3$ but no better result than this can be hoped for. This falls short of the hypothesis of Theorem 4.1.

If it is not possible to prove the existence of a complete n-point on every line of a graph, it may still be possible to prove the existence of one n-point, somewhere in the graph. It will now be shown that this weaker result would actually have been sufficient in the case of Theorem 4.2 for a proof of the remainder of the Latin square structure. In other words, an association scheme with L_2 parameter values either has a complete n-point on each line, implying the existence of the orthogonal squares, or it has no n-points at all. It will be possible in Section 4.3 to extend this part of a uniqueness proof to some L_3 schemes not covered by the main theorem of that section. The present result, while vacuous for L_2 schemes with most values of n , does show that for n = 4 , the one case in which a non-Latin square scheme can have L_2 parameter values, the graph of such a scheme cannot contain any complete 4-points. It was verified for the first counter-example of Section 4.1

that a particular line was not contained in any complete 4-point; the following theorem shows that the same is true for each of the 48 lines of the graph.

THEOREM 4.3. If an association scheme with two associate classes has parameter values $v = n^2$, $n_1 = 2(n-1)$, $p_{11}^1 = n-2$ and there exists a set of n treatments which are pairwise first associates, then every pair of first associates is in such a set.

PROOF: Denote the set of n treatments in an n-point by A and an arbitrary treatment of A by θ. Of the $2(n-1)$ first associates of θ, $n-1$ are in A; denote the set of the remaining $n-1$ first associates of θ by B and an arbitrary treatment of B by ϕ. It follows from Lemma 4.1 that ϕ has no first associates in set A except θ, so the $n-2$ first associates which ϕ has in common with θ must be the remaining $n-2$ treatments of set B. Since ϕ was an arbitrary treatment of set B, it follows that each treatment of the set must be a first associate of each of the others, meaning that the set consisting of θ and its first associates not in A form a complete n-point. θ was an arbitrary treatment of set A and the same argument applies to each of the n treatments of A, proving the existence of n additional n-points, each having one treatment in common with A. Since any treatment in two of the additional n-points would be a first associate of two treatments of A, Lemma 4.1 shows that these n-points are disjoint, exhausting the n^2 treatments of the scheme. They may be called parallel n-points. The same reasoning applied to the original n-point A may now be applied to any one of the new ones to show the existence of another set of n parallel n-points, of which one is A . This shows that each treatment of the scheme is in two complete n-points, the 2(n-1) other treatments of the two n-points being first associates of the treatment. The first associates of all treatments are accounted for by the two sets of n-points, showing that every pair of first associates is contained in an n-point.

This completes the proof of the theorem. Theorem 4.1 may then be applied to show that the scheme must be of L_2 type.

4.3 On the Uniqueness of L_g Association Schemes, $g \geq 3$.

In this section methods will be developed by which Theorem 4.2 can be extended to an infinite class of Latin square type association schemes with 3 or more constraints. Theorems 4.4 to 4.6 are devoted to obtaining a lower bound analogous to (4.11) for the number Z_{11} of off-diagonal 0's in the submatrix A_{11} defined by (4.1). The bounds obtained apply to a wider class of incidence matrices and are in a form which gives direct information on the value of k for which a complete k-point is known to exist. Theorem 4.7 applies the results to association schemes. Theorem 4.8 and Lemma 4.5, also concerned with association schemes and valid for all schemes with two associate classes, introduce a different line of reasoning concerning the existence of complete k-points in association schemes and are somewhat similar to Lemma 4.2. Finally in Theorem 4.9 the case of association schemes with parameter values of the Latin square series is taken up and it is shown that for a fixed number g of constraints and sufficiently large n, the Latin square type scheme is unique in the sense used in this chapter; that is, the scheme can only be constructed by the use of some set of g mutually orthogonal $n \times n$ squares to define first associates. In Corollary 4.9 the sufficiently large values of n are stated explicitly. Theorem and Corollary 4.9 are the main results of the section and the chapter. The chapter is concluded by a discussion of some extensions and possible extensions. The most important of these, Theorem 4.10, is analogous to Theorem 4.3.

Some ideas to be used in Theorem 4.4 to 4.6 will now be given in two definitions.

DEFINITION 4.1. An incidence matrix A will be said to satisfy this definition if it is a $t \times t$ symmetric matrix with 0's on the main diagonal and if it satisfies the requirement that if any two rows contain a pair of 0's which are symmetrically located with respect to the main diagonal, then the inner product of those two rows must not exceed D. Z will denote the number of off-diagonal 0's in A.

If A is the incidence matrix of a linear graph, the rows being identified with points, then two rows containing a pair of symmetrically located 0's represent two points not joined by a line, and the requirement on inner products has the interpretation that two such points are joined by at most D 2-chains. If A is a principal minor submatrix of an association matrix of a PBIB design, the rows being identified with treatments, then two rows containing a pair of symmetrically located 0's represent treatments which are not associates, while the inner product of two rows is equal to the number of treatments (of the set corresponding to the submatrix) which are common associates of the two treatments.

DEFINITION 4.2. This paragraph constitutes the definition of a set of integers $s_1, s_2, \ldots, s_f$ and a set of submatrices Q_{ij} of a symmetric incidence matrix A with 0's on the main diagonal. s_1 will denote the maximum order for a principal minor submatrix of A which has 1's in all off-diagonal positions. If there are no 1's, $s_1 = 1$. Denote an $s_1 \times s_1$ minor of this form by this form Q_{11}. The value of s_1 is uniquely determined but possibly not the set of rows and columns in Q_{11}. It is not essential but will simplify the later discussion of rows and columns of A are permuted (simultaneously) so that Q_{11} is the leading principal minor. In the remaining principal minor submatrix, determined by the remaining $t-s_1$ rows and $t-s_1$ columns, let s_2 denote the maximum order for a principal minor submatrix with 1's in all off-diagonal positions, denote such a submatrix by Q_{22}, and permute these rows and columns so that Q_{22} is in the next diagonal position. Clearly $s_1 \geq s_2$. Different choices of Q_{11} may lead to different values of s_2; for a particular choice of Q_{11}, the value of s_2 is determined but possibly not the $s_2 \times s_2$ submatrix Q_{22}. Repeat for the remaining diagonal submatrix, and so on until A has diagonal blocks of order $s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_f$, where $s_1 + s_2 + \cdots + s_f = t$, each s_i has the maximal property described for s_1 and s_2, and all off-diagonal elements of each block are 1's. The i^{th} diagonal block will be denoted by Q_{ii}; the submatrix determined by the rows of Q_{ii} and the columns of Q_{jj} will be denoted by Q_{ij}. This partition of A is illustrated in the following diagram.

$$A = \begin{array}{c} \left[\begin{array}{c:c:c:c} Q_{11} & Q_{12} & \cdots & Q_{1f} \\ \hdashline Q_{21} & Q_{22} & \cdots & Q_{2f} \\ \hdashline \cdots & \cdots & \cdots\cdots & \cdots \\ \hdashline Q_{f1} & Q_{f2} & \cdots & Q_{ff} \end{array}\right] \end{array} \begin{array}{l} s_1 \text{ rows}, \\ s_2 \text{ rows}, \\ \cdots \\ s_f \text{ rows}. \end{array} \tag{4.13}$$

It is desired to investigate the possible values of Z in the matrix A specified in Definition 4.1. A matrix with $Z = t(t-1)$, containing 0's everywhere or a matrix with $Z = 0$, with 1's in all off-diagonal positions, satisfies the requirement, but there may be intermediate values of Z which are impossible. If the matrix is partitioned according to Definition 4.2, the fact that the diagonal blocks Q_{ii} contain no off-diagonal 0's will provide an upper bound for Z. On the other hand, it is evident that the off-diagonal blocks Q_{ij} must contain some 0's, or it would be possible to form some larger diagonal blocks full of 1's, violating the maximal property of the s_i. This type of reasoning is put into a definite form in the following theorem.

THEOREM 4.4. If A is an incidence matrix satisfying definition 4.1 and $s_1, s_2, \ldots, s_f$ are determined according to Definition 4.2, then the total number Z of off-diagonal 0's of A satisfies the following inequality.

$$\sum_{i=1}^{f}\sum_{j=i+1}^{f} s_j \operatorname{Max}(s_i + s_j - D,\ 2s_i - 2D,\ 2) \leq Z \leq t(t-1) - \sum_{i=1}^{f} s_i(s_i - 1) \quad (4.14)$$

PROOF: The partition of A depicted in (4.13) will be used.

Take any submatrix Q_{ij} with $i < j$, an $s_i \times s_j$ matrix, and consider any $m \times k$ submatrix of Q_{ij} which contains 1's everywhere. This submatrix, the symmetrically located portion of Q_{ji}, an $m \times m$ submatrix of Q_{ii} and a $k \times k$ submatrix of Q_{jj} can be combined to form a symmetric $(m+k) \times (m+k)$ matrix with no off-diagonal 0's. By the maximal property of s_i, it is necessary that $m + k \leq s_i$. Next consider the set of s_i rows of Q_{ij}, and for each column define a subset consisting of all the rows which contain 0's in that column. There are s_j subsets in all. Take any k of these subsets, corresponding to a k-columned submatrix of Q_{ij} . By the inequality just proved for submatrices containing 1's everywhere, at most $s_i - k$ rows of this submatrix contain 1's everywhere, meaning that at least k rows contain 0's . This means that any k of the subsets of rows of Q_{ij} contain between them at least k distinct rows. This is true for $k = 1, 2, \ldots, s_j$. By the theorem of P. Hall [22],[27] on representatives of subsets, there exists a system of distinct representatives of the s_j subsets. That is, there are s_j distinct rows of Q_{ij} which may be ordered so that the μ^{th} row contains a 0 in the μ^{th} column of Q_{ij} .

So far we have shown that Q_{ij} contains at least s_j 0's no two of which are in the same row or column. This seems to be about the best possible result using nothing but the condition that the s_i are maximal, but the condition on inner products of A still has not been applied. This will be done next.

We will use the submatrix consisting of the blocks Q_{ii} , Q_{ij} , Q_{ji} , Q_{jj} , still with $i < j$. This is a symmetric $(s_i + s_j) \times (s_i + s_j)$ matrix. It has been shown that Q_{ij} contains a set of s_j 0's , no two of which are in the same row or column. Consider one of these 0's and its symmetrically placed 0 in Q_{ji} . The two rows containing these rows must have inner product $\leq D$. Therefore at most D 1's may occur in the remaining cells of these two rows of Q_{ij} and Q_{ji} , meaning that there are at least $s_i + s_j - D - 2$ additional 0's in these two rows. This can be repeated for each of the initial 0's ,

$$\left[\begin{array}{ccc|ccc}
0 & \vdots & Q_{ii} & & \vdots & Q_{ij} \\
\ddots & \vdots & & & \vdots & \\
1\,1\,\cdot\,1 & 0 & 1\,\cdot\,1 & \cdots & 0 & \cdots \\
 & \vdots & \ddots & & \vdots & \\
 & \vdots & 0 & & \vdots & \\
\hline
 & \vdots & & 0 & \vdots & \\
 & \vdots\; Q_{ji} & & \ddots & \vdots & Q_{jj} \\
\cdots & 0 & \cdots & 1\,\cdot\,1 & 0 & 1\,\cdot\,1 \\
 & \vdots & & & \vdots & \ddots \\
 & \vdots & & & \vdots & 0
\end{array}\right]$$

and since they were in distinct rows and columns, the additional 0's all fall in different rows and are therefore distinct. This proves the existence of at least

$$s_j(s_i + s_j - D - 2)$$

0's in blocks Q_{ij} and Q_{ji}, in addition to the $2s_j$ already discussed. If $s_i + s_j \leq D + 2$, this is vacuous, but in any case the existence of the $2s_j$ 0's has been proved.

Therefore a lower bound for the number of 0's in blocks Q_{ij} and Q_{ji} is

$$(4.15) \qquad \operatorname{Max}\Big(s_j(s_i + s_j - D) \,,\; 2s_j \Big) .$$

Again considering the initial 0 in Q_{ij}, note that the number of additional 0's in the row of Q_{ji} which contains the symmetrically placed initial 0 is by symmetry equal to the number of additional 0's in the <u>column</u> of Q_{ij} containing the initial 0. There are therefore at least $s_i + s_j - D - 2$ additional 0's in the row and column of Q_{ij}

containing any of the s_j initial 0's. If these s_j totals are combined, some of the additional 0's may be counted more than once, but since no two of the initial 0's were in the same row or column, none are counted more than twice, and the number of duplications is at most equal to the number of cells of the submatrix which are in the same row as one of the initial 0's and in the same column as another. The number of such cells is $s_j(s_j-1)$. A lower bound for the number of additional 0's in Q_{ij} is obtained by subtracting this from the combined total,

$$s_j(s_i + s_j-D-2) - s_j(s_j-1) = s_j(s_i-D-1) .$$

If $s_i \leq D+1$ this is vacuous, but in any case the existence of the s_j initial 0's in Q_{ij} has been proved. The number of 0's in Q_{ji} is equal to the number in Q_{ij} , giving as a lower bound for the number of 0's in both blocks

$$\text{(4.16)} \qquad \text{Max}\Big(s_j(2s_i-2D) \ , \ 2s_j \Big) .$$

This may be combined with (4.15) to show that blocks Q_{ij} and Q_{ji} contain at least

$$s_j \ \text{Max}\Big(s_i + s_j-D \ , \ 2s_i-2D \ , \ 2 \Big)$$

0's. Summation over all the off-diagonal submatrices Q_{ij} gives the lower bound stated in the theorem. The upper bound in the theorem is the total number of cells in the submatrices Q_{ij} , since all the off-diagonal 0's are in these submatrices. This completes the proof of Theorem 4.4.

The set of values of Z satisfying (4.14) includes the set of possible values of Z for all matrices A satisfying Definition 4.1 and for which the procedure described in Definition 4.2 leads to a particular partition of t into positive summands s_i. The union of the sets obtained from (4.14) for all admissible partitions of t then includes all possible values of Z for a given order $t \times t$ of matrix A and a given value of D. The class of admissible partitions of t may be restricted. For example, if A is the incidence matrix of a graph which is known to contain complete 3-points but no complete configurations with as many as $t-1$ points, then $3 \leq s_1 \leq t-2$. On the other hand, if restrictions on the value of Z are known, this theorem may restrict the possible partitions of t and in particular the value of s_1.

It may be possible to obtain a slightly better lower bound than that given by the theorem. One of the lower bounds for the number of 0's in submatrices Q_{ij} and Q_{ij}, which is twice the number of 0's in Q_{ij} and is therefore an even number, is $s_j(s_i + s_j - D)$. If this product is odd in any term of the sum, it may therefore be replaced by the next larger even number.

The following numerical examples illustrate the theorem and the remark just made.

NUMERICAL EXAMPLES

Each term of the double sum in the lower bound given by (4.14) is a function of D and of two of the s_i values, and is independent of t. A preliminary table of values of $s_j \operatorname{Max}\left(s_i + s_j - D , 2s_i - 2D , 2 \right)$, computed for a fixed value of D and a suitable range of values $s_i \geq s_j$, is convenient for use in evaluating the sum and can be used for any value of t. A table of this kind follows, computed for the case $D = 4$.

$s_i \backslash s_j$	1	2	3	4	5
1	2	-	-	-	-
2	2	4	-	-	-
3	2	4	6	-	-
4	2	4	<u>10</u>	10	-
5	2	6	12	20	30
6	4	8	<u>16</u>	24	<u>36</u>

The underlined entries in this table replace the computed values 9, 15 and 35,

For a particular value t and partition $s_1 + s_2 + \cdots + s_f$ the summation over pairs s_i, s_j, $i < j$ is quickly carried out. For $t = 7$ and the partition $3 + 3 + 1$, the sum includes three terms. The term resulting from the pair $s_i, s_j = 3,3$ is 6; the pair $3,1$ occurs twice, each time contributing the term 2; and the total is $6 + 2(2) = 10$. Therefore if the process described in Definition 4.2 leads to diagonal blocks of the following form in a 7×7 matrix, the number Z

$$\left[\begin{array}{ccc|ccc|c} 0 & 1 & 1 & & & & \\ 1 & 0 & 1 & & & & \\ 1 & 1 & 0 & & & & \\ \hline & & & 0 & 1 & 1 & \\ & & & 1 & 0 & 1 & \\ & & & 1 & 1 & 0 & \\ \hline & & & & & & 0 \end{array}\right]$$

of off-diagonal 0's in the matrix must be at least 10. The upper bound given by (4.14) is 30, the total number of cells in the off-diagonal blocks. Similar bounds for all of the partitions of 7 are listed as further examples.

Partition of t	Lower bound on Z	Upper bound on Z
7	0	0
6 1	4	12
5 2	6	20
5 1 1	2(2) + 2 = 6	22
4 3	10	24
4 2 1	4 + 2 + 2 = 8	28
4 1 1 1	3(2) + 3(2) = 12	30
3 3 1	6 + 2(2) = 10	30
3 2 2	2(4) + 4 = 12	32
3 2 1 1	4 + 2(2) + 2(2) + 2 = 14	34
3 1 1 1 1	4(2) + 6(2) = 20	36
2 2 2 1	3(4) + 3(2) = 18	36
2 2 1 1 1	4 + 6(2) + 3(2) = 22	38
2 1 1 1 1 1	5(2) + 10(2) = 30	40
1 1 1 1 1 1 1	21(2) = 42	42

Of the conclusions which can be drawn from these results, the following are typical.

(a) The value $Z = 2$ is impossible.

(b) If the graph contains no 4-points, meaning $s_1 \leq 3$, then $Z \geq 10$. Note that the proof of this requires consideration of the lower bounds for all partitions with $s_1 \leq 3$, and does not follow from the particular result obtained for the partition 3 3 1.

(c) If Z is known to satisfy $Z < 8$, then $s_1 \geq 5$, proving that the graph contains a complete 5-point. The restriction on inner products is essential, as shown by the following example, in which $Z = 6$ and $s_1 = 4$ but inner products such as that of rows 1 and 5 are not ≤ 4.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The lower bound on Z given by Theorem 4.4 is rather complicated, depending as it does on all the terms $s_1, s_2, \ldots, s_f$ in a partition of t, and it suffers from the disadvantage that it applies only to a particular partition. In order to get a lower bound which depends only on t and D it is necessary to minimize over a class of partitions of t which may be very large. Four lower bounds will now be derived which involve s_1 but none of the other s_i. It is simple to apply these formulas and take the maximum of the values obtained as a lower bound on Z, valid for all partitions in which the largest term is s_1. A lower bound which depends only on t and D may then be obtained by minimizing over admissible values of s_1. Lower bounds which are independent of $s_2, \ldots, s_g$ are not only simpler but more useful. In the present application of Theorem 4.4 and the following theorems, the object will be to prove that the linear graph formed from the incidence matrix A has a complete subgraph whose order exceeds a certain minimum value.

The value s_1 is important here, since it may be interpreted as the maximum order of a complete subgraph. The values $s_2, \dots, s_f$ will be of less interest and their interpretation is not so simple.

The first simplified lower bound on Z follows directly from Theorem 4.4.

COROLLARY 4.4. If an incidence matrix A satisfies Definition 4.1 and if s_1 is defined by Definition 4.2, then the total number Z of off-diagonal 0's in A satisfies the inequality

$$(4.17) \qquad Z \geq 2(t-s_1)(s_1-D) \; .$$

PROOF: This inequality is obtained from (4.14) by taking only the terms of double sum corresponding to $i = 1$ and taking the second of the three expressions in parentheses. The sum then reduces to

$$\sum_{j=2}^{f} s_j(2s_1-2D) = 2(s_1-D) \sum_{j=2} s_j \; .$$

Since $s_1 + s_2 + \dots + s_f = t$, the sum in the right member reduces to $t-s_1$ and the result is proved.

The 0's enumerated in this corollary are those in the first s_1 rows of A, that is, in submatrices $Q_{12}, Q_{13}, \dots, Q_{1f}$, and the symmetrically located 0's in the first s_1 columns of A. If s_1 is nearly as large as t, these rows and columns will contain most of the off-diagonal 0's, and inequality (4.17) may be nearly as strong as (4.14). For small values of s_1, it becomes much weaker, collapsing completely for $s_1 \leq D$.

The two lower bounds for Z given in Theorem 4.5 do not follow from the statement of Theorem 4.4 but use some of its proof.

THEOREM 4.5. When A is an incidence matrix satisfying Definition 4.1 and $s_1, s_2, \ldots, s_f$ are determined according to Definition 4.2, then the total number Z of off-diagonal 0's of A satisfies both of the following inequalities.

$$(4.18) \qquad Z \geq \tfrac{1}{2}(t-D)(t-s_1) \; ;$$

$$(4.19) \qquad Z \geq \tfrac{1}{2}(t - s_1)(t + s_1 - 2D) \; .$$

PROOF: The symbol Y_i will be used to denote the total number of off-diagonal 0's in the $s_i \times t$ submatrix of A consisting of blocks $Q_{i1}, Q_{i2}, \ldots, Q_{if}$. In this notation the proof of Corollary 4.4 implies the statement

$$(4.20) \qquad Y_1 \geq (t - s_1)(s_1 - D) \; .$$

If two rows of A contain a pair of 0's which are located symmetrically with respect to the main diagonal, then by the restriction on inner products of rows, there can be at most D columns of A which contain 1's in both of these rows, and the two rows together must contain at least $t-D$ off-diagonal 0's, including the original pair. In the proof of Theorem 4.4 it was shown that for $i < j$, submatrix Q_{ij} contained a set of s_j 0's, no two of which were in the same row or column. These 0's and the symmetrically located 0's in Q_{ji} therefore lie in $2s_j$ distinct rows, forming s_j pairs of rows, each pair satisfying the inner product condition and containing at least $t-D$

off-diagonal 0's . The $2s_j$ rows are all contained in the $(s_i + s_j) \times t$ submatrix whose rows are determined by submatrices Q_{ij} and Q_{ji}, giving the result

$$(4.21) \qquad Y_i + Y_j \geq s_j(t - D) \; , \quad i < j \; .$$

Several inequalities of this kind will now be added.

$$\begin{aligned} Y_1 + Y_2 &\geq s_2(t - D) \\ Y_2 + Y_3 &\geq s_3(t - D) \\ &\cdots\cdots\cdots \\ Y_{f-1} + Y_f &\geq s_f(t - D) \end{aligned}$$

Adding, and noting that $\sum_{i=1}^{f} Y_i = Z$,

$$(4.22) \qquad 2Z - Y_1 - Y_f \geq (t - s_1)(t - D) \; .$$

Dropping the non-negative terms Y_1 and Y_f strengthens the inequality and leads at once to (4.18) . Adding (4.20) and (4.22) gives

$$(4.23) \qquad 2Z - Y_f \geq (t - s_1)(t + s_1 - 2D) \; .$$

The term Y_f is dropped again and (4.19) is obtained, completing the proof.

The two inequalities of this theorem are weaker than (4.17) for large values of s_1 , but give better results when s_1 is small. The expression in the right member of (4.19) has its maximum value for $s_1 = D$, the value for which (4.17) gives the trivial lower bound 0 . For $s_1 < D$, (4.18) is the strongest of the three inequalities.

These three lower bounds on Z fill the needs of the present section, but may fall far short of the actual minimum value of Z for many values of s_1. For example, when $s_1 = 1$, the best result obtained from any of the three is $Z \geq \frac{1}{2}(t-1)(t-D)$, a very conservative underestimate, since $s_1 = 1$ means that A contains no 1's and the actual value of Z is $t(t-1)$. A fourth lower bound, which will be given in Theorem 4.6, is stronger for very small values of s_1, but makes no use of the restriction on inner products and is of little use for large values of s_1. It is closely related to a known result in graph theory which will be mentioned following the proof.

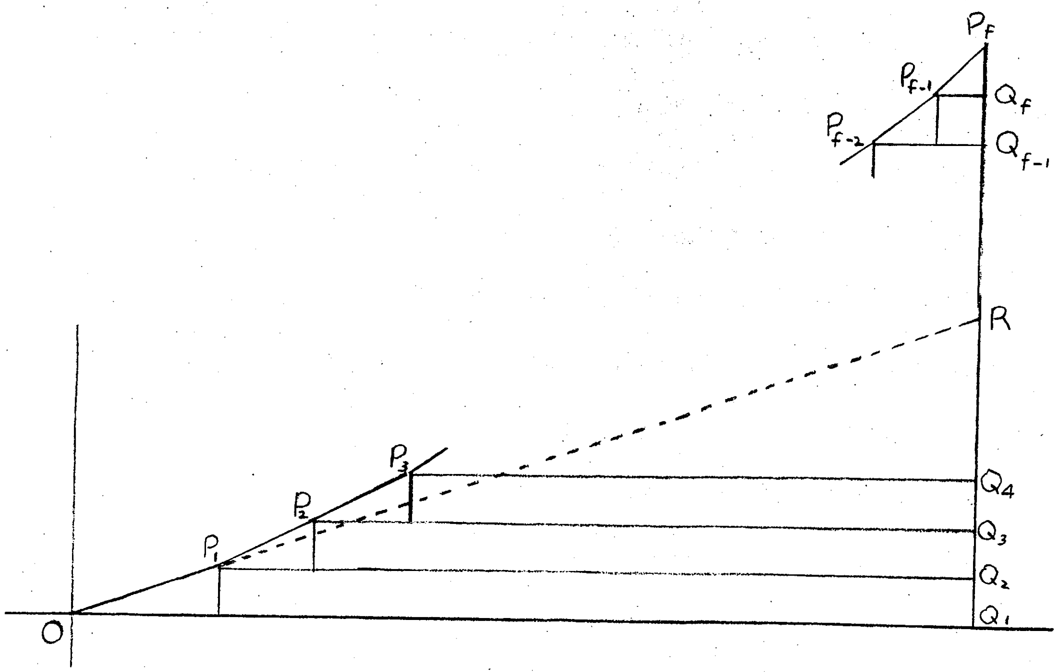
THEOREM 4.6. If A is a symmetric incidence matrix with 0's on the main diagonal, and if s_1 is determined according to Definition 4.2, then the total number Z of off-diagonal 0's of A satisfies the inequality

$$(4.24) \qquad Z \geq \frac{t^2}{s_1} - t \ .$$

PROOF: Inequality (4.14) of Theorem 4.4 is used, taking the third of the three expressions in parentheses. The inequality reduces to

$$(4.25) \qquad Z \geq 2 \sum_{i=1}^{f} \sum_{j=i+1}^{f} s_j \ .$$

This sum is represented graphically by the sum of the areas of the rectangles in the figure below, which is located with reference to a rectangular coordinate system with origin O.



Vertex P_i has coordinates $\left(\sum_{j=1}^{i} s_j \,,\, i \right)$;

vertex Q_i has coordinates $(t \,,\, i-1)$.

In particular, the coordinates of P_1 and P_f are $(s_1 \,,\, 1)$ and $(t \,,\, f)$, respectively. Rectangle P_iQ_i then has altitude 1 and area equal to the sum $\sum_{j=j+1}^{f} s_j$, and the sum of the areas of all the rectangles is equal to the double sum in (4.25) . It will be convenient to deal with the area of the polygon $OP_1P_2 \ldots P_fQ_1$, which exceeds the combined area of the rectangles by exactly $\frac{1}{2}t$, the combined area of triangles whose altitudes are equal to 1 and the sum of whose bases is t .

$$(4.26) \qquad \sum_{i=1}^{f} \sum_{j=i+1}^{f} s_j = \text{Area } OP_1P_2 \ldots P_fQ_1 - \tfrac{1}{2}t .$$

Since $s_1 \geq s_2 \geq \ldots \geq s_f$, the polygonal line $OP_1P_2 \ldots P_f$ is concave upward and the area of the polygon is not less than the area of triangle ORQ_1, where R lies on OP_1 extended. $OQ_1 = t$ and by similar triangles $Q_1R = \frac{t}{s_1}$, giving the result

$$\text{Area } ORQ_1 = \frac{t^2}{2s_1} ,$$

which is enough to prove the theorem. However, it will be of interest to prove (4.28) below, a slightly stronger result which calls for closer study of the figure.

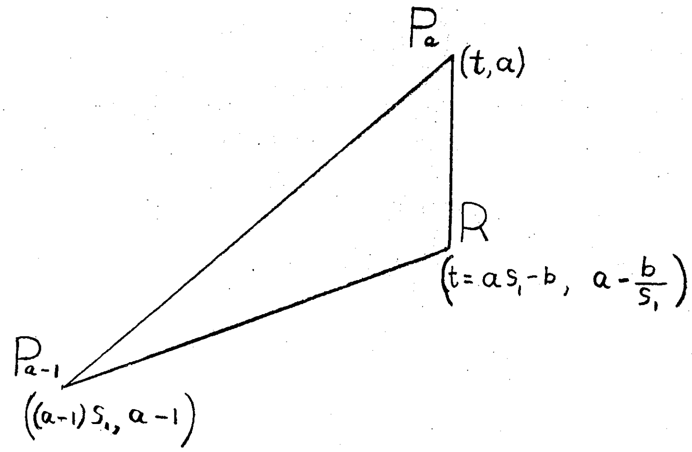
Q_1R cannot exceed the integral length f of the segment Q_1P_f; if t is written in the form

$$t = as_1 - b , \quad a \text{ and } b \text{ integers} , \quad 0 \leq b < s_1 ,$$

then

$$Q_1R = \frac{t}{s_1} = a - \frac{b}{s_1} > a-1 ,$$

implying $f \geq a$. For a fixed value of s_1, the minimum possible value of f is achieved if $s_2 = s_3 = \ldots = s_{a-1} = s_1$, $f = a$ and $s_a = s_1 - b$. Then for $i = 1, 2, \ldots, a-1$, the vertices P_i will have coordinates (is_1, i) and will lie on the line OR, and the only portion of the polygon which lies outside of triangle ORQ_1 will be the small triangle $P_{a-1}P_aR$, which has base $\frac{b}{s_1}$, altitude $s_1 - b$, and area $\frac{b(s_1-b)}{2s_1}$.



For the same value of s_1 and any other choice of $s_2, s_3, \ldots, s_f$, additional vertices of the polygon will lie above the line OR and a greater area of the polygon will lie outside the triangle. Therefore

$$(4.27) \qquad \text{Area } OP_1P_2 \ldots P_fQ_1 \geq \frac{t^2}{2s_1} + \frac{b(s_1 - b)}{2s_1} .$$

Combining (4.25), (4.26) and (4.27),

$$(4.28) \qquad Z \geq \frac{t^2}{s_1} - t + \frac{b(s_1 - b)}{s_1} .$$

When the non-negative final term is dropped for simplicity, Theorem 4.6 is proved. The lower bound in (4.28) may be written in either of the forms in the following statement.

$$(4.29) \qquad Z \geq (a - 1)(as_1 - 2b) = (a - 1)(t - b) .$$

When matrix A is taken as the incidence matrix of a linear graph on t points, each pair of off-diagonal 0's corresponds to a pair of points of the graph which are not joined by a line. The number of lines in the graph is $\binom{t}{2} - \frac{1}{2}Z$. s_1 is the maximal number of points in a complete subgraph. A problem of considerable interest in graph theory is to find the maximal number of lines for a graph on t points which does not contain any complete subgraph with $s_1 + 1$ points. Turan solved this problem in 1941 [36] [37] by deriving an upper bound for the number of lines, then constructing an example in which the number of lines is equal to the upper bound. His upper bound is equivalent to the lower bound (4.29) for Z, which is therefore very nearly a new proof of his result. The remaining step in such a proof is to show that the lower bound of (4.29) or (4.28) is monotone decreasing in s_1, so that the number of off-diagonal 0's can be less than the bound only if there is a complete subgraph with more than s_1 points. The proof is not difficult but is not needed here and will be omitted. In the graph constructed by Turan, the points are divided into s_1 disjoint sets, b sets having $a-1$ points and s_1-b sets having a points, then all pairs of points in different sets are joined. Any subgraph with $s_1 + 1$ points must contain two points which are in the same set and are therefore not joined. The number of pairs of points not joined is $b\binom{a-1}{2} + (s_1-b)\binom{a}{2}$. The number of off-diagonal 0's in the incidence matrix is twice this total and reduces to $Z = (a-1)(as_1-2b)$, the expression appearing in (4.29).

The minimum value of Z, given t, D, and s_1, will be denoted by $m(t, D, s_1)$. To summarize, Z denotes the number of off-diagonal 0's in an incidence matrix, and $m(t, D, s_1)$ is the value obtained by minimizing Z over the class of all symmetric $t \times t$ incidence matrices with 0's on the main diagonal, satisfying the condition that any two rows containing a pair of symmetrically located off-diagonal 0's must have inner product $\leq D$, and having an $s_1 \times s_1$ principal minor submatrix with 1's in all off-diagonal positions but no such submatrix of any larger order. Since there are only finitely many $t \times t$ incidence matrices, this minimum value exists. The value of $m(t , D , s_1)$ is not known in general, though it was pointed out in the previous paragraph that when the restriction on inner products is relaxed, which may be done by taking $D \geq t-2$, the exact value is given by

$$(4.30) \qquad m(t , D = t-2 , s_1) = (a-1)(t-b) \quad .$$

It was also remarked that in this case and for fixed t, the function is monotone decreasing in s_1. Not even this is known for most values of D, though it may be conjectured that increasing the order of the largest complete subgraph of a graph will necessitate an increase in the total number of lines. The lower bounds on Z derived in Corollary 4.4, Theorem 4.5 and Theorem 4.6 are of course lower bounds on $m(t , D , s_1)$. The following notation will be useful in discussing these bounds.

$$(4.31) \qquad B_1(t , D , s_1) = 2(t - s_1)(s_1 - D) \ ,$$

$$(4.32) \qquad B_2(t , D , s_1) = \tfrac{1}{2}(t-s_1)(t + s_1 - 2D) \ ,$$

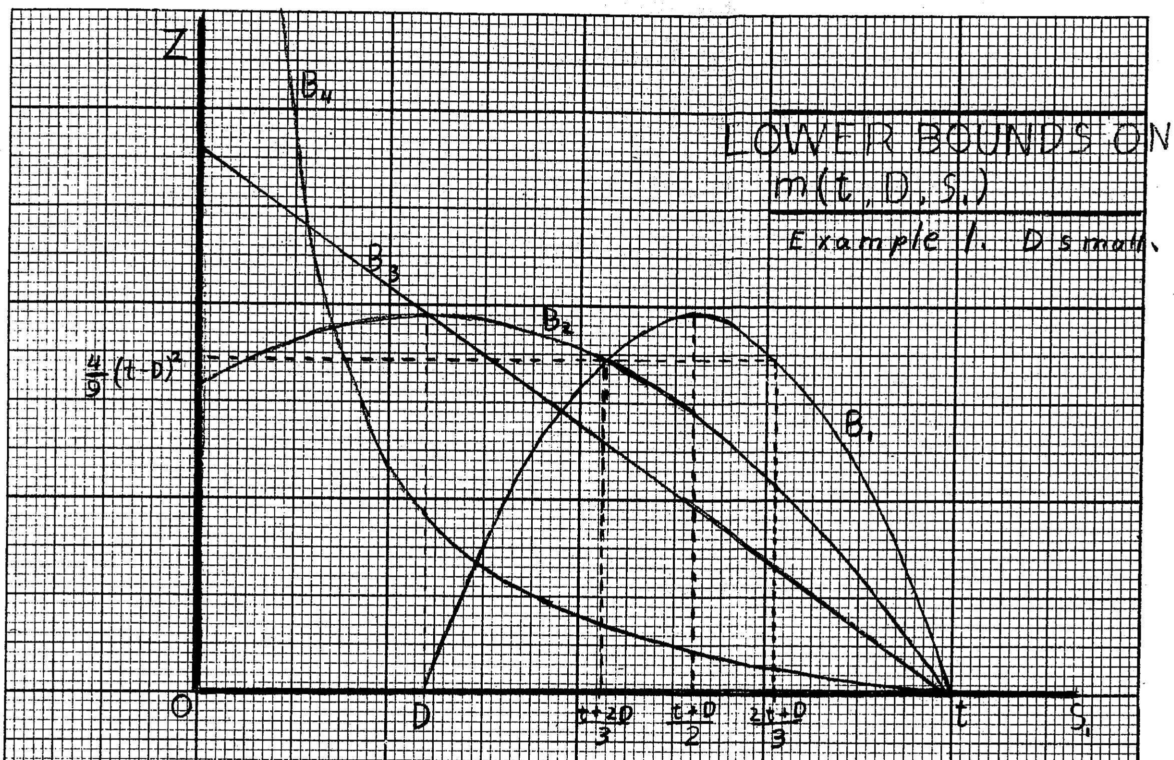
$$(4.33) \qquad B_3(t , D , s_1) = \tfrac{1}{2}(t - D)(t - s_1) \ ,$$

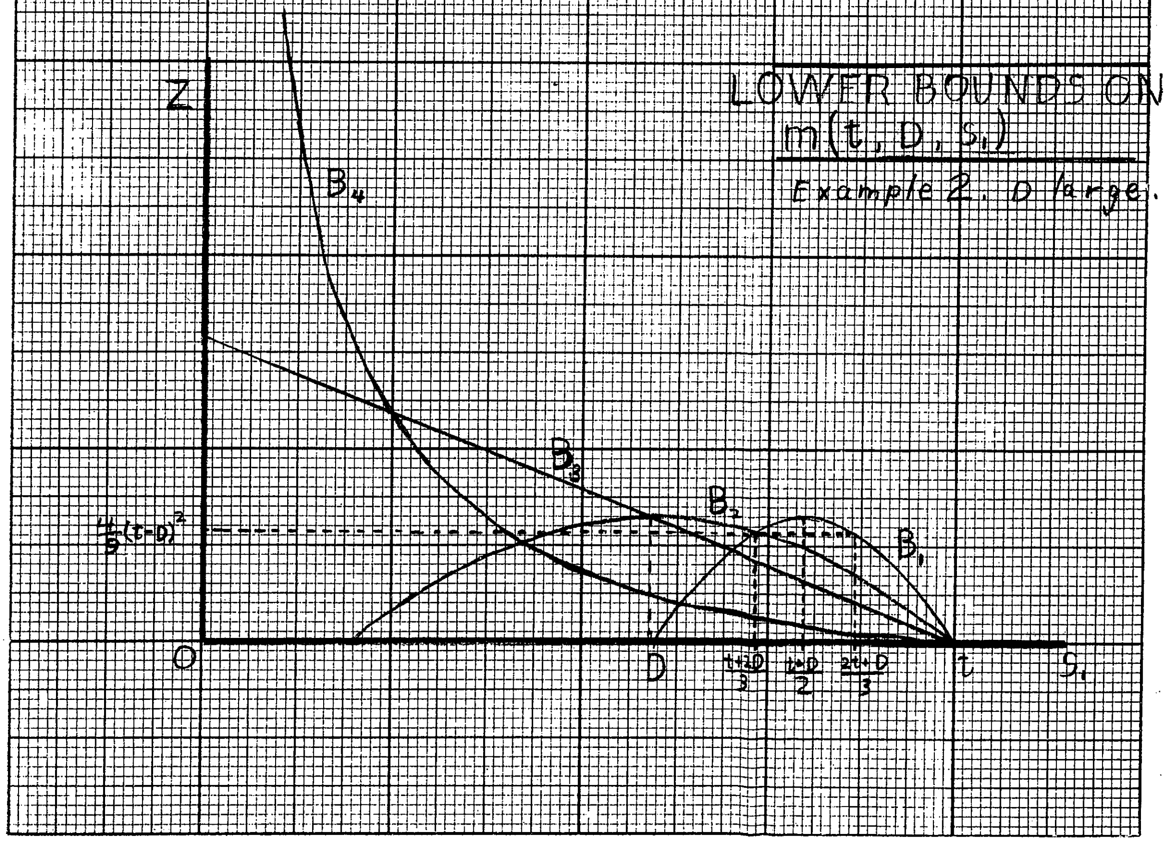
$$(4.34) \qquad B_4(t , D , s_1) = \frac{t^2}{s_1} - t \ .$$

Then the fact that these four expressions are lower bounds on Z may be expressed

$$(4.35) \qquad m(t , D , s_1) \geq B_i(t , D , s_1) \ , \quad i = 1 , 2 , 3 , 4 \ .$$

The dependence of the four bounds on s_1 for fixed t and D is shown schematically in the following figures.


Z
B_4
LOWER BOUNDS ON
$m(t, D, S_1)$
Example 1. D small.
B_3
B_2
$\frac{4}{9}(t-D)^2$
B_1
O
D
$\frac{t+2D}{3}$
$\frac{t+D}{2}$
$\frac{2t+D}{3}$
t
S_1


Z
LOWER BOUNDS ON
$m(t, D, S_1)$
Example 2. D large.
B_4
B_3
B_2
$\frac{4}{9}(t-D)^2$
B_1
O
D
$\frac{t+2D}{3}$
$\frac{t+D}{2}$
$\frac{2t+D}{3}$
t
S_1

As indicated by the figures, for fixed t the bounds B_1, B_2 and B_3 become weaker as D is increased. It is probably to be expected that an increase in D will permit an increase in the inner product of some pairs of rows of the incidence matrix, allowing some 0's in such rows to be replaced by 1's and decreasing the value of Z. In the extreme case $D \geq t-2$, the bound B_4 is the only one needed, as mentioned above. In the cases shown in the figures, each of the four bounds is stronger than the others for certain values of s_1. The figures indicate that for $s_1 = \frac{2t+D}{3}$, the lower bound for Z is $\frac{4}{9}(t-D)^2$, and that Z can be less than this value only for $s_1 > \frac{2t+D}{3}$. This observation is essential in the proof of the next lemma. The information needed to show that it is true in general is contained in the following table.

RANGE OF VALUES OF s_1	LOWER BOUND ON $m(t, D, s_1)$	VARIATION OF BOUND (monotone within each interval)
$1 \leq s_1 \leq D$ (1.)	B_3	Decreases from $\frac{1}{2}(t-D)(t-1)$ to $\frac{1}{2}(t-D)^2$
$D \leq s_1 \leq \frac{t+2D}{3}$	B_2	Decreases from $\frac{1}{2}(t-D)^2$ to $\frac{4}{9}(t-d)^2$
$\frac{t+2D}{3} \leq s_1 \leq \frac{t+D}{2}$	B_1	Increases from $\frac{4}{9}(t-D)^2$ to $\frac{1}{2}(t-D)^2$
$\frac{t+D}{2} \leq s_1 \leq \frac{2t+D}{3}$	B_1	Decreases from $\frac{1}{2}(t-D)^2$ to $\frac{4}{9}(t-D)^2$
$\frac{2t+D}{3} \leq s_1 \leq t$	B_1	Decreases from $\frac{4}{9}(t-D)^2$ to 0.

1. This line of the table is omitted in the case $D = 0$.

The table show that $m(t, D, s_1)$ can be less than $\frac{4}{9}(t-D)^2$ only for $s_1 > \frac{2t+D}{3}$, and that for s_1 in this range, $m(t, D, s_1)$ is bounded below by $B_1(t, D, s_1)$, which is monotone decreasing in s_1. This implies the following lemma.

LEMMA 4.4. If A is an incidence matrix satisfying Definition 4.1 and for some $\sigma > \frac{2t+D}{3}$ the number Z satisfies the inequality

$$(4.36) \qquad Z \leq B_1(t, D, \sigma) = 2(t-\sigma)(\sigma - D) ,$$

then $s_1 \geq \sigma$.

The known lower bounds on $m(t, D, s_1)$ admit the possibility that Z is less than $\frac{1}{2}(t-D)^2$, (but not less than $\frac{4}{9}(t-D)^2$) for $D < s_1 < \frac{t \quad D}{2}$. If $m(t, D, s_1)$ were known to be monotone decreasing in s_1, Z would be known to be at least equal to $\frac{1}{2}(t-D)^2$ for all s_1 in this interval and the restriction on σ in the statement of the lemma could be weakened to $\sigma > \frac{t+D}{2}$.

The purpose of Theorem 4.4, Corollary 4.4, Theorem 4.5 and Theorem 4.6 has been to provide methods of proof strong enough to extend Theorem 4.2 to Latin square type association schemes with more than two constraints. It may be recalled that it was desired to prove that the submatrix A_{11}, shown in (4.1), contains a complete $(n-2)$-point. The proof that this is true for most L_g schemes will be completed in the next 3 theorems, the first of which requires Lemma 4.4. The other theorems and corollary which have been proved in this section will not be used explicitly. Submatrix A_{11} was defined by (4.1) but it will be convenient to recall the definition here. Where A_1 is the matrix of first associates in an association scheme with two classes, and two initial treatments which are first associates are chosen, A_{11} is the submatrix whose rows and columns are determined by the p^1_{11} common first associates of the two chosen treatments. It is a $p^1_{11} \times p^1_{11}$ symmetric matrix with 0's on the main diagonal.

THEOREM 4.7. In any association scheme with two associate classes, define

$$(4.37) \qquad \sigma_1 = \tfrac{1}{2}(p^1_{11} + p^2_{11} - 2) + \tfrac{1}{2}\sqrt{(p^1_{11} - p^2_{11} + 2)^2 - 2p^1_{12}(p^2_{11} - 1)}$$

and

$$(4.38) \qquad \sigma_2 = \frac{2p^1_{11} + p^2_{11} - 2}{3}$$

Then if σ_1 is real and $\sigma_1 > \sigma_2$, each pair of first associates in the scheme is contained in a set of k treatments which are pairwise first associates, where $k \geq \sigma_1 + 2$.

PROOF: Lemma 4.4 will be applied to submatrix A_{11} of the incidence matrix A_1 of first associates in the scheme. Two rows of A_{11} containing a pair of symmetrically located off-diagonal 0's correspond to two second associates, and there must be exactly p^2_{11} columns of A_1 which contain 1's in both of the two rows. Columns 1 and 2 are of this form. Therefore there can be at most $p^2_{11} - 2$ such columns in A_{11}, and A_{11} satisfies the conditions of Definition 4.1, with $t = p^1_{11}$, $D = p^2_{11} - 2$.

By Lemma 4.3, the number z_{11} of off-diagonal 0's in A_{11} satisfies inequality (4.2),

$$z_{11} \leq p_{12}^1(p_{11}^2 - 1) \quad .$$

σ_1 will be defined by setting $B_1(t, D, \sigma_1) = p_{12}^1(p_{11}^2 - 1)$, whereupon (4.2) assumes the form of (4.36) in the statement of Lemma 4.4. Using the given values for t and D, the definition of σ_1 may be written

$$2(p_{11}^1 - \sigma_1)(\sigma_1 - p_{11}^2 + 2) = p_{12}^1(p_{11}^2 - 1)$$

and solved for σ_1 to give the definition (4.37) in the statement of this theorem. The other root of the quadratic equation, if real, will be less than $\frac{t+D}{2}$ and cannot meet the conditions of Lemma 4.4.

Definition (4.38) for σ_2 is equivalent to $\sigma_2 = \frac{2t+D}{3}$ and the hypothesis $\sigma_1 > \sigma_2$ of this theorem is identical with the condition placed on σ in Lemma 4.4, where σ_1 here plays the role of σ in the lemma. The lemma may then be applied to show that A_{11} contains a principal minor submatrix of order $s_1 \geq \sigma_1$ with 1's in all off-diagonal positions. The s_1 corresponding treatments of the association scheme, together with treatments 1 and 2, form a set of $k = s_1 + 2 \geq \sigma_1 + 2$ treatments which are pairwise first associates. Since treatments 1 and 2 were taken as an arbitrary pair of first associates in the definition of the submatrix A_{11}, this completes the proof of the theorem.

The inequality $\sigma_1 > \sigma_2$ can be transformed by straightforward algebra to the form

$$(4.38a) \qquad 3\sqrt{(p^1_{11} - p^2_{11} + 2)^2 - 2p^1_{12}(p^2_{11} - 1)} > p^1_{11} - p^2_{11} + 2$$

Squaring both sides gives the following inequality, which is true only if (4.38a) is true, is equivalent to it if $p^1_{11} - p^2_{11} + 2 \geq 0$, and is somewhat simpler to apply.

$$(4.39) \qquad 4(p^1_{11} - p^2_{11} + 2)^2 - 9p^1_{12}(p^2_{11} - 1) > 0 .$$

It will be shown later that for fixed g, association schemes with parameter values of the Latin square type with g constraints satisfy the conditions of Theorem 4.7 if the number n^2 of treatments is sufficiently large, proving the existence of a complete k-point on every line of the graph, with each $k \geq \sigma_1 + 2$. However, for $g \geq 3$, σ_1 is too small for the existence of a complete n-point to be proved by this theorem alone. The next theorem and lemma bridge the gap in the argument.

THEOREM 4.8. In any association scheme with two associate classes, let there be a set A of k_1 treatments which are pairwise first associates, and a set B of k_2 treatments which are pairwise first associates, and let the intersection of the two sets contain u treatments.

(i) If $u \geq 2$, then

$$(4.40) \qquad k_1 + k_2 - u \leq p^1_{11} + 2 .$$

(ii) If there is a treatment in either set which is a second associate of a treatment in the other set, then

(4.41) $u \leq p_{11}^2$.

(iii) If $k_1 + k_2 > p_{11}^1 + p_{11}^2 + 2$, then either $u \leq 1$ or all treatments in the union of sets A and B are pairwise first associates.

PROOF: If $u \geq 2$, then there are at least two treatments which are in both of sets A and B , meaning that each is the first associate of each of the $k_1 + k_2 - u - 2$ remaining treatments in the union of A and B . But no two first associates can have more than p_{11}^1 first associates in common. Therefore $k_1 + k_2 - u - 2 \leq p_{11}^1$, proving statement (i).

If there is a treatment θ in set A and a treatment ϕ in set B which are second associates, they can have at most p_{11}^2 first associates in common. But θ is a first associate of all the remaining treatments in set A , ϕ is a first associate of all the remaining treatments in set B , and the u treatments which are in both sets are common first associates of both θ and ϕ . Therefore $u \leq p_{11}^2$, proving statement (ii).

If the hypotheses of both of statements (i) and (ii) are satisfied, then inequalities (4.40) and (4.41) are both true and may be added to give

$$k_1 + k_2 \leq p_{11}^1 + p_{11}^2 + 2 .$$

If the contrary is true, then one of the hypotheses of statements (i) and (ii) must be false, meaning either that $u \leq 1$ or that each treatment in each set is a first associate of all treatments in both sets, which means that all treatments in the union of the two sets are pairwise first associates. This proves statement (iii) .

In terms of the linear graph whose incidence matrix is A_1, statement (iii) of Theorem 4.8 means that if the graph contains two complete configurations, or k-points, of orders k_1 and k_2, and if $k_k + k_2 > p_{11}^1 + p_{11}^2 + 2$, then either the two configurations have no line in common or the graph contains a complete configuration of which both are subgraphs.

LEMMA 4.5. In any association scheme with two classes, if for any treatment θ there exists an integer k_o satisfying

$$k_o > \tfrac{1}{2}(p_{11}^1 + p_{11}^2 + 2)$$

such that every pair of first associates including θ is contained in a set of $k \geq k_o$ treatments which are pairwise first associates, then the n_1 first associates of θ fall into disjoint sets, each set together with θ forming a complete configuration with at least k_o treatments.

PROOF: Each of the n_1 first associates of θ forms with θ a pair of first associates which by hypothesis are contained in a complete configuration of $k \geq k_o$ treatments. Form one such configuration on θ and each of its first associates and consider the sets of treatments in the n_1 configurations. These are subsets of the set consisting of θ and its first associates. If any of the sets are identical, drop the duplicates. Since each set contains more than $\tfrac{1}{2}(p_{11}^1 + p_{11}^2 + 2)$ treatments, any two of them satisfy the hypothesis of statement (iii) of Theorem 4.8. θ is in each of the sets, and if any first associate is in two of the sets, the two sets have $u \geq 2$ first associates in common and by Theorem 4.8 their union forms a complete configuration. In this case, drop both sets and use their union instead. This process may be repeated as long as any of the first associates of θ are in more than one set. After a finite number of repetitions the result will be a set of disjoint sets of first associates, each set together with θ forming a complete configuration. Each configuration contains at least k_o treatments, since it is formed by union of sets having at least k_o treatments.

THEOREM 4.9. If an association scheme with two associate classes and $v = n^2$ treatments has the parameter values of a Latin square type scheme with g constraints, and if n exceeds the larger root of each of the equations

$$(4.42) \qquad 4n^2 - (g-1)(9g^2-9g+7)n + (g-1)^2(9g^2-9g+7) = 0 \ ,$$

$$(4.43) \qquad 2gn^2 - (g^5-2g^4+3g^3-g^2-2g+1)n - (g^6-3g^5+3g^4+2g^3-3g^2+g+1 = 0 \ ,$$

then there exists a set of g ~~mutually~~ orthogonal $n \times n$ squares which may be used to define first associates in the scheme, and the scheme is of Latin square type.

PROOF: The parameters of a Latin square type scheme with g constraints include the following.

$$n_1 = g(n-1) \ ,$$
$$p_{11}^1 = n + g^2 - 3g \ ,$$
$$p_{11}^2 = g^2 - g \ ,$$
$$p_{12}^1 = (g-1)(n-g+1) \ .$$

σ_1 and σ_2 will be defined as in Theorem 4.7 and σ_1 has the form

$$\sigma_1 = \tfrac{1}{2}(n+2g^2-4g-2) + \tfrac{1}{2}\sqrt{(n-2g+2)^2 - 2(g-1)(n-g+1)(g^2-g-1)}\ .$$

Statement (4.42) will be needed in the application of Theorem 4.7 and Lemma 4.5, while (4.43) will be needed in the final part of the proof. As a preliminary step, it will now be shown that the hypotheses imply $n \geq 2g$, a fact which will be used to simplify the application of Theorem 4.7. When $n = 2g$, the expression in equation (4.43) reduces to

$$-3g^6 + 7g^5 - 9g^4 + 8g^3 + 7g^2 - 3g - 1\ ,$$

which is easily shown to be negative for all $g \geq 2$, showing that the larger root of (4.43) is greater than $2g$ for all $g \geq 2$. It is no restriction to take $n \geq 2$ in the special case $g = 1$. It may therefore be assumed for any g that $n \geq 2g$. Since $p^1_{11} - p^2_{11} + 2 = n - 2g + 2$, this implies

$$(4.44) \qquad p^1_{11} - p^2_{11} + 2 > 0\ .$$

It was pointed out following the proof of Theorem 4.7 that if (4.44) holds, the inequality $\sigma_1 > \sigma_2$ is equivalent to (4.39). In the present case, (4.39) has the form

$$(4.45) \qquad 4(n-2g+2)^2 - 9(g-1)(n-g+1)(g^2-g-1) > 0\ ,$$

reducing to

$$(4.46) \qquad 4n^2 - (g-1)(9g^2-9g+7)n + (g-1)^2(9g^2-9g+7) > 0\ .$$

If n exceeds the larger root of (4.42), this inequality will be satisfied, implying $\sigma_1 > \sigma_2$, and by Theorem 4.7 each pair of first associates in the scheme is contained in a set of k pairwise first associates for some $k \geq k_o = \sigma_1 + 2$. The relations $\sigma_1 > \sigma_2$ and (4.44) are used in the following inequalities on k_o.

$$k_o = \sigma_1 + 2 > \sigma_2 + 2 = \frac{2p_{11}^1 + p_{11}^2 + 4}{3} = \frac{4p_{11}^1 + 2p_{11}^2 + 8}{6} =$$

$$= \frac{3p_{11}^1 + 3p_{11}^2 + 6}{6} + \frac{p_{11}^1 - p_{11}^2 + 2}{6} > \frac{3p_{11}^1 + 3p_{11}^2 + 6}{6} .$$

$$(4.47) \qquad k_o > \tfrac{1}{2}(p_{11}^1 + p_{11}^2 + 2) .$$

Therefore the conditions of Lemma 4.5 are met for any treatment θ, proving that the n_1 first associates of any treatment θ fall into disjoint sets, which will be referred to as <u>special sets</u>, each special set containing at least $\sigma_1 + 1$ treatments and forming with θ a complete configuration of at least $\sigma_1 + 2$ treatments. σ_1 will now be required to satisfy the condition

$$(4.48) \qquad \sigma_1 + 1 > \frac{n_1}{g+1} = \frac{g(n-1)}{g+1} .$$

This reduces to

$$(g+1)\sqrt{(n-2g+2)^2 - 2(g-1)(n-g+1)(g^2-g-1)} > (g-1)n - 2g^3 + 2g^2 - 2$$

and is satisfied if the following inequality, obtained by squaring and simplifying, is satisfied.

$$2gn^2 - (g^5 - 2g^4 + 3g^3 - g^2 - 2g + 1)n - (g^6 - 3g^5 + 3g^4 + 2g^3 - 3g^2 + g + 1) > 0 .$$

This in turn is satisfied if n exceeds the larger root of (4.43), so that the hypothesis of the theorem implies (4.48). It follows from (4.48) that $(g+1)(\sigma_1+1) > n_1$. Since the number of treatments in each special set is at least $\sigma_1 + 1$ and the sum of the numbers of treatments is n_1 , this implies that the number of sets must be less than $g+1$. By corollary 4.1 , θ does not lie in any complete configuration with more than n treatments and none of the special sets can contain more than $n-1$ treatments, and in order for the sum of the numbers to be $n_1 = g(n-1)$ there must be at least g special sets. Therefore there must be exactly g special sets, each containing exactly $n-1$ treatments, meaning that θ , which was an arbitrary treatment, lies in exactly g complete configurations of n treatments, and each pair of first associates lies in such a configuration. Then by Theorem 4.1, there exists a set of g mutually orthogonal $n \times n$ squares which may be used to define first associates in the scheme, completing the proof that the scheme is of Latin square type. The requirements placed on n by Theorem 4.9 will now be examined more closely, in a few cases by using the exact solutions of equations (4.42) and (4.43) .

When $g = 2$, equation (4.42) becomes $4n^2 - 25n + 25 = 0$ and the larger root is $n = 5$; equation (4.43) becomes $4n^2 - 17n - 43 = 0$ and the larger root is $n = 6.03$; the theorem applies for $n \geq 7$. A better result has already been obtained in Theorem 4.2.

When $g = 3$, equation (4.42) becomes $4n^2 - 122n + 244 = 0$ and the larger root is $n = 28.3$; equation (4.43) becomes $6n^2 - 148n - 319 = 0$ and the larger root is $n = 26.6$; the theorem applies for $n \geq 29$.

When $g = 4$, equation (4.42) becomes $4n^2 - 345n + 1035 = 0$ and the larger root is $n = 83.2$; equation (4.43) becomes $8n^2 - 681n - 1957 = 0$ and the larger root is $n = 87.9$; the theorem applies for $n \geq 88$.

When $g = 5$, equation (4.42) becomes $4n^2 - 748n + 2992 = 0$ and the larger root is $n = 182.9$; equation (4.43) becomes $10n^2 - 2216n - 8431 = 0$ and the larger root is $n = 225.3$; the theorem applies for $n \geq 226$.

For use with larger values of g, the general solution of (4.42) is easily obtained, giving the inequality

$$(4.49) \qquad n > \frac{(g-1)(9g^2-9g+7) + (g-1)\sqrt{(9g^2-9g+7)^2 - 16(9g^2-9g+7)}}{8}.$$

For $g \geq 2$, the expression $9g^2 - 9g + 7$ is positive and dropping the second term in the radicand increases the value of the right member of (4.49), showing that it is sufficient for n to satisfy

$$n > \frac{(g-1)(9g^2 - 9g + 7)}{4} = \frac{9g^3 - 18g^2 + 16g - 7}{4}$$

Still for $g \geq 2$, an even stronger requirement on n is

$$(4.50) \qquad n > \frac{9g^3}{4}.$$

The general solution of (4.43) leads to

$$(4.51) \qquad n > \frac{g^5 - (2g^4 - 3g^3 + g^2 + 2g - 1) + \sqrt{g^{10} - (4g^9 - 10g^8 + 6g^7 + 15g^6 - 28g^5 - g^4 - 26g^3 - 10g^2 - 4g - 1)}}{4g}$$

For $g \geq 2$, the two expressions in parentheses are easily shown to be positive, and a sufficient condition for n to satisfy (4.51) is obtained by dropping them, giving

$$(4.52) \qquad n > \tfrac{1}{2}g^4$$

Therefore, for $g \geq 2$, any n satisfying (4.50) and (4.52) will exceed the larger root of each of the equations (4.42) and (4.43), permitting the theorem to be applied. For $g \geq 5$, (4.50) is a weaker requirement than (4.52) and may be dropped if (4.52) is used. The results of the last few paragraphs are summarized in the following corollary.

COROLLARY 4.9. If an association scheme with two associate classes and $v = n^2$ treatments has the parameter values of a Latin square type scheme with g constraints, then the following conditions are sufficient that the scheme be of Latin square type.

If $g = 3$, $n \geq 29$;
if $g = 4$, $n \geq 88$;
if $g = 5$, $n \geq 226$;
if $g \geq 6$, $n > \tfrac{1}{2}g^4$.

Theorem 4.9 shows that for any fixed g and for all values of n except a finite number of possible exceptions, the Latin square type association scheme is unique in the sense that it can be constructed only by means of a set of g orthogonal squares. Corollary 4.9 gives explicit upper bounds below which any exceptional values of n must lie. It may be noted that in the cases $g = 2$, 3 , 4 and 5 the bound given by the simplified inequalities (4.50) and (4.52) is considerably larger than the one obtained from the original equations (4.42) and (4.43). The bounds approach each other in an asymptotic sense, as illustrated by two more special cases.

When $g = 10$, $\frac{1}{2}g^4 = 5000$ and the larger root of (4.43) is 4152.9 .

When $g = 100$, $\frac{1}{2}g^4 = 5 \times 10^7$ and the larger root of (4.43) is 4.90×10^7.

The difference between the bounds is unimportant in any study of designs within the useful range, since no Latin square type designs used in any ordinary statistical experiment at the present time require a value of n larger than 20 . For $g \geq 3$ the question of exactly which values of n admit non-Latin square type designs with Latin square parameter values is still far from solved. For $g = 3$, the scheme with $n = 4$ is easily shown to be unique, it was shown in counter-example 1 of section 4.1 that the scheme with $n = 5$ is not unique, and the question of uniqueness has not been answered for the schemes with $6 \leq n \leq 28$.

In each of the three examples in Section 4.1, second associates could be defined by a set of orthogonal squares, suggesting that in any scheme with L_g parameter values, either first or second associates can be so defined. No proof or disproof of this statement is known, but its implications may be illustrated by a numerical example. If a scheme with L_4 parameter values exists in which first associates cannot be defined by any set of 4 orthogonal squares, and if the statement is true, then second associates can be defined by a set of $n-3$ orthogonal squares, meaning that a set of $n-5$ orthogonal Latin squares exists, and showing incidentally that

L_g schemes exist for all $g \leq n-3$. This is far more than is known for any values of $n \geq 10$ which are not prime powers.

A number of methods have been employed by the writer in an attempt to find a better result than Theorem 4.9, but without much success. The nature of some of these methods will be mentioned as a guide to possible future work on the problem. The crucial step in the proof is to show that each pair of first associates, together with $n-2$ of their common first associates, form a set of n treatments which are pairwise first associates. This means for the incidence matrix A_1 of first associates that the submatrix A_{11}, defined in (4.1), contains an $(n-2) \times (n-2)$ submatrix with 1's in all off-diagonal positions, or in terms of linear graphs, that in the graph whose incidence matrix is A_1, every line is contained in a complete n-point. This step of the proof, which took one paragraph in the case of Theorem 4.2 and L_2 schemes, has occupied most of the present section in the general case, and has been divided into three phases, as follows.

Part 1: Lemma 4.3. The number Z_{11} of off-diagonal 0's in A_{11} is small.

Part 2: Theorem 4.7; Lemma 4.4 and Theorems 4.4 to 4.6. If Z_{11} is small, then each line is contained in a complete k-point, where k is fairly large.

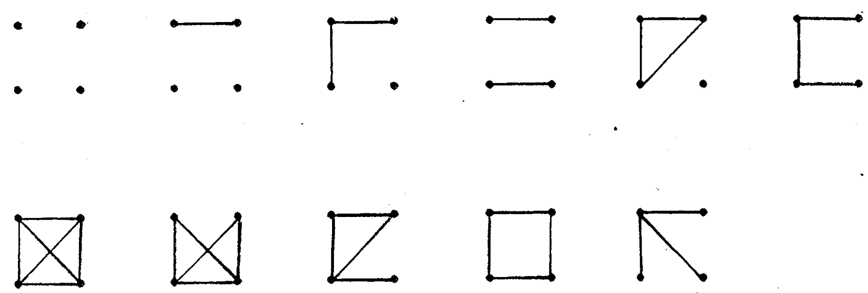
Part 3: Theorem 4.8 and Lemma 4.5; parts of proof of Theorem 4.9. If the k-points are sufficiently large, they can fit into the graph only if the scheme has Latin square structure.

The remarks to be made about Parts 2 and 3 are brief and will precede the discussion of Part 1.

The principal result used in Part 2 is Lemma 4.4, which makes use of the lower bounds on $m(t, D, s_1)$ developed in Theorems 4.4 to 4.6. If a proof along the lines used in Theorem 4.9 is to be improved at this point, it would seem that the thing to look for is a stronger lower bound. As pointed out following the proof of Lemma 4.4, a proof that $m(t, D, s_1)$ is monotone decreasing in s_1 would immediately permit a stronger statement of the lemma. While there is considerable literature on linear graphs and their incidence matrices, the case in which there is a restriction on the inner product of rows of the matrix does not seem to have received much attention and it is possible that more information on the number of 0's in A could be obtained.

In Part 3, it is shown without difficulty that the first associates of each treatment fall into disjoint sets, each forming with the given treatment a set corresponding to a complete k-point of the graph. Then in the proof of Theorem 4.9 the condition (4.48) is imposed. This insures that each of the disjoint sets of first associates is large enough that the number of sets can be at most g, making it easy to prove that the number of sets must be exactly g. It turns out that (4.48) requires much larger values of n for all $g \geq 4$ than any of the other conditions imposed. If additional information could be obtained about the k-points in the graph, or about the number of disjoint sets of first associates, it might be possible to replace condition (4.48) by some weaker requirement.

Most of the writer's attempts to generalize Theorem 4.9 were concentrated on Part 1, the derivation of an upper bound for the number Z_{11} of off-diagonal 0's in submatrix A_{11}, or equivalently, of a lower bound for the number T_{11} of 1's in A_{11}. The definition of an association scheme for a PBIB design is enough to determine the number of lines in the corresponding graph, the number of triangles on a line, and the number of occurrences of other configurations which involve three points of the graph. The definition does not determine the frequencies with which any subgraphs having four or more points occur. The number of complete 4-points which include treatments 1 and 2 in (4.1) is identical with the number of pairs of symmetrically located 1's in submatrix A_{11}, and is therefore very closely related to the problem. In an effort to determine the total number of complete 4-points in the graph, the more general problem of classifying the $\binom{n^2}{4}$ 4×4 principal minor submatrices of A_{11}, which determine the subgraphs having 4 points, was begun. Apart from permutations of rows and columns, there are 11 distinct symmetric 4×4 incidence matrices with 0's on the main diagonal, corresponding to the 11 distinct graphs on 4 points.



Several equations were found in the 11 frequencies with which the 4×4 submatrices occurred in A_1, for example by computing the total number of triangles of the graph in terms of the frequencies of the 4-graphs containing triangles, and equating it to the known total number of triangles in the graph. The sum of the determinants of the $\binom{n^2}{4}$ submatrices is equal to a known coefficient of the characteristic equation of A_1 and led to another equation. The 11 frequencies are expressible in terms of the total numbers $T_{\mu\nu}$ of 1's in the submatrices $A_{\mu\nu}$ of A_1 and it was found advantageous to set up all the equations in terms of the $T_{\mu\nu}$. There are 16 submatrices but symmetry of A_1 gives 6 equations of the form $T_{\mu\nu} = T_{\nu\mu}$ and reduces the number of independent $T_{\mu\nu}$ to 10. Other methods were used to obtain equations in the $T_{\mu\nu}$, in particular an enumeration of the 3-chains joining points 1 and 2 of the graph. The number could be expressed in terms of certain of the $T_{\mu\nu}$ and could be computed directly in terms of products and other operations on the matrix A_1, using methods of Katz [25] and Ross and Harary [29]. A similar enumeration of chains of 4 or more lines was investigated. In all, over 20 equations were obtained, reducing to a set of 9 independent linear equations in the 10 totals $T_{\mu\nu}$, and a one-parameter family of solutions was obtained. For reasons which will be stated in the following paragraph, it was not to be expected that a tenth independent equation could be determined, but the non-negative nature of the $T_{\mu\nu}$ provides some inequalities and leads to upper and lower bounds on the totals $T_{\mu\nu}$ and on the frequencies of the 11 types of 4 x 4 submatrices. No relations leading to further inequalities were found, and the best result obtained for T_{11} was equivalent to inequality (4.2) of Lemma 4.3. The other information obtained on the $T_{\mu\nu}$ appears to contribute nothing to the problems of this thesis, and will not be discussed.

All of these results apply to any association scheme with two classes.

A comparison of inequality (4.2) with the situation in an actual association scheme will show why an improved inequality was hoped for, as well as a possible reason why one was not found. The case of an L_3 scheme with 25 treatments will be taken as an example.

Suppose that the treatments are represented by the numbers in the following array, and that first associates are defined by rows and columns of the array and the letters of a 5 x 5 Latin square with A B C D E as its first row.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

The $p_{11}^1 = 5$ common first associates of treatments 1 and 2 are treatments 3 , 4 and 5 , the treatment occurring with the letter A in column 2 and the treatment occurring with the letter B in column 1 . Treatments 3 , 4 and 5 are pairwise first associates, accounting for six 1's in the 5 x 5 submatrix A_{11} . Neither of the two remaining treatments is a first associate of any of treatments 3 , 4 and 5 . If they are first associates of each other they lead to two more 1's in A_{11} and $T_{11} = 8$; if they are second associates, $T_{11} = 6$. The value of T_{11} depends on whether or not the array $\begin{matrix} A & B \\ B & A \end{matrix}$ occurs in the first two columns of the Latin square. The following two examples show that either situation is possible.

A	B	C	D	E		A	B	C	D	E
B	A	E	C	D		B	E	A	C	D
C	E	D	A	B		C	A	D	E	B
D	C	B	E	A		D	C	E	B	A
E	D	A	B	C		E	D	B	A	C

Both structures can occur in $n \times n$ Latin squares for many values of n, and probably for all $n > 3$, and it is easy to show that in the two possible cases the number T_{11} of 1's in the $n \times n$ submatrix A_{11} is either n^2-5n+6 or n^2-5n+8. The number Z_{11} of off-diagonal 0's is then $4n-6$ or $4n-8$ respectively. Thus T_{11} is not determined uniquely by the parameter values of the association schemes, and no system of equations can lead to a unique value for it. On the other hand, the assertion of Lemma 4.3 in an L_3 scheme is $Z_{11} \leq 10n-20$. This upper bound is considerably larger than either of the possible values. In view of the large discrepancy between this upper bound and either of the possible values, there appear to be good grounds to be dissatisfied with it, at least until a thorough search has been made for a better one. The efforts of the writer to improve Lemma 4.3 have already been described.

While the proof of Theorem 4.9 included a demonstration that in the schemes involved, <u>every</u> pair of first associates is contained in an n-point, the weaker result that <u>one</u> n-point exists in the scheme would have been sufficient in the case of some schemes. This was shown for L_2 schemes in Theorem 4.3, and is shown for a class of L_3 schemes in the following theorem. The parameter values of L_3 schemes will now be listed for easy reference.

$$v = n^2 , \qquad P_1 = \begin{bmatrix} n & 2n-4 \\ 2n-4 & (n-2)(n-3) \end{bmatrix} ,$$
$$n_1 = 3(n-1) ,$$
$$n_2 = (n-1)(n-2) , \qquad P_2 = \begin{bmatrix} 6 & 3n-9 \\ 3n-9 & n^2-6n+10 \end{bmatrix} .$$

THEOREM 4.10. If an association scheme with two associate classes has parameter values $v = n^2$, $n_1 = 3(n-1)$, $p^1_{11} = n$, where $n \geq 14$, and there exists a set of n treatments which form a complete n-point, then every pair of first associates is in such a set and the scheme is of L_3 type.

PROOF: Relations (2.2) to (2.5) may be used with the given parameter values to show that the remaining parameter values are those of an L_3 scheme. Number treatments so that the set of treatments in the complete n-point receives numbers 1 to n , with an arbitrary treatment of the set designated as treatment n . Let θ be a first associate of treatment n which is not in the n-point but is otherwise arbitrary. Next consider the $p^1_{12} = 2n-4$ treatments which are first associates of treatment n and second associates of θ . Fewer than n of these can be in the n-point, so that (for $n \geq 4$) one such treatment not in the n-point can be chosen. A treatment chosen in this way will be numbered $n + 1$.

The pair of first associates n , $n+1$ will now be used in an indirect method of obtaining information about treatment θ . The first step is a classification of the treatments other than $1, 2, \ldots, n+1$ into four mutually exclusive sets. Choose notation so that the treatments in each set have consecutive numbers and the four sets are numbered in the order listed.

Set 1: common first associates of treatments n and $n+1$,

Set 2: the remaining first associates of treatment n (including θ) ,

Set 3: the remaining first associates of treatment $n+1$,

Set 4: common second associates of treatments n and $n+1$.

By Lemma 4.1, treatment $n+1$ has exactly $g-1 = 2$ first associates in the n-point, of which one is treatment n and the other must be one of the common first associate of treatments n and $n+1$. Set 1 consists of the rest of the $p_{11}^{1} = n$ common first associates. Therefore Set 1 contains $n-1$ treatments.

Treatment n has $n_1 = 3(n-1)$ first associates, of which $n-1$ are in the n-point, one is treatment $n+1$, and $n-1$ are in set 1 . By difference, Set 2 contains $n-2$ treatments.

It may be shown that Sets 3 and 4 contain $2n-4$ and n^2-5n+6 treatments respectively, but these facts will not be used.

The rows and columns corresponding to Sets 1 to 4 determine submatrices which will be denoted by $D_{\mu\nu}$ as indicated in (4.53) below. $t_{\mu\nu}$ and $z_{\mu\nu}$ will denote the number of 1's and the number of off-diagonal 0's , respectively, in a row of $D_{\mu\nu}$; $T_{\mu\nu}$ and $Z_{\mu\nu}$ will denote the number of 1's and the number of off-diagonal 0's , respectively, in the entire submatrix $D_{\mu\nu}$. Unless otherwise specified, statements made for $t_{\mu\nu}$ and $z_{\mu\nu}$ will be true for each row involved.

$$(4.53) \qquad A_1 = \left[\begin{array}{ccccc|c|c|c|c|c|c}
0 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\
1 & 0 & \cdot & \cdot & 1 & 1 & & & & & \\
\cdot & \cdot & & & \cdot & \cdot & & & & & \\
\cdot & \cdot & & & \cdot & \cdot & & & & & \\
1 & 1 & \cdot & \cdot & 0 & 1 & & & & & \\ \hline
1 & 1 & \cdot & \cdot & 1 & 0 & 1 & 1\ .\ 1 & 1\ .\ 1 & 0\ .\ 0 & 0\ .\ 0 \\ \hline
 & & & & & 1 & 0 & 1\ .\ 1 & 0\ .\ 0 & 1\ .\ 1 & 0\ .\ 0 \\ \hline
\multicolumn{5}{c|}{E_1} & \begin{matrix}1\\ \cdot\\ 1\end{matrix} & \begin{matrix}1\\ \cdot\\ 1\end{matrix} & D_{11} & D_{12} & D_{13} & \\ \hline
\multicolumn{5}{c|}{E_2} & \begin{matrix}1\\ \cdot\\ 1\end{matrix} & \begin{matrix}0\\ \cdot\\ 0\end{matrix} & D_{21} & D_{22} & D_{23} & \\ \hline
\multicolumn{5}{c|}{} & \begin{matrix}0\\ \cdot\\ 0\end{matrix} & \begin{matrix}1\\ \cdot\\ 1\end{matrix} & D_{31} & D_{32} & D_{33} & \\ \hline
\multicolumn{5}{c|}{} & \begin{matrix}0\\ \cdot\\ 0\end{matrix} & \begin{matrix}0\\ \cdot\\ 0\end{matrix} & & & &
\end{array}\right]
\begin{array}{l}
\text{Treatments} \\ \text{1 to n-1 .} \\ \text{Treatment n .} \\ \text{Treatment n+1 .} \\ \text{Set 1.} \\ \text{n-1 rows.} \\ \text{Set 2.} \\ \text{n-2 rows.} \\ \text{Set 3.} \\ \text{Set 4.}
\end{array}$$

By Lemma 4.1, any treatment not in the n-point is the first associate of exactly two treatments of the n-point. If it is in Set 1 or Set 2, one of these two treatments is treatment n , and one is among treatments 1 to $n-1$. This implies that every row of E_1 and E_2 contains exactly one 1 .

The inner product of two rows of A_1 is equal to $p^1_{11} = n$ or $p^2_{11} = 6$, according as the two rows correspond to first or second associates. The inner product of row n with the row corresponding to a treatment of Sets 1 to 4 may be expressed in terms of the row totals $t_{\mu\nu}$. Among the first $n-1$ elements in row $n + 1$ is a single 1 which may contribute to the inner product of this row with other rows, and the inner product cannot be expressed exactly in terms of the $t_{\mu\nu}$. However, inequalities can be obtained. The following relations are obtained in this way.

Using inner products of row n with rows of Set 2,

$$(4.54) \qquad t_{21} + t_{22} = n - 1 \ .$$

Using inner products of row $n+1$ with rows of Set 2,

$$t_{21} + t_{23} \leq 5 \ .$$

Because the row totals $t_{\mu\nu}$ are non-negative, the last statement implies

$$(4.55) \qquad t_{21} \leq 5 \ .$$

Each row of D_{22} has $n-2$ elements, or which $n-3$ are not on the main diagonal of A_1 , giving

$$(4.56) \qquad t_{22} + z_{22} = n - 3 \ .$$

Statements (4.54), (4.55) and (4.56) may be solved simultaneously to give

$$(4.57) \qquad z_{22} \leq 3 \ ,$$

a statement which holds for every row of submatrix D_{22} .

Assume that $Z_{22} > 0$, which is equivalent to saying that submatrix D_{22} contains off-diagonal 0's . Consider two such 0's which are symmetrically located with respect to the diagonal. The rows of A_1

containing these 0's correspond to a pair of second associates and have inner product equal to 6, meaning that exactly 6 columns of A_1 contain 1's in both of these columns. Column n is one such column. Therefore at most 5 columns of D_{22} contain 1's in both of these rows. D_{22} contains n-2 columns, and each of the remaining n-7 columns must contain a 0 in at least one of the two rows. This total includes the two 0's first considered. One of the rows must contain at least half of this total, which may be expressed $\left[\frac{n-6}{2}\right]$, where $[x]$ denotes the greatest integer $\leq x$. Therefore, under the assumption that $Z_{22} > 0$, there must be <u>at least one</u> row for which

$$(4.58) \qquad z_{22} \geq \left[\frac{n-6}{2}\right] .$$

This violates (4.57) for all $n \geq 14$. Therefore for $n \geq 14$, D_{22} contains 1's in all off-diagonal positions. The treatments of Set 2 together with treatment n are therefore pairwise first associates and form a complete (n-1)-point. Treatment θ is therefore contained with treatment n in a complete (n-1)-point. But θ is an arbitrary treatment of the set of first associates of treatment n which are not in the initial n-point. Therefore every first associate of treatment n is contained with treatment n in a complete configuration with at least n-1 treatments. n-1 exceeds $\frac{1}{2}(p_{11}^1 + p_{11}^2 + 2) = \frac{1}{2}(n + 6 + 2)$ for all $n \geq 11$, and for such values of n, Lemma 4.5 may be applied to show that the 3n-3 first associates of treatment n fall into <u>disjoint</u> sets of at least n-2 treatments, each set forming a complete configuration with treatment n. By lemma 4.1, none of the disjoint sets can contain more than $n-1$ treatments. It is easily verified that these conditions can be satisfied only if there are 3 sets, each with exactly $n-1$ first associates of treatment n. Therefore treatment n is in 3 n-points which have no other treatment in common, and is contained with every one of its first associates in a complete n-point. But in the numbering of treatments, treatment n was taken as an arbitrary treatment of the given n-point. Therefore every treatment of the given n-point is contained with each of its first associates in a complete n-point. Every one of the n^2-n treatments not in the initial set of n is a first associate of two treatments of the set and is therefore in at least one complete n-point. Finally, since every treatment in the scheme is contained in an n-point, the argument used here shows that every treatment is contained with each of its first associates in an n-point. Theorem 4.1 then shows that the scheme is of Latin square type and the proof is complete.

Theorem 4.10 is vacuous for all values of $n \geq 29$, for which Theorem 4.9 gives the stronger result that any scheme with L_3 parameter values must have L_3 structure. Theorem 4.10 shows that for $14 \leq n \leq 28$, if the existence of one n-point in an association scheme with L_3 parameter values can be demonstrated, the scheme must have L_3 structure. The possibilities for extending Theorem 4.10 to Latin square type schemes with more than three constraints or to smaller values of n in the case of three constraints appear to be about as good as the possibilities already discussed for the extension of Theorem 4.9. Example 2 of Section 4.1, which shows that non-L_3 schemes with L_3 parameter values can exist, is not known to be a counter-example to Theorem 4.10, because it is not known whether the association scheme contains any complete 5-point. It was verified that a particular pair of first associates was not contained in a 5-point, but the scheme contains 150 pairs of first associates, most of which have not been investigated.

All of the theorems and lemmas of this section except Theorems 4.9 and 4.10 apply to any association scheme with two associate classes. The same is true of Lemma 4.3 in the previous section. It is possible to use them to investigate the structure of association schemes not in the Latin square series. Theorem 4.7 provides a sufficient condition for the existence of a complete k-point, or set of k treatments which are pairwise first associates, and is easily applied to any association scheme. This was done with the schemes listed in Table II and it was found that in most cases σ_1 is imaginary and the theorem proves nothing. However, σ_1 is real and satisfies the required inequality for schemes of the Triangular series with 66 or more treatments. It is possible to use Theorem 4.7 as the basis of a proof that for $n \geq 12$, the only association scheme with $v = \binom{n}{2}$ treatments and the parameter values of the Triangular series is the scheme whose construction was described in Section 2.1. It is not known whether this is a new result. In any case, the proof will not be given here. Speaking rather loosely, the thing which is needed to make Theorem 4.7 work is a large value of p^1_{11} and a small value of p^2_{11}. A small value of p^2_{11} means a small value for the inner product of two rows of the association matrix which correspond to second associates and is closely related to the restriction on such inner products stated in Definition 4.1 and used in several theorems of this section. In most of the schemes of Table II which are not in the Triangular or Latin Square series, p_{11}^2 is at least as large as p_{11}^1 and it can probably not be expected that the methods of this section will show the existence in these schemes of k-points for any large k. A more significant fact for many association schemes may be the non-existence of k-points. The methods of this chapter were not designed to prove this.

V. SUMMARY

CHAPTER I. GENERAL PROPERTIES OF PARTIALLY BALANCED DESIGNS AND ASSOCIATION SCHEMES

Section 1.1. Introduction.

This section gives some simple examples of incomplete block designs and partially balanced incomplete block designs in particular, followed by a formal definition of PBIB designs and a basic list of relations satisfied by the parameters of the designs.

Section 1.2. Association Schemes and Incidence Matrices.

In this section association schemes are defined with some simple examples, and the incidence matrices of association schemes, denoted by A_i, are introduced. The relation of these matrices to the more familiar incidence matrix N of the blocks of the design is discussed briefly.

Section 1.3. Applications and Algebraic Properties of the Matrices A_i.

Theorem 1.1 gives a rule (1.16) for forming products of the matrices A_i. This result is used in several parts of Chapters II and III, and has other applications which are not treated in this dissertation. Theorem 1.2 shows that any set of matrices satisfying (1.16) and a few light restrictions may be used to define an association scheme satisfying the conditions of partial balance. This theorem is used in the proof of Theorem 3.3.

Association incidence matrices do not seem to have received much study. Nearly all of the results presented or mentioned in this section were obtained as original results by the writer, but some of them have recently been obtained by others, to whom credit is given in the discussion following Theorem 1.2.

CHAPTER II. ENUMERATION OF POSSIBLE DESIGNS AND ASSOCIATION SCHEMES WITH TWO ASSOCIATE CLASSES

Section 2.1. The Class of PBIB Designs with Two Associate Classes.

The general expressions given in Chapter I for partially balanced designs are specialized in (2.1) to (2.5) to the important special case of designs with two associate classes. It is convenient to classify PBIB designs according to the method of defining the association relation, and a classification due to Bose and Shimamoto of the known association schemes with two classes is adopted here. Four of the types, group divisible, triangular, simple, and cyclic, are described briefly, and the fifth type, Latin square, is discussed in considerable detail. The association scheme of a design of Latin square type with g constraints, briefly denoted by L_g, is ordinarily defined in terms of a set of $g-2$ mutually orthogonal Latin squares; the present treatment is based instead on a set of g mutually orthogonal squares which do not necessarily have the Latin square property. The symmetry of an L_g association scheme is emphasized by this point of view, which is not new but does not seem to have been discussed much in the available literature. Expressions for the parameter values of L_g schemes are derived, first in expressions (2.9), then in a new notation in (2.12) for use in Chapter III. It is pointed out that for certain negative values of the arguments, these expressions give sets of parameter values which are different from those for any of the schemes classified by Bose and Shimamoto. The possible new schemes are given the name "negative Latin square" and the brief notation L_g^*, where g is a negative integer, and are studied at some length in Chapter III.

This section is primarily a collection of known results, with the addition of some new notation and the definition of negative Latin square designs.

Section 2.2. Enumeration of Association Schemes.

An enumeration of association schemes may be considered a preliminary step in the enumeration of combinatorially possible PBIB designs and is carried out in this section for designs with two associate classes. Group divisible schemes are easily enumerated and are omitted from the present list. The enumeration is arbitrarily limited to schemes with $v \leq 100$, a figure which was chosen to include most of the schemes within the range useful to experimenters, and to include schemes related to 10×10 Latin squares.

Some notation of Connor and Clatworthy is adopted and one of their results is listed as Theorem 2.0. This theorem specifies a one-parameter family of non-group-divisible schemes, whose parameter values are listed in Table Ia of the Appendix. All other non-group-divisible schemes are shown in Theorem 2.1 to be contained in a larger family whose parameter values can be listed systematically. Table Ib of the

Appendix is a working table in which this listing is carried out. This list is shortened somewhat by omitting the complement of each scheme listed, that is, the scheme obtained by changing the designations of first and second associates. Table II collects the results of Tables Ia and Ib in an orderly arrangement. The parameter values of known association schemes are identified in this table, along with some which are proved impossible by later theorems of this section. Table II lists 101 sets of parameter values, of which four are shown to be impossible, 50 were already known, 6 are constructed for the first time in this dissertation, and the remaining 41 are still unknown.

Theorems 2.2 and 2.3 show that if the number of treatments in a PBIB design with two associate classes is of the form $p+1$ or p for any prime p, then the only possible association schemes are of group divisible type or the type specified by Theorem 2.0, respectively. Theorems 2.4 to 2.8 state additional necessary conditions for the existence of association schemes with two associate classes. The condition stated in Theorem 2.5 is used to shorten the computation of Table Ib. The other theorems provide the four impossibility proofs mentioned in connection with Table II, and give some information about the structure of any possible scheme in approximately 12 of the unknown cases. Lemma 2.2, used in the proof of Theorem 2.8, specializes Theorem 1.2 to the case of two associate classes, giving a simple condition that a given matrix be the incidence matrix of first associates. It is used again in Section 3.3.

An exhaustive list of possible partially balanced designs was often promised by the earlier writers in the field but does not seem to have appeared, although Bose, Clatworthy and Shrikhande have published tables which include virtually all designs within the practical range known up to 1953. The present tabulation is believed to be new. It should be of some use in the application of PBIB designs to experiments, and of further use in later studies of the structure of designs and association schemes. Also new in this section are most of the details of Theorem 2.1 and all of Theorems 2.2 to 2.8. Several immediate additions to the tables given here are possible, including an extension to some values of $v > 100$, and further investigation of the 41 schemes which are unknown. Another question to be discussed in some aspects for Latin square type designs in Chapter IV but considered only incidentally for other designs, is the question of the number of solutions of a constructible association scheme.

Section 2.3. Enumeration of Possible Designs for Particular Association Schemes.

Several known facts about PBIB designs are reviewed and used to develop a systematic method of enumerating all possible designs for a given association scheme. The method is outlined in this section and carried out in Tables III and IV of the Appendix. The enumeration is limited to constructed association schemes of the L_g and L_g^* series, and for each association scheme is limited to designs with $r \leq 10$ and $k \leq 10$. Many of the designs in Table IV are easily constructed and a few are easily shown to be impossible; all designs either constructed or known to be impossible are identified in the table. Many of them are easily enumerated by a few methods which are listed for convenience as Theorems 2.9 to 2.15 in this section. Two designs which have been constructed by the author by other methods are listed in Section A.3 of the Appendix. Enumeration proofs of impossibility of three designs appear in the same section. Section 2.3 concludes with a brief mention of singular designs.

The author is not sure that any of the material in Section 2.3 is new, though no list of possible designs as inclusive as Table IV seems to have appeared and Theorems 2.12 to 2.14 may be new. Tables III and IV could easily be extended to designs with association schemes of other types, and to designs with $r > 10$. The latter extension would be of dubious value to experimenters but might give a useful background for further theoretical studies. The large number of unknown designs in any list such as Table IV suggests a comparably large collection of potential theorems on the construction or impossibility of designs.

CHAPTER III. NEGATIVE LATIN SQUARE TYPE ASSOCIATION SCHEMES

Section 3.1. Relationships Between Latin Square and Negative Latin Square Association Schemes.

It is pointed out that the Negative Latin square schemes share with the ordinary Latin square schemes the property that the multiplicities α_1 and α_2 of the characteristic roots of NN' are equal in some order to the numbers n_1 and n_2 of first and second associates of a treatment. It is shown in Theorem 3.1 that the only other schemes with this property are the one-parameter family specified by Theorem 2.0. There is some discussion of two alternate notations for the parameter values of the negative Latin square series. In one notation, negative integer parameters are used, n^* the negative square root of $v = n^2$ and g^*, the negative integer which is used as a subscript in the symbol L_g^*. In this notation, the expressions for the parameter values have the same form as those for the Latin square series. The other notation is based on the positive square root of v and the numerical value of the subscript in the symbol L_g^* and does not lead to expressions of the same form but is more convenient for some purposes. The section concludes with some remarks about the relation between negative Latin square schemes and finite Euclidean plane geometries. The existence of the geometry is a sufficient but not a necessary condition for the existence of an ordinary Latin square scheme. The existence of a connection either way between the geometry and the negative Latin square scheme has not been proved or disproved.

The computation of the multiplicities α_i of the characteristic roots of NN' was first carried out for L_g designs by Connor and Clatworthy, using a method which immediately applies to L_g^* designs. The class of negative Latin square designs and association schemes was defined in Section 2.1 and the study of the connection between the α_i and the n_i is new in this section.

Section 3.2. Construction of Negative Latin Square Type Association Schemes by a Method Based on Finite Fields.

Theorem 3.2 provides a method of constructing a wide class of association schemes from finite fields. In general the schemes have more than two associate classes. Methods are described for setting down the association scheme and for computing the values of the parameters n_i and p^i_{jk}. Following an illustrative example using the field with 16 elements there is a discussion of two families of schemes which can be constructed when the order of the finite field is a perfect square n^2 (which requires that it be an even power of some prime). The simpler of these two schemes is shown to be equivalent to the finite Euclidean plane with n points on a line, and the parameter values are computed. The same computation for the later scheme is completed later in the section for several particular values of n, but is not carried out in general.

An association relation defined by combining associate classes in a scheme with three or more classes will not in general satisfy the conditions of partial balance. Theorem 3.3 states necessary and sufficient conditions for a relation defined in this way to satisfy the definition of an association scheme. In Corollary 3.3 a simplified form of the conditions is stated for the case in which the new scheme has two classes. The proof of the theorem makes use of association matrices and applies Theorems 1.1 and 1.2.

The method of Corollary 3.3 is then applied to the schemes constructed for n^2 treatments by the method of Theorem 3.2. It is shown that L_g schemes for any $g \leq n$ can be constructed in this way from the schemes of the first family for each value of $v = n^2$ which is a prime power. The second family of schemes is related to negative Latin square schemes, four of which are constructed in this section. The method either fails or is not applicable to the remaining L_g^* schemes taken up in the present study. As a result of their common origin from a finite field, the L_g^* scheme with n^2 treatments constructed here and the finite Euclidean plane with n^2 points are related in a way which is shown to permit a geometrical interpretation of the scheme.

Theorems 3.2 and 3.3 both have applications beyond those developed in this section. Both theorems were derived by the writer but the equivalent of Theorem 3.2 was published independently by Sprott in 1955, before the writing of this dissertation was completed. A comparison of the present work with that of Sprott appears in the concluding paragraph of the section. The L_g^* schemes constructed here are believed to be new.

Section 3.3. Construction of a Negative Latin Square Type Scheme with 100 Treatments by Enumeration.

In this section a detailed study is made of a particular association scheme with 100 treatments. The 100×100 incidence matrix A_1 is studied in detail and because of results proved in earlier chapters of this dissertation and because of some simplifying circumstances for the particular scheme, it is possible to obtain rather complete information about properties of certain submatrices of A_1. It is shown in particular that one 22×77 submatrix S is the incidence matrix for the blocks of a balanced incomplete block (BIB) design and that if the design is constructed the entire matrix A_1 can be constructed from it. The construction of the balanced design is the part of the section which uses empirical methods, and even though some effective shortcuts are used, the reader has to put up with the individual examination of about half of the 77 blocks of the design, following some of them through several stages of incompletion and false starts. Once the design is constructed, the association matrix A_1 can be constructed in short order. The balanced design itself is a by-product, and not the only one. The dual of the design, obtained by interchanging the notions of treatment and block, is found to be a PBIB design with a previously unknown association scheme whose matrix of first associates appears as another submatrix of A_1 and which is constructed here for the first time. Other submatrices of A_1 are related to still other designs and to some interesting arrangements of 4×4 orthogonal squares.

This section applies several methods of Chapters I and II which may be new, results in two association schemes which are believed to be new, and gives constructions of several other incomplete block designs and other combinatorial arrangements which may be of interest. The scheme with 100 treatments is in the negative Latin square series and cannot be constructed by the method of Section 3.2 because there is no finite field with 100 elements. The scheme may possibly have a connection with the unsolved question of the existence of orthogonal 10×10 squares, but the author has no conjecture as to what sort of connection there might be.

CHAPTER IV. THE STRUCTURE OF LATIN SQUARE TYPE ASSOCIATION SCHEMES

Section 4.1 Preliminary Discussion of Uniqueness, and Some Counter-examples.

Given any set of g mutually orthogonal $n \times n$ squares, an L_g scheme can be constructed by using rows of the squares to define first associates. It is not obviously true that all schemes with the parameter values of the Latin square series can be constructed in this way. If it is true for a particular pair of values of n and g, so that the existence of a scheme with the appropriate parameter values implies the existence of the set of orthogonal squares, we shall say that the L_g scheme for n^2 treatments is unique. The term **unique** will be used in this situation whether or not the set of orthogonal squares is unique, and questions of enumeration of Latin squares are not taken up here. An L_g association scheme will be said not to be unique if there exists a scheme having the same parameter values but no set of orthogonal squares exists by which first associates in the scheme can be defined. Three examples are given in this section of L_g schemes which are not unique.

If first associates in a scheme cannot be defined by orthogonal squares, it may be that second associates can; in fact, this is the case in each of the three examples. It may be conjectured that in any scheme with Latin square parameter values, either first or second associates may be defined by a suitable set of Latin squares. No proof or disproof of this conjecture is attempted in this chapter. Instead it is proved in Sections 4.2 and 4.3 that for a fixed number g of constraints and sufficiently large n, the L_g scheme for n^2 treatments is unique in the sense defined above. An alternate statement is that for a fixed number n^2 of treatments and a sufficiently small number g of constraints, the existence of the association scheme is equivalent to the existence of the set of orthogonal squares. For a comparison of these results with the conjecture just stated, the reader is referred to the discussion following Corollary 4.9 in Section 4.3.

Section 4.1 contains a statement of some terminology of linear graphs which is used throughout Chapter IV and in this summary.

Section 4.2. On the Uniqueness of L_2 Association Schemes.

The uniqueness of Latin square type association schemes with two constraints is taken up in this section, though some of the theorems and lemmas apply more generally. The uniqueness of an L_g association scheme for n^2 treatments is proved if it can be shown that each treatment is contained in g complete n-points which have no treatments in common in addition to the initial one. In Theorem 4.1 it is proved that for $n > (g-1)^2$ it is sufficient to show that each pair of first associates is contained in one complete n-point. This is preceded by two lemmas. If a scheme with the parameter values of the Latin square series contains n treatments forming a complete configuration, then Lemma 4.1 reveals a good deal of uniformity in the association relations of the n treatments with the remaining n^2-n treatments. It is an immediate corollary that no complete configuration having more than n points can occur in a scheme with Latin square parameter values. Both Lemma 4.1 and its corollary are repeatedly useful in this chapter. Lemma 4.2 deals with the number of treatments which two complete configurations can have in common in an L_g scheme, and is slightly stronger in this case than Theorem 4.8 and Lemma 4.5, which apply to a wider class of schemes.

Unlike the other theorems and lemmas of this section, Lemma 4.3 is not restricted to schemes with L_g parameter values. It states an upper bound for the number of off-diagonal 0's in a specified submatrix of the association matrix A_1 in any scheme with two associate classes. This of course is equivalent to a lower bound on the number of 1's. In Theorem 4.2, the principal result of the section, the same submatrix is examined in the case of L_2 schemes, and it is shown that with the single exception of the scheme with 16 treatments, the lower bound of Lemma 4.3 is inconsistent with the presence of any off-diagonal 0's in the submatrix. The portion of the linear graph corresponding to the submatrix is then a complete configuration and it follows easily that every pair of first associates is contained in a complete n-point. Theorem 4.1 then shows that the L_2 scheme is unique. The scheme with 16 treatments had already been shown by one of the examples of Section 4.1 not to be unique. Additional information in this exceptional case is given by Theorem 4.3.

In a passage following the proof of Theorem 4.2 it is shown that unless the methods used in this section can be improved, it will not be possible to generalize Theorem 4.2 to other L_g schemes. The new methods and the generalization appear in Section 4.3.

Section 4.3. On the Uniqueness of L_g Association Schemes, $g \geq 3$.

The principal results of this section are Theorem and Corollary 4.9, in which are established the uniqueness of an infinite class of Latin square type association schemes. The preparation for this theorem is long and somewhat indirect, involving five theorems and two lemmas in this section, as well as some of the material of Section 4.2.

Theorems 4.4 to 4.6 and Lemma 4.4 are general results on the structure of incidence matrices, all with a bearing on the existence of complete configurations, or equivalently the existence of principal minor submatrices with 1's in all off-diagonal positions. These theorems are arranged in order of decreasing generality, Lemma 4.4 stating a particular fact which is used in Theorem 4.7.

A property of association matrices of PBIB designs and of their submatrices is that the inner products of rows or columns taken as vectors are subject to restrictions. In this series of theorems the requirement is imposed that certain inner products must not exceed a fixed value D. While rectangular incidence matrices can be studied from this point of view, the present investigation is limited to symmetric incidence matrices with 0's on the main diagonal, which will be taken in the applications to be principal minor submatrices of association matrices. The pairs of rows subject to the restriction on inner products are those which contain a pair of off-diagonal 0's symmetrically located with respect to the main diagonal; in the matrix A_1 of first associates in an association scheme, such a pair of rows corresponds to a pair of second associates, and the inner product of the two rows of A_1 is equal to p_{11}^2. The inner product of the same two rows of any submatrix cannot be larger than p_{11}^2 and may be known in some cases to be bounded by some definite smaller value. In this application of Theorems 4.4 to 4.6 the least upper bound that can be established for the inner product of such rows is taken as the value D.

This series of theorems takes up the connection between the number of 1's in a matrix of the form considered and the order of submatrices which have 1's in all off-diagonal positions. When the matrices are interpreted as linear graphs, this becomes a connection between the number of lines in the graph and the order of complete subgraphs.

Numerous theorems of this kind are already known but not for the case in which row inner products are restricted. Theorem 4.6 is very closely related to one of these theorems. A number of other approaches to the study of incidence matrices having restrictions on row inner products are possible, and some are discussed in a passage following Corollary 4.9.

Definition 4.2, applying to symmetric incidence matrices with 0's on the main diagonal, describes a certain permutation of rows and columns and a partition of the matrix into blocks in such a way that the blocks lying on the main diagonal are square and contain no other 0's. Notation is introduced including notation for the orders of the diagonal blocks. For a matrix partitioned in this form, Theorem 4.4 expresses upper and lower bounds for Z, the total number of off-diagonal 0's, as functions of the orders of the diagonal blocks, the order t of the matrix, and the upper bound D on the restricted inner products. Application of this theorem is complicated by the fact that a particular partition may involve a large number of diagonal blocks and by the fact that to obtain results of any generality it may be necessary to consider a large number of possible partitions. Some numerical examples illustrate the application of the theorem.

The lower bounds on Z are the ones of greatest interest in this study, and more useful lower bounds are obtained in Corollary 4.4, Theorem 4.5 and Theorem 4.6. In each of these the lower bound is expressed in terms of the order t of the matrix, the bound D on inner products, and the <u>maximum</u> order s_1 for a principal minor submatrix without off-diagonal 0's. The minimum value of Z for given t, D and

s_1 is denoted by $m(t, D, s_1)$ and is primarily considered for fixed t and D, in which case it is a function of s_1. The lower bounds found for Z may also be regarded as functions of s_1, and are lower bounds for $m(t, D, s_1)$. They are illustrated for two typical cases in two figures. The exact nature of $m(t, D, s_1)$ is not known. It may be conjectured that it is monotone decreasing in s_1. If this function or a lower bound for it which is a function of s_1 is monotone decreasing, then certain inequalities on Z are sufficient to imply certain inequalities on s_1. In Lemma 4.4, which applies to matrices satisfying certain specified conditions and is used directly in the proof of Theorem 4.7, an implication of this kind is used to establish a lower bound on s_1. This amounts to a lower bound on the order of the maximal complete configuration of the graph. The proof of the lemma includes a demonstration that the lower bounds on Z are monotone decreasing for a certain range of values of s_1.

Theorem 4.7, the first theorem of this section which applies only to the incidence matrices of association schemes, defines a quantity σ_1 in terms of the parameters of the association scheme and states sufficient conditions that the scheme contain a complete k-point of order $k \geq \sigma_1 + 2$. The proof deals with a submatrix A_{11} of the incidence matrix A_1 and makes use of Lemmas 4.3 and 4.4. σ_1 plays the role of the lower bound on s_1 in Lemma 4.4, and is defined in such a way in (4.37) that the result of Lemma 4.3 provides the inequality on Z needed as a hypothesis. The other hypotheses of Theorem 4.7 guarantee that s_1 falls in the range for which Lemma 4.4 is valid. It can be shown that Theorem 4.7 applies to many association schemes with L_g parameter values and to some schemes of other types. The conclusion of the theorem for these schemes is that every pair of first associates is contained in a complete configuration of order at least equal to a value which is specified.

Theorem 4.8 applies to complete configurations in an association scheme. The principal result is that if two complete configurations have at least two treatments in common and if the numbers of treatments in the sets are sufficiently large, then all of the treatments in their union form a complete configuration. This theorem may be described in another way by borrowing a term used in the sociometric applications of linear graphs and referring to a complete configuration as a clique. In this terminology, Theorem 4.8 states that if two sufficiently large cliques have more than one member in common, they must merge. Lemma 4.5 states a further result which in the same language has the following wording: if all of an individual's associates are fellow members with him in cliques having more than a specified critical number of members, then none of the associates are members of more than one of the cliques. (Two people who meet in a certain clique never meet anywhere else.) The proofs of Theorem 4.8 and Lemma 4.5 make use of properties of association schemes and would of course apply to a social group only if they met the rather stringent requirements of partial balance, as defined in Section 1.1.

Theorem 4.9, applying several of the preceding results, finally establishes that for any fixed number g of constraints and for all except a finite number of possible exceptional numbers n^2 of treatments, the association scheme of Latin square type with g constraints is unique in the sense that there exists no other type of scheme having the same parameter values. Corollary 4.9 uses numerical computations to give explicit lower bounds below which any exceptional values of n must lie. The proof of Theorem 4.9 is summarized and discussed in some detail in a passage following the proof of Corollary 4.9.

Theorem 4.10 furnishes some additional information about some of the exceptional cases not covered by Theorem 4.9. Applying to L_3 schemes, it is analogous to Theorem 4.3. The proof is more difficult than that of Theorem 4.3, illustrating the increasing complexity of Latin square type association schemes as the number of constraints increases.

The section concludes with a statement without proof of a uniqueness theorem very similar to those of this chapter, applying to a class of triangular type association schemes. It appears that the methods of this chapter will not apply to the remaining types of association schemes without some modification . Reference has already been made to a passage following Corollary 4.9 in which possible further results are discussed. The opening paragraph of Section 4.1 contains some remarks on the significance of the uniqueness proofs of this chapter. The writer believes that most of the theorems and proofs are new.

APPENDIX

A.1 Tables of Parameter Values of Association Schemes.

The tables in this section are constructed by methods developed in Section 2.2. Table II gives values of the parameters v, n_i, p^i_{jk} and α_i for all PBIB designs with two associate classes, not of group divisible type, and having $v \leq 100$. The parameter values listed are determined by the association scheme of a design and are independent of the values of r, k, b and λ_i. Tables Ia and Ib show the preliminary computation used in constructing Table II. Each table is preceded by an explanation of the notation used.

TABLE Ia. PRELIMINARY COMPUTATION OF THE PARAMETER VALUES OF ASSOCIATION SCHEMES BY MEANS OF THEOREM 2.0. Theorem 2.0, due to Connor and Clatworthy [17], specifies a class of association schemes whose parameter values may all be expressed in terms of a positive integral parameter t. Values of t from 1 to 24 are listed in the first column of this table. The values in the next eight columns are obtained from the following equations, stated in Theorem 2.0.

$$p^1_{12} = p^2_{12} = p^1_{22} = p^2_{11} = t \quad ,$$

$$n_1 = n_2 = \alpha_1 = 2t \quad ,$$

$$v = 4t + 1 \quad .$$

The final column of the table, headed #, gives the serial number by which the scheme is identified in Table II.

TABLE Ia

PRELIMINARY COMPUTATION OF PARAMETER VALUES OF ASSOCIATION SCHEMES BY MEANS OF THEOREM 2.0

t	p^1_{12}	p^2_{12}	p^1_{22}	p^2_{11}	n_1	n_2	α_1	v	#
1	1	1	1	1	2	2	2	5	1
2	2	2	2	2	4	4	4	9	2
3	3	3	3	3	6	6	6	13	4
4	4	4	4	4	8	8	8	17	8
5	5	5	5	5	10	10	10	21	10
6	6	6	6	6	12	12	12	25	12
7	7	7	7	7	14	14	14	29	17
8	8	8	8	8	16	16	16	33	18
9	9	9	9	9	18	18	18	37	24
10	10	10	10	10	20	20	20	41	26
11	11	11	11	11	22	22	22	45	29
12	12	12	12	12	24	24	24	49	33
13	13	13	13	13	26	26	26	53	37
14	14	14	14	14	28	28	28	57	43
15	15	15	15	15	30	30	30	61	44
16	16	16	16	16	32	32	32	65	54
17	17	17	17	17	34	34	34	69	57
18	18	18	18	18	36	36	36	73	59
19	19	19	19	19	38	38	38	77	65
20	20	20	20	20	40	40	40	81	73
21	21	21	21	21	42	42	42	85	78
22	22	22	22	22	44	44	44	89	80
23	23	23	23	23	46	46	46	93	82
24	24	24	24	24	48	48	48	97	89

TABLE Ib. PRELIMINARY COMPUTATION OF THE PARAMETER VALUES OF ASSOCIATION SCHEMES BY MEANS OF THEOREM 2.1. Theorem 2.1 specifies all association schemes with two associate classes which are not of group divisible type and are not given by Theorem 2.0. The additional restriction (2.25),

$$p_{12}^{1} \leq p_{12}^{2} \quad ; \quad \text{if} \quad p_{12}^{1} = p_{12}^{2} \, , \quad \text{then} \quad n_1 \leq n_2$$

is imposed to avoid duplication in the table. The schemes are listed in order of increasing values of $\sqrt{\Delta}$, a parameter which was introduced in [17] and is used in this dissertation. For the schemes being tabulated and a fixed value of $\sqrt{\Delta}$, all possible pairs of values p_{12}^{1}, p_{12}^{2} appear as consecutive entries in row $\sqrt{\Delta}$ of the table of Figure 6 in Section 2.2. The consecutive values of the column index σ in the same table are denoted by s and $s+1$ and are used in the computation of columns 9 and 10 of Table Ib. The values p_{12}^{1} and p_{12}^{2} are listed in columns 2 and 3.

The parameter γ appearing in column 4 is defined by

$$\gamma = p_{12}^{2} - p_{12}^{1} \, .$$

The parameters p_{22}^{1} and p_{11}^{2}, appearing in columns 5 and 6, satisfy (2.24),

$$p_{22}^{1} p_{11}^{2} = p_{12}^{1} p_{12}^{2} \, .$$

The value p_{22}^{1} must therefore be a divisor of the product of the entries in columns 2 and 3 . It is also required to satisfy (2.27),

$$p_{22}^{1} \geq \gamma + 1 \ .$$

Values of p_{22}^{1} and p_{11}^{2} satisfying these requirements but for which

$$p_{12}^{1} + p_{12}^{2} + p_{22}^{1} + p_{11}^{2} = v - 1 > 99$$

are also omitted and the omission is indicated by a row of dashes. Finally, if $p_{12}^{1} = p_{12}^{2}$, the restriction $n_1 \leq n_2$ implies $p_{22}^{1} \leq p_{11}^{2}$, and values such that $p_{22}^{1} > p_{11}^{2}$ are omitted in this case.

The values of n_2 and n_1 , appearing in the next columns, are determined by (2.4),

$$n_2 = p_{12}^{1} + p_{22}^{1} \ ,$$

$$n_1 = p_{12}^{2} + p_{11}^{2} \ .$$

If n_1 and n_2 are both odd, Theorem 2.5 shows that no association scheme can exist. In this case, the word "odd" is entered in each of the two columns and the rest of the row of the table is left blank.

The quantities $(s+1)n_2$ and sn_1 , listed in the next columns of the table, are used in the computation of α_1 .

The parameter α_1 which appears in column 11 is computed by (2.28),

$$\alpha_1 = \frac{(s+1)n_2 + sn_1}{\sqrt{\Delta}}$$

and must be a positive integer for any association scheme. If α_1 is fractional the letter "f" is listed in column 11 and the rest of the row is left blank.

If α_1 is an integer, the value v is listed in the next column. It is determined by (2.2),

$$v = n_1 + n_2 + 1 \quad .$$

The final column of Table Ib, headed #, lists for each complete set of parameter values the serial number by which the set is identified in Table II.

TABLE Ib

PRELIMINARY COMPUTATION OF PARAMETER VALUES OF ASSOCIATION SCHEMES BY MEANS OF THEOREM 2.1

$\sqrt{\Delta}$	p^1_{12}	p^2_{12}	γ	p^1_{22}	p^2_{11}	n_2	n_1	$(s+1)n_2$	sn_1	α_1	v	#
3	2	2	0	1	4	3	6	6	6	4	10	3
				2	2	4	4	8	4	4	9	2
4	3	4	1	2	6	5	10	10	10	5	16	6
				3	4	6	8	12	8	5	15	5
				4	3	odd	odd					
				6	2	9	6	18	6	6	16	7
				12	1	odd	odd					
5	4	6	2	3	8	7	14	14	14	f		
				4	6	8	12	16	12	f		
				6	4	10	10	20	10	6	21	9
				8	3	12	9	24	9	f		
				12	2	16	8	32	8	8	25	11
				24	1	28	7	56	7	f		
5	6	6	0	1	36	7	42	21	84	21	50	34
				2	18	8	24	24	48	f		
				3	12	9	18	27	36	f		
				4	9	10	15	30	30	12	26	13
				6	6	12	12	36	24	12	25	12
6	5	8	3	4	10	9	18	18	18	6	28	15
				5	8	10	16	20	16	6	27	14
				8	5	odd	odd					
				10	4	15	12	30	12	7	28	16
				20	2	25	10	50	10	10	36	20
				40	1	45	9	90	9	f		
6	8	9	1	2	36	10	45	30	90	20	56	39
				3	24	odd	odd					
				4	18	12	27	36	54	15	40	25
				6	12	14	21	42	42	14	36	22
				8	9	16	18	48	36	14	35	19
				9	8	odd	odd					
				12	6	20	15	60	30	15	36	23
				18	4	26	13	78	26	f		
				24	3	32	12	96	24	20	45	27
				36	2	44	11	132	22	f		
				72	1	80	10	240	20	f		

TABLE Ib (continued)

$\sqrt{\Delta}$	p^1_{12}	p^2_{12}	γ	p^1_{22}	p^2_{11}	n_2	n_1	$(s+1)n_2$	sn_1	α_1	v	#
7	6	10	4	5	12	11	22	22	22	f		
				6	10	12	20	24	20	f		
				10	6	16	16	32	16	f		
				12	5	18	15	36	15	f		
				15	4	21	14	42	14	8	36	21
				20	3	26	13	52	13	f		
				30	2	36	12	72	12	12	49	30
				60	1	66	11	132	11	f		
7	10	12	2	3	40	13	52	39	104	f		
				4	30	14	42	42	84	18	57	41
				5	24	15	36	45	72	f		
				6	20	16	32	48	64	16	49	31
				8	15	18	27	54	54	f		
				10	12	20	24	60	48	f		
				12	10	22	22	66	44	f		
				15	8	25	20	75	40	f		
				20	6	30	18	90	36	18	49	32
				24	5	34	17	102	34	f		
				30	4	40	16	120	32	f		
				40	3	50	15	150	30	f		
				60	2	70	14	210	28	34	85	75
				- -	- -							
7	12	12	0	- -	- -							
				2	72	14	84	56	252	44	99	90
				3	48	15	60	60	180	f		
				4	36	16	48	64	144	f		
				6	24	18	36	72	108	f		
				8	18	20	30	80	90	f		
				9	16	21	28	84	84	24	50	35
				12	12	24	24	96	72	24	49	33
8	7	12	5	6	14	13	26	26	26	f		
				7	12	14	24	28	24	f		
				12	7	odd	odd					
				14	6	21	18	42	18	f		
				21	4	28	16	56	16	9	45	28
				28	3	35	15	70	15	f		
				42	2	49	14	98	14	14	64	47
				- -	- -							

TABLE Ib (continued)

$\sqrt{\Delta}$	p^1_{12}	p^2_{12}	γ	p^1_{22}	p^2_{11}	n_2	n_1	$(s+1)n_2$	sn_1	α_1	v	#
8	12	15	3	4	45	16	60	48	120	21	77	64
				5	36	odd	odd					
				6	30	18	45	54	90	18	64	48
				9	20	odd	odd					
				10	18	22	33	66	66	f		
				12	15	24	30	72	60	f		
				15	12	odd	odd					
				18	10	30	25	90	50	f		
				20	9	32	24	96	48	18	57	42
				30	6	42	21	126	42	21	64	49
				36	5	48	20	144	40	23	69	56
				45	4	odd	odd					
				60	3	72	18	216	36	f		
				- -	- -							
8	15	16	1	- -	- -							
				4	60	19	76	76	228	38	96	84
				5	48	20	64	80	192	34	85	76
				6	40	21	56	84	168	f		
				8	30	23	46	92	138	f		
				10	24	25	40	100	120	f		
				12	20	27	36	108	108	27	64	51
				15	16	30	32	120	96	27	63	46
				16	15	odd	odd					
				20	12	35	28	140	84	28	64	52
				24	10	39	26	156	78	f		
				30	8	45	24	180	72	f		
				40	6	55	22	220	66	f		
				48	5	odd	odd					
				60	4	75	20	300	60	45	96	85
				- -	- -							
9	8	14	6	7	16	15	30	30	30	f		
				8	14	16	28	32	28	f		
				14	8	22	22	44	22	f		
				16	7	24	21	48	21	f		
				28	4	36	18	72	18	10	55	38
				56	2	64	16	128	16	16	81	67
				- -	- -							

TABLE Ib (continued)

$\sqrt{\Delta}$	p^1_{12}	p^2_{12}	γ	p^1_{22}	p^2_{11}	n_2	n_1	$(s+1)n_2$	sn_1	α_1	v	#
9	14	18	4	6	42	20	60	60	120	20	81	68
				7	36	21	54	63	108	19	76	61
				9	28	23	46	69	92	f		
				12	21	26	39	78	78	f		
				14	18	28	36	84	72	f		
				18	14	32	32	96	64	f		
				21	12	35	30	105	60	f		
				28	9	42	27	126	54	20	70	58
				36	7	50	25	150	50	f		
				42	6	56	24	168	48	24	81	69
				63	4	77	22	231	44	f		
				- -	- -							
9	18	20	2	8	45	26	45	104	135	f		
				9	40	27	60	108	180	32	88	79
				10	36	28	56	112	168	f		
				12	30	30	50	120	150	30	81	70
				15	24	33	44	132	132	f		
				18	20	36	40	144	120	f		
				20	18	38	38	152	114	f		
				24	15	42	35	168	105	f		
				30	12	48	32	192	96	32	81	71
				36	10	54	30	216	90	34	85	77
				40	9	58	29	232	87	f		
				45	8	63	28	252	84	f		
				- -	- -							
9	20	20	0	- -	- -							
				8	50	28	70	140	280	f		
				10	40	30	60	150	240	f		
				16	25	36	45	180	180	40	82	74
				20	20	40	40	200	160	40	81	73
10	9	16	7	8	18	17	34	34	34	f		
				9	16	18	32	36	32	f		
				12	12	21	28	42	28	7	50	36
				16	9	odd	odd					
				18	8	27	24	54	24	f		
				24	6	33	22	66	22	f		
				36	4	45	20	90	20	11	66	55
				48	3	odd	odd					
				72	2	81	18	162	18	18	100	93
				- -	- -							

TABLE Ib (continued)

$\sqrt{\Delta}$	p^1_{12}	p^2_{12}	γ	p^1_{22}	p^2_{11}	n_2	n_1	$(s+1)n_2$	sn_1	α_1	v	#
10	16	21	5	6	56	22	77	66	154	22	100	94
				7	48	odd	odd					
				8	42	24	63	72	126	f		
				12	28	28	49	84	98	f		
				14	24	30	45	90	90	18	76	62
				16	21	32	42	96	84	18	75	60
				21	16	odd	odd					
				24	14	40	35	120	70	19	76	63
				28	12	44	33	132	66	f		
				42	8	58	29	174	58	f		
				48	7	64	28	192	56	f		
				56	6	72	27	216	54	27	100	95
				- -	- -							
10	21	24	3	- -	- -							
				12	42	33	66	132	198	33	100	98
				14	36	35	60	140	180	32	96	86
				18	28	39	52	156	156	f		
				21	24	42	48	168	144	f		
				24	21	odd	odd					
				28	18	49	42	196	126	f		
				36	14	57	38	228	114	f		
				42	12	63	36	252	108	36	100	99
				- -	- -							
10	24	25	1	- -	- -							
				20	30	44	55	220	220	44	100	100
				24	25	48	50	240	200	44	99	92
				25	24	odd	odd					
				30	20	54	45	270	180	45	100	101
				- -	- -							
11	10	18	8	9	20	19	38	38	38	f		
				10	18	20	36	40	36	f		
				12	15	22	33	44	33	7	56	40
				15	12	25	30	50	30	f		
				18	10	28	28	56	28	f		
				20	9	30	27	60	27	f		
				30	6	40	24	80	24	f		
				36	5	46	23	92	23	f		
				45	4	55	22	110	22	12	78	66
				60	3	70	21	140	21	f		
				- -	- -							

TABLE Ib (continued)

$\sqrt{\Delta}$	p^1_{12}	p^2_{12}	γ	p^1_{22}	p^2_{11}	n_2	n_1	$(s+1)n_2$	sn_1	α_1	v	#
11	18	24	6	- -	- -							
				9	48	27	72	81	144	f		
				12	36	30	60	90	120	f		
				16	27	34	51	102	102	f		
				18	24	36	48	108	96	f		
				24	18	42	42	126	84	f		
				27	16	45	40	135	80	f		
				36	12	54	36	162	72	f		
				48	9	66	33	198	66	24	100	97
				- -	- -							
12	11	20	9	10	22	21	42	42	42	7	64	50
				11	20	22	40	44	40	7	63	45
				20	11	odd	odd					
				22	10	33	30	66	30	8	64	53
				44	5	55	25	110	25	f		
				55	4	66	24	132	24	13	91	81
				- -	- -							
12	20	27	7	- -	- -							
				15	36	odd	odd					
				18	30	38	57	114	114	19	96	87
				20	27	40	54	120	108	19	95	83
				27	20	odd	odd					
				30	18	50	45	150	90	20	96	88
				36	15	56	42	168	84	21	99	91
				- -	- -							
13	12	22	10	11	24	23	46	46	46	f		
				12	22	24	44	48	44	f		
				22	12	34	34	68	34	f		
				24	11	36	33	72	33	f		
				33	8	45	30	90	30	f		
				44	6	56	28	112	28	f		
				- -	- -							
14	13	24	11	12	26	25	50	50	50	f		
				13	24	26	48	52	48	f		
				24	13	odd	odd					
				26	12	39	36	78	36	f		
				39	8	52	32	104	32	f		
				52	6	65	30	130	30	f		
				- -	- -							

TABLE Ib (continued)

$\sqrt{\Delta}$	p_{12}^1	p_{12}^2	γ	p_{22}^1	p_{11}^2	n_2	n_1	$(s+1)n_1$	sn_1	α_1	v	#
15	14	26	12	13	28	27	54	54	54	f		
				14	26	28	52	56	52	f		
				26	14	40	40	80	40	8	81	72
				28	13	42	39	84	39	f		
				52	7	66	33	132	33	11	100	96
				- - - -								
16	15	28	13	14	30	29	58	58	58	f		
				15	28	30	56	60	56	f		
				20	21	odd	odd					
				21	20	36	48	72	48	f		
				28	15	odd	odd					
				30	14	45	42	90	42	f		
				35	12	50	40	100	40	f		
				42	10	57	38	114	38	f		
				- - - -								
17	16	30	14	15	32	31	62	62	62	f		
				16	30	32	60	64	60	f		
				20	24	36	54	72	54	f		
				24	20	40	50	80	50	f		
				30	16	46	46	92	46	f		
				32	15	48	45	96	45	f		
				40	12	56	42	112	42	f		
				- - - -								
18	17	32	15	16	34	33	66	66	66	f		
				17	32	34	64	68	64	f		
				32	17	odd	odd					
				34	16	51	48	102	48	f		
				- - - -								
19	18	34	16	- - - -								

TABLE II. PARAMETER VALUES OF ASSOCIATION SCHEMES NOT OF GROUP DIVISIBLE TYPE. This table is restricted to schemes in which the number of treatments v does not exceed 100. Schemes are listed in order of increasing values of v, and for fixed v, increasing values of n_1. Duplication is avoided in this table by the condition

$$n_1 \leq n_2 \; ; \quad \text{if } n_1 = n_2 \, , \quad \text{then} \quad p^1_{12} \leq p^2_{12}.$$

Because this differs from condition (2.25) used in Table Ib, it has been necessary to change the designation of first and second associates in about 40 sets of parameter values. The same values occur, but with the indices 1 and 2 interchanged wherever they appear. The entries in most columns of Table II are copied directly from Tables Ia and Ib. The remaining numerical values are obtained by the relations

$$\begin{aligned} p^1_{11} &= n_1 - p^1_{12} - 1 \, , \\ p^2_{22} &= n_2 - p^2_{12} - 1 \, , \\ \alpha_2 &= v - \alpha_1 - 1 \, , \end{aligned}$$

and by the remark that for the schemes listed in Table Ia , $\Delta = v$. The parameter $\sqrt{\Delta}$ will be found convenient in locating a particular set of parameter values in Tables Ia and Ib, which are arranged in order of increasing values of $\sqrt{\Delta}$. Non-integral values of $\sqrt{\Delta}$ occur only in Table Ia.

Two columns of Table II are included under the heading "remarks". In the first of these, schemes which are known to have been constructed or to have been proved impossible are indicated by the letter "C" or "X" respectively. In the second, schemes of triangular, simple and or cyclic types are identified by name, and schemes in the Latin square series are identified by the symbol L_g , where g is the number of constraints. The schemes of these classes which have been constructed are either tabulated by Bose, Connor and Clatworthy, or are easily constructed. Schemes of the negative Latin square series introduced in Section 2.1 are identified by the symbol L_g^* and, if constructed, by a reference to the section in which the construction is described. One constructed scheme, #64, does not fall in any of the categories mentioned and is identified simply by a reference to the section in which it is constructed. The four schemes whose impossibility has been proved by theorems in Section 2.2 are identified by the numbers of the theorems.

TABLE II

#	v	n_1	n_2	p^1_{11}	p^1_{12}	p^1_{22}	p^2_{11}	p^2_{12}	p^2_{22}	α_1	α_2	$\sqrt{\Delta}$	Remarks	
1	5	2	2	0	1	1	1	1	0	2	2	$\sqrt{5}$	C	Cyclic
2	9	4	4	1	2	2	2	2	1	4	4	3	C	L_2
3	10	3	6	0	2	4	1	2	3	5	4	3	C	Triangular
4	13	6	6	2	3	3	3	3	2	6	6	$\sqrt{13}$	C	Cyclic
5	15	6	8	1	4	4	3	3	4	9	5	4	C	Triangular
6	16	5	10	0	4	6	2	3	6	10	5	4	C	L^*_{-1}, Sec. 3.2
7	16	6	9	2	3	6	2	4	4	6	9	4	C	L_2
8	17	8	8	3	4	4	4	4	3	8	8	$\sqrt{17}$	C	Cyclic
9	21	10	10	5	4	6	4	6	3	6	14	5	C	Triangular
10	21	10	10	4	5	5	5	5	4	10	10	$\sqrt{21}$		
11	25	8	16	3	4	12	2	6	9	8	16	5	C	L_2
12	25	12	12	5	6	6	6	6	5	12	12	5	C	L_3, L^*_{-2}
13	26	10	15	3	6	9	4	6	8	13	12	5	C	Simple
14	27	10	16	1	8	8	5	5	10	20	6	6	C	Simple
15	28	9	18	0	8	10	4	5	12	21	6	6	X	Theorem 2.6
16	28	12	15	6	5	10	4	8	6	7	20	6	C	Triangular
17	29	14	14	6	7	7	7	7	6	14	14	$\sqrt{29}$	C	Cyclic
18	33	16	16	7	8	8	8	8	7	16	16	$\sqrt{33}$		
19	35	16	18	6	9	9	8	8	9	20	14	6	C	Simple
20	36	10	25	4	5	20	2	8	16	10	25	6	C	L_2
21	36	14	21	7	6	15	4	10	10	8	27	7	C	Triangular
22	36	14	21	4	9	12	6	8	12	21	14	6		L^*_{-2}
23	36	15	20	6	8	12	6	9	12	15	20	6	C	L_3
24	37	18	18	8	9	9	9	9	8	18	18	$\sqrt{37}$	C	Cyclic
25	40	12	27	2	9	18	4	8	18	24	15	6	C	Simple
26	41	20	20	9	10	10	10	10	9	20	20	$\sqrt{41}$	C	Cyclic
27	45	12	32	3	8	24	3	9	22	20	24	6	C	Simple
28	45	16	28	8	7	21	4	12	15	9	35	8	C	Triangular
29	45	22	22	10	11	11	11	11	10	22	22	$\sqrt{45}$		
30	49	12	36	5	6	30	2	10	25	12	36	7	C	L_2
31	49	16	32	3	12	20	6	10	21	32	16	7		L^*_{-2}
32	49	18	30	7	10	20	6	12	17	18	30	7	C	L_3
33	49	24	24	11	12	12	12	12	11	24	24	7	C	L_4, L^*_{-3}
34	50	7	42	0	6	36	1	6	35	28	21	5		
35	50	21	28	8	12	16	9	12	15	25	24	7	C	Simple
36	50	21	28	4	16	12	12	9	18	42	7	10	X	Theorem 2.4
37	53	26	26	12	13	13	13	13	12	26	26	$\sqrt{53}$	C	Cyclic
38	55	18	36	9	8	28	4	14	21	10	44	9	C	Triangular
39	56	10	45	0	9	36	2	8	36	35	20	6		
40	56	22	33	3	18	15	12	10	22	48	7	11	X	Theorem 2.4
41	57	14	42	1	12	30	4	10	31	38	18	7		

TABLE II (continued)

#	v	n_1	n_2	p^1_{11}	p^1_{12}	p^1_{22}	p^2_{11}	p^2_{12}	p^2_{22}	α_1	α_2	$\sqrt{\Delta}$	Remarks
42	57	24	32	11	12	20	9	15	16	18	38	8	C Simple
43	57	28	28	13	14	14	14	14	13	28	28	$\sqrt{57}$	
44	61	30	30	14	15	15	15	15	14	30	30	$\sqrt{61}$	C Cyclic
45	63	22	40	1	20	20	11	11	28	55	7	12	
46	63	30	32	13	16	16	15	15	16	35	27	8	C Simple
47	64	14	49	6	7	42	2	12	36	14	49	8	C L_2
48	64	18	45	2	15	30	6	12	32	45	18	8	L^*_{-2}
49	64	21	42	8	12	30	6	15	26	21	42	8	C L_3
50	64	21	42	0	20	22	10	11	30	56	7	12	X Theorem 2.6
51	64	27	36	10	16	20	12	15	20	36	27	8	C L^*_{-3} Sec. 3.2
52	64	28	35	12	15	20	12	16	18	28	35	8	C L_4
53	64	30	33	18	11	22	10	20	12	8	55	12	
54	65	32	32	15	16	16	16	16	15	32	32	$\sqrt{65}$	
55	66	20	45	10	9	36	4	16	28	11	54	10	C Triangular
56	69	20	48	7	12	36	5	15	32	23	45	8	
57	69	34	34	16	17	17	17	17	16	34	34	$\sqrt{69}$	
58	70	27	42	12	14	28	9	18	23	20	49	9	C Simple
59	73	36	36	17	18	18	18	18	17	36	36	$\sqrt{73}$	C Cyclic
60	75	32	42	10	21	21	16	16	25	56	18	10	
61	76	21	54	2	18	36	7	14	39	56	19	9	
62	76	30	45	8	21	24	14	16	28	57	18	10	
63	76	35	40	18	16	24	14	21	18	19	56	10	
64	77	16	60	0	15	45	4	12	47	55	21	8	C Sec. 3.3
65	77	38	38	18	19	19	19	19	18	38	38	$\sqrt{77}$	
66	78	22	55	11	10	45	4	18	36	12	65	11	C Triangular
67	81	16	64	7	8	56	2	14	49	16	64	9	C L_2
68	81	20	60	1	18	42	6	14	45	60	20	9	C L^*_{-2} Sec. 3.2
69	81	24	56	9	14	42	6	18	37	24	56	9	C L_3
70	81	30	50	9	20	30	12	18	31	50	30	9	C L^*_{-3} Sec. 3.2
71	81	32	48	13	18	30	12	20	27	32	48	9	C L_4
72	81	40	40	25	14	26	14	26	13	8	72	15	
73	81	40	40	19	20	20	20	20	19	40	40	9	C L_5, L^*_{-4}
74	82	36	45	15	20	25	16	20	24	41	40	9	C Simple
75	85	14	70	3	10	60	2	12	57	34	50	7	
76	85	20	64	3	16	48	5	15	48	50	34	8	C Simple
77	85	30	54	11	18	36	10	20	33	34	50	9	
78	85	42	42	20	21	21	21	21	20	42	42	$\sqrt{85}$	
79	88	27	60	6	20	40	9	18	41	55	32	9	
80	89	44	44	21	22	22	22	22	21	44	44	$\sqrt{89}$	C Cyclic
81	91	24	66	12	11	55	4	20	45	13	77	12	C Triangular

TABLE II (continued)

#	v	n_1	n_2	p^1_{11}	p^1_{12}	p^1_{22}	p^2_{11}	p^2_{12}	p^2_{22}	α_1	α_2	$\sqrt{\Delta}$	Remarks
82	93	46	46	22	23	23	23	23	22	46	46	$\sqrt{93}$	
83	95	40	54	12	27	27	20	20	33	75	19	12	
84	96	19	76	2	16	60	4	15	60	57	38	8	
85	96	20	75	4	15	60	4	16	58	45	50	8	
86	96	35	60	10	24	36	14	21	38	63	32	10	
87	96	38	57	10	27	30	18	20	36	76	19	12	
88	96	45	50	24	20	30	18	27	22	20	75	12	
89	97	48	48	23	24	24	24	24	23	48	48	$\sqrt{97}$	C Cyclic
90	99	14	84	1	12	72	2	12	71	54	44	7	
91	99	42	56	21	20	36	15	27	28	21	77	12	
92	99	48	50	22	25	25	24	24	25	54	44	10	
93	100	18	81	8	9	72	2	16	64	18	81	10	C L_2
94	100	22	77	0	21	56	6	16	60	77	22	10	C L^*_{-2} Sec. 3.3
95	100	27	72	10	16	56	6	21	50	27	72	10	C L_3
96	100	33	66	18	14	52	7	26	39	11	88	15	
97	100	33	66	14	18	48	9	24	41	24	75	11	
98	100	33	66	8	24	42	12	21	44	66	33	10	L^*_{-3}
99	100	36	63	14	21	42	12	24	38	36	63	10	L_4
100	100	44	55	18	25	30	20	24	30	55	44	10	L^*_{-4}
101	100	45	54	20	24	30	20	25	28	45	54	10	L_5

A.2. Tables of Parameter Values of Possible Designs for Particular Association Schemes.

The tables in this section are constructed by methods developed in Section 2.3. Table IV gives values of the parameters v, r, k, b, λ_i and z_i for all possible designs with $v \leq 100$, $r \leq 10$ and $k \leq 10$ and having known association schemes in the Latin square or negative Latin square series. Table III illustrates the preliminary computation used in the construction of Table IV. Each table is preceded by an explanation of the notation used.

TABLE III. PRELIMINARY COMPUTATIONS OF THE PARAMETER VALUES OF POSSIBLE DESIGNS ILLUSTRATED FOR SEVERAL ASSOCIATION SCHEMES. The method of computation used here requires a separate section of the table for each association scheme, and is presented in this table for schemes #2 and 32 and a portion of #6. For use in the computation, numerical values of several parameters of the association scheme are listed at the beginning of the section, along with expressions for the quantities m, M, M', z_1, z_2 and $r(k-1)$. These expressions are given in (2.47) to (2.53) and (2.3) in Chapter II.

Non-negative integral values of λ_2 are listed in numerical order in the first column of the table. For a particular value of λ_2, the lower bound m on λ_1 is listed if positive, and the smaller of the upper bounds M and M' is listed. Values of λ_1 between the bounds are then listed in column 5, with the omission of the value $\lambda_1 = \lambda_2$ and of values $\lambda_1 > \lambda_2$ in case $n_1 = n_2$. When a value of λ_2 is reached for which the bounds admit no integral value of λ_1, a row of dashes is entered in the λ_1 column. Because the quantities m, M, M' are linear in λ_2, no further values of λ_2 need to be considered. For each pair of values λ_1, λ_2, the quantities z_1, z_2 and $r(k-1)$ are listed in the next columns. When the last of these is expressed in every possible way as the product of two positive integers, the two factors may be taken as values of r and $k-1$ and lead to all possible pairs of values of r and k, which are then listed in the next columns. The list is shortened by the restrictions

$$r \leq 10 , \quad k \leq 10 ,$$

and by conditions (2.43) and (2.44) ,

$$r \geq z_i , \quad i = 1, 2 ,$$

$$\text{if } r > z_i , \; i = 1 \text{ and } 2 , \text{ then } r \geq k .$$

Factorizations of $r(k-1)$ which violate any of these conditions are omitted without comment. The last one is illustrated by the row near the end of the computations shown for scheme #6 , with the entries

1 3 -1 25 -- -- . For each pair r , k , the value of b is computed from (2.1), $b = vr/k$, and entered in the next column if integral; fractional values of b are indicated by the letter f. Finally, if b is integral, condition (2.45)

$$\text{if } r = z_i , \; i = 1 \text{ or } 2 , \text{ then } b \geq v - \alpha_i$$

is imposed in cases where it applies, eliminating a few more sets of parameter values.

TABLE III

PRELIMINARY COMPUTATIONS OF PARAMETER VALUES OF POSSIBLE DESIGNS ILLUSTRATED FOR SEVERAL ASSOCIATION SCHEMES

<u>Scheme #2</u>, L_2, $n = 3$, $v = 9$, $g = 2$, $f = 2$, $n_1 = 4 = \alpha_1$, $n_2 = 4 = \alpha_2$,
$m = 2\lambda_2 - 10$, $M = \frac{1}{2}\lambda_2 + 5$, $M' = 22\frac{1}{2} - \lambda_2$,
$z_1 = -\lambda_1 + 2\lambda_2$, $z_2 = 2\lambda_1 - \lambda_2$, $r(k-1) = 4\lambda_1 + 4\lambda_2$.

λ_2	m	M	M'	λ_1	z_1	z_2	r(k-1)	r	k	b
1	--	$5\frac{1}{2}$	--	0	2	-1	4	2	3	6 $> v-\alpha_1$, OK.
								4	2	18
2	--	6	--	0	4	-2	8	4	3	12 $> v-\alpha_1$, OK.
								8	2	36
				1	3	0	12	3	5	f
								4	4	9
								6	3	18
3	--	$6\frac{1}{2}$	--	0	6	-3	12	6	3	18 $> v-\alpha_1$, OK.
				1	5	-1	16	8	3	24
				2	4	1	20	4	6	6 $> v-\alpha_1$, OK.
								5	5	9
								10	3	30
4	--	7	--	0	8	-4	16	8	3	24 $> v-\alpha_1$, OK.
				1	7	-2	20	10	3	30
				2	6	0	24	6	5	f
								8	4	18
				3	5	2	28	7	5	f
5	--	$7\frac{1}{2}$	--	0	10	-5	20	10	3	30 $> v-\alpha_1$, OK.
				1	9	-3	24	—	—	
				2	8	-1	28	--	--	
				3	7	1	32	8	5	f
				4	6	3	36	6	7	f
								9	5	f
6	2	8	--	2	10	-2	32	--	--	
				3	9	0	36	9	5	f
				4	8	2	40	8	6	12 $> v-\alpha_1$, OK.
								10	5	18
				5	7	4	44	--	--	

TABLE III (continued)

λ_2	m	M	M'	λ_1	z_1	z_2	r(k-1)	r	k	b
7	4	$8\frac{1}{2}$	--	4	10	1	44	--	--	
				5	9	3	48	--	--	
				6	8	5	52	--	--	
8	6	9	--	6	10	4	56	--	--	
				7	9	6	60	10	7	f
9	8	$9\frac{1}{2}$	--	8	10	7	68	--	--	
10	10	10	--	--						

<u>Scheme #6</u>, L^*_{-1}, $n = -4$, $v = 16$, $g = -1$, $f = -2$, $n_1 = 5 = \alpha_2$, $n_2 = 10 = \alpha_1$,

$m = 2\lambda_2 - 10$, $M = \frac{2}{3}\lambda_2 + 3\frac{1}{3}$, $M' = 18 - 2\lambda_2$,

$z_1 = -\lambda_1 + 2\lambda_2$, $z_2 = 3\lambda_1 - 2\lambda_2$, $r(k-1) = 5\lambda_1 + 10\lambda_2$.

λ_2	m	M	M'	λ_1	z_1	z_2	r(k-1)	r	k	b
0	--	$3\frac{1}{3}$	--	1	-1	3	5	5	2	40
				2	-2	6	10	10	2	80
				3	-3	9	15	--	--	
1	--	4	--	0	2	-2	10	2	6	f
								5	3	f
				2	0	4	20	4	6	f
								5	5	16
								10	3	f
				3	-1	7	25	--	--	
				4	-2	10	30	10	4	40 $> v - \alpha_2$, OK.
2	--	$4\frac{2}{3}$	--	0	4	-4	20	4	6	f
								5	5	16
								10	3	f
				1	3	-1	25	--	--	
				3	1	5	35	5	8	10 $< v - \alpha_2$, impossible.
								7	6	f
				4	0	8	40	8	6	f
								10	5	32

TABLE III (continued)

Scheme #32, L_3, $n = 7$, $v = 49$, $g = 3$, $f = 5$, $n_1 = 18 = \alpha_1$, $n_2 = 30 = \alpha_2$,
$m = \frac{5}{4}\lambda_2 - 2\frac{1}{2}$, $M = \frac{2}{3}\lambda_2 + 3\frac{1}{3}$, $M' = 5 - \frac{5}{3}\lambda_2$,
$z_1 = -4\lambda_2 + 5\lambda_2$, $z_2 = 3\lambda_1 - 2\lambda_2$, $r(k-1) = 18\lambda_1 + 30\lambda_2$.

λ_2	m	M	M'	λ_1	z_1	z_2	r(k-1)	r	k	b
0	--	$3\frac{1}{3}$	--	1	-4	3	18	3	7	$21 > v-\alpha_2$, OK.
								6	4	f
								9	3	147
				2	-8	6	36	6	7	$42 > v-\alpha_2$, OK.
								9	5	f
				3	-12	9	54	9	7	$63 > v-\alpha_2$, OK.
1	--	--	$3\frac{1}{3}$	0	5	-2	30	5	7	$35 > v-\alpha_1$, OK.
								6	6	49
								10	4	f
				2	-3	4	66	--	--	
				3	-7	7	84	--	--	
2	--	--	$1\frac{2}{3}$	0	10	-4	60	10	7	$70 > v-\alpha_1$, OK.
				1	6	-1	78	--	--	
3	$2\frac{1}{2}$	--	0	--						

TABLE IV. PARAMETER VALUES OF POSSIBLE DESIGNS. This is a list of those parameter values which satisfy all the conditions applied in Table III. The table is limited to designs with known Latin square and negative Latin square association schemes and with $v \leq 100$, $r \leq 10$ and $k \leq 10$. If $n_1 = n_2$, duplication is avoided by the restriction $\lambda_1 < \lambda_2$; designs with $\lambda_1 > \lambda_2$ can then be obtained by changing the designation of first and second associates. The design parameters for each association scheme are listed together, preceded by a list of parameter values of the scheme. Designs are identified by the numbers given to the scheme in Table II, and by a serial numbering of the designs for each scheme. Designs which are known to have been constructed or have been proved impossible are marked by the letter C or X respectively, followed by an explanatory remark or reference.

The phrase "Pairs of first associates" indicates that all such pairs of treatments are taken as blocks; designs of this kind are described in Theorem 2.9. The word "Lattice" indicates a well-known type of design whose structure is stated in Theorem 2.10. Some of the designs may be formed by replicating other designs. The procedure is justified in Theorem 2.11 and the designs are identified by the letter "R", followed by the serial numbers of the other design or designs used. The statement "$N = A_i$" or "$N = A_i + I$" indicates a way in which the incidence matrix of the design may be formed from the association matrix. Further details are given in Theorems 2.12 and 2.13. Some designs are identified as the complements of other designs in the table. Two designs are complements if each block of one contains exactly the treatments not contained in the corresponding block of the other. In some cases there is a direct reference to a theorem of Section 2.3 or a section of the Appendix in which the design is constructed.

TABLE IV

Scheme #2, L_2, $n = 3$, $v = 9$, $g = 2$, $n_1 = 4$, $f = 2$, $n_2 = 4$, $P_1 = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$,

$NN' = rk(r - z_1)^4(r - z_2)^4$.

#	r	k	b	λ_1	λ_2	z_1	z_2		Remarks
2-1	4	2	18	0	1	2	-1	C	Pairs of first associates.
2-2	8	2	36	0	2	4	-2	C	R: 1;1.
2-3	2	3	6	0	1	2	-1	C	Lattice; based on g squares.
2-3a	2	3	6	1	0	-1	2	C	Lattice; based on f squares.
2-4	4	3	12	0	2	4	-2	C	R: 3;3.
2-5	6	3	18	0	3	6	-3	C	R: 3;3;3.
2-6	6	3	18	1	2	3	0	C	R: 3;3;3a.
2-7	8	3	24	0	4	8	-4	C	R: 3;3;3;3.
2-8	8	3	24	1	3	5	-1	C	R: 3;3;3;3a.
2-9	10	3	30	0	5	10	-5	C	R: 3;3;3;3;3.
2-10	10	3	30	1	4	7	-2	C	R: 3;3;3;3;3a.
2-11	10	3	30	2	3	4	1	C	R: 3;3;3;3a;3a.
2-12	4	4	9	1	2	3	0	C	$N = A_1$.
2-13	8	4	18	2	4	6	0	C	R: 12;4.
2-14	5	5	9	2	3	4	1	C	$N = A_2 + I$ (Complement of 12).
2-15	10	5	18	4	6	8	2	C	R: 14;14.
2-16	4	6	6	2	3	4	1	C	Complement of 3.
2-17	8	6	12	4	6	8	2	C	R: 16;16.

TABLE IV (continued)

Scheme #6, L_{-1}^{*}, $n = -4$, $g = -1$, $f = -2$, $v = 16$, $n_1 = 5$, $n_2 = 10$, $P_1 = \begin{bmatrix} 0 & 4 \\ 4 & 6 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{10}(r - z_2)^5 .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
6-1	5	2	40	1	0	-1	3	C Pairs of first associates.
6-2	10	2	80	2	0	-2	6	C R: 1;1.
6-3	10	2	80	0	1	2	-2	C Pairs of second associates.
6-4	10	4	40	0	3	6	-6	
6-5	10	4	40	4	1	-2	10	
6-6	5	5	16	0	2	4	-4	C $N = A_1$.
6-7	5	5	16	2	1	0	4	C Appendix A.3.
6-8	10	5	32	0	4	8	-8	C R: 6;6.
6-9	10	5	32	2	3	4	0	C R: 6;7.
6-10	10	5	32	4	2	0	8	C R: 7;7.
6-11	6	6	16	0	3	6	-6	
6-12	9	6	24	1	4	7	-5	
6-13	5	8	10	1	3	5	-3	C Appendix A.3.
6-14	10	8	20	4	5	6	2	
6-15	10	8	20	6	4	2	10	
6-16	10	10	16	4	7	10	-2	

TABLE IV (continued)

Scheme #7 , L_2 , $n = 4$, $g = 2$, $f = 3$, $v = 16$, $n_1 = 6$, $n_2 = 9$, $P_1 = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}$,

$$NN' = rk(r - z_1)^6(r - z_2)^9 .$$

#	r	k	b	λ_1	λ_2	z_1	z_2		Remarks
7-1	6	2	48	1	0	-2	2	C	Pairs of first associates.
7-2	9	2	72	0	1	3	-1	C	Pairs of second associates.
7-3	3	3	16	1	0	-2	2	C	Equivalent to LS14 of [6].
7-4	6	3	32	2	0	-4	4	C	Theorem 2.14.
7-5	9	3	48	3	0	-6	6	C	R: 3;3;3.
7-6	9	3	48	0	2	6	-2	C	Theorem 2.14.
7-7	2	4	8	1	0	-2	2	C	Lattice.
7-8	3	4	12	0	1	3	-1	C	Lattice.
7-9	4	4	16	2	0	-4	4	C	R: 7;7.
7-10	6	4	24	0	2	6	-2	C	R: 8;8.
7-11	6	4	24	3	0	-6	6	C	R: 7;7;7.
7-12	7	4	28	2	1	-1	3	C	R: 7;7;8.
7-13	8	4	32	4	0	-8	8	C	R: 7;7;7;7.
7-14	8	4	32	1	2	4	0	C	R: 7;8;8.
7-15	9	4	36	0	3	9	-3	C	R: 8;8;8.
7-16	9	4	36	3	1	-3	5	C	R: 7;7;7;8.
7-17	10	4	40	5	0	-10	10	C	R: 7;7;7;7;7.
7-18	7	7	16	4	2	-2	6	C	$N = A_1 + I$.
7-19	6	8	12	4	2	-2	6	C	Theorem 2.15.
7-20	9	8	18	6	3	-3	9		
7-21	9	8	18	3	5	9	1	C	Theorem 2.15.
7-22	9	9	16	6	4	0	8	C	$N = A_2$.

TABLE IV (continued)

Scheme #11, L_2, $n = 5$, $g = 2$, $f = 4$, $v = 25$, $n_1 = 8$, $n_2 = 16$, $P_1 = \begin{bmatrix} 3 & 4 \\ 4 & 12 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 6 \\ 6 & 9 \end{bmatrix}$,

$$NN' = rk(r - z_1)^8(r - z_2)^{16}.$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
11-1	8	2	100	1	0	-3	2	C Pairs of first associates.
11-2	8	4	50	3	0	-9	6	C Theorem 2.14.
11-3	2	5	10	1	0	-3	2	C Lattice.
11-4	4	5	20	2	0	-6	4	C R: 3;3.
11-5	4	5	20	0	1	4	-1	C Lattice.
11-6	6	5	30	3	0	-9	6	C R: 3;3;3.
11-7	8	5	40	4	0	-12	8	C R: 3;3;3;3.
11-8	8	5	40	2	1	-2	3	C R: 3;3;5.
11-9	8	5	40	0	2	8	-2	C R: 5;5.
11-10	10	5	50	5	0	-15	10	C R: 3;3;3;3;3.
11-11	10	5	50	3	1	-5	5	C R: 3;3;3;5.
11-12	10	5	50	1	2	5	0	C R: 3;5;5.
11-13	8	8	25	3	2	-1	4	C $N = A_1$.
11-14	9	9	25	5	2	-7	8	C $N = A_1 + I$.
11-15	8	10	20	5	2	-7	8	C Theorem 2.15.

Scheme #12, L_3, $n = 5$, $g = 3$, $f = 3$, $v = 25$, $n_1 = 12$, $n_2 = 12$, $P_1 = \begin{bmatrix} 5 & 6 \\ 6 & 6 \end{bmatrix}$, $P_2 = \begin{bmatrix} 6 & 6 \\ 6 & 5 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{12}(r - z_2)^{12}.$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
12-1	6	3	50	1	0	-2	3	C LS14, [6].
12-2	4	4	25	1	0	-2	3	
12-3	8	4	50	2	0	-4	6	
12-4	3	5	15	1	0	-2	3	C Lattice.
12-4a	3	5	15	0	1	3	-2	C Lattice.
12-5	6	5	30	2	0	-4	6	C R: 4;4.
12-6	9	5	45	3	0	-6	9	C R: 4;4;4.
12-7	9	5	45	2	1	-1	4	C R: 4;4;4a.
12-8	4	10	10	2	1	-1	4	
12-9	8	10	20	4	2	-2	8	

TABLE IV (continued)

Scheme #20 , L_2 , $n = 6$, $v = 36$, $g = 2$, $n_1 = 10$, $f = 5$, $n_2 = 25$, $P_1 = \begin{bmatrix} 4 & 5 \\ 5 & 20 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 8 \\ 8 & 16 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{10}(r - z_2)^{25} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks	
20-1	10	2	180	1	0	-4	2	C	Pairs of first associates.
20-2	5	3	60	1	0	-4	2	X	Appendix A.3.
20-3	10	3	120	2	0	-8	4	C	Theorem 2.14.
20-4	10	4	90	3	0	-12	6		
20-5	5	5	36	2	0	-8	4		
20-6	10	5	72	4	0	-16	8	C	Theorem 2.14.
20-7	2	6	12	1	0	-4	2	C	Lattice.
20-8	4	6	24	2	0	-8	4	C	R: 7;7.
20-9	5	6	30	0	1	5	-1		
20-10	6	6	36	3	0	-12	6	C	R: 7;7;7.
20-11	8	6	48	4	0	-16	8	C	R: 7;7;7;7.
20-12	9	6	54	2	1	-3	3		
20-13	10	6	60	0	2	10	-2		
20-14	10	6	60	5	0	-20	10	C	R: 7;7;7;7;7.
20-15	10	9	40	3	2	-2	4		
20-16	10	10	36	4	2	-6	6	C	$N = A_1$.

TABLE IV (continued)

Scheme #23, L_3, $n = 6$, $v = 36$, $g = 3$, $n_1 = 15$, $f = 4$, $n_2 = 20$, $P_1 = \begin{bmatrix} 6 & 8 \\ 8 & 12 \end{bmatrix}$, $P_2 = \begin{bmatrix} 6 & 9 \\ 9 & 10 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{15}(r - z_2)^{20} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
23-1	10	3	120	0	1	4	-2	
23-2	5	4	45	1	0	-3	3	
23-3	10	4	90	2	0	-6	6	
23-4	5	5	36	0	1	4	-2	
23-5	10	5	72	0	2	8	-4	
23-6	3	6	18	1	0	-3	3	C Lattice.
23-7	4	6	24	0	1	4	-2	
23-8	6	6	36	2	0	-6	6	C R: 6;6.
23-9	8	6	48	0	2	8	-4	
23-10	9	6	54	3	0	-9	9	C R: 6;6;6.
23-11	10	6	60	2	1	-2	4	
23-12	10	9	40	4	1	-8	10	
23-13	10	10	36	2	3	6	0	

Scheme #30, L_2, $n = 7$, $v = 49$, $g = 2$, $n_1 = 12$, $f = 6$, $n_2 = 36$, $P_1 = \begin{bmatrix} 5 & 6 \\ 6 & 30 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 10 \\ 10 & 25 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{12}(r - z_2)^{30} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
30-1	6	3	98	1	0	-5	2	C Theorem 2.14.
30-2	4	4	49	1	0	-5	2	X Appendix A.3.
30-3	8	4	98	2	0	-10	4	C Theorem 2.14.
30-4	2	7	14	1	0	-5	2	C Lattice.
30-5	4	7	28	2	0	-10	4	C R: 4;4.
30-6	6	7	42	3	0	-15	6	C R: 4;4;4.
30-7	6	7	42	0	1	6	-1	C Lattice.
30-8	8	7	56	4	0	-20	8	C R: 4;4;4;4.
30-9	10	7	70	5	0	-25	10	C R: 4;4;4;4;4.
30-10	10	7	70	2	1	-4	3	C R: 4;4;7.
30-11	9	9	49	3	1	-9	5	

TABLE IV (continued)

Scheme #32, L_3, $n = 7$, $v = 49$, $g = 3$, $n_1 = 18$, $f = 5$, $n_2 = 30$, $P_1 = \begin{bmatrix} 7 & 10 \\ 10 & 20 \end{bmatrix}$, $P_2 = \begin{bmatrix} 6 & 12 \\ 12 & 17 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{18}(r - z_2)^{30} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
32-1	9	3	147	1	0	-4	3	C Theorem 2.14.
32-2	6	6	49	0	1	5	-2	
32-3	3	7	21	1	0	-4	3	C Lattice.
32-4	5	7	35	0	1	5	-2	C Lattice.
32-5	6	7	42	2	0	-8	6	C R: 3;3.
32-6	9	7	63	3	0	-12	9	C R: 3;3;3.
32-7	10	7	70	0	2	10	-4	C R: 4;4.

Scheme #33, L_4, $n = 7$, $v = 49$, $g = 4$, $n_1 = 24$, $f = 4$, $n_2 = 24$, $P_1 = \begin{bmatrix} 11 & 12 \\ 12 & 12 \end{bmatrix}$, $P_2 = \begin{bmatrix} 12 & 12 \\ 12 & 11 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{24}(r - z_2)^{24} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
33-1	8	4	98	0	1	4	-3	
33-2	4	7	28	0	1	4	-3	C Lattice.
33-3	8	7	56	0	2	8	-6	C R: 2;2.
33-4	9	9	49	1	2	5	-2	

Scheme #47, L_2, $n = 8$, $v = 64$, $g = 2$, $n_1 = 14$, $f = 7$, $n_2 = 49$, $P_1 = \begin{bmatrix} 6 & 7 \\ 7 & 42 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 12 \\ 12 & 26 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{14}(r - z_2)^{49} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
47-1	7	7	64	3	0	-18	6	
47-2	2	8	16	1	0	-6	2	C Lattice.
47-3	4	8	32	2	0	-12	4	C R: 2;2.
47-4	6	8	48	3	0	-18	6	C R: 2;2;2.
47-5	7	8	56	0	1	7	-1	C Lattice.
47-6	8	8	64	4	0	-24	8	C R: 2;2;2;2.
47-7	10	8	80	5	0	-30	10	C R: 2;2;2;2;2.

TABLE IV (continued)

Scheme #49 , L_3 , $n = 8$, $v = 64$, $g = 3$, $n_1 = 21$, $f = 6$, $n_2 = 42$, $P_1 = \begin{bmatrix} 8 & 12 \\ 12 & 30 \end{bmatrix}$, $P_2 = \begin{bmatrix} 6 & 15 \\ 15 & 26 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{21}(r - z_2)^{42} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
49-1	7	4	112	1	0	-5	3	
49-2	7	7	64	0	1	6	-2	
49-3	7	7	64	2	0	-10	6	
49-4	3	8	24	1	0	-5	3	C Lattice.
49-5	6	8	48	2	0	-10	6	C R: 4;4.
49-6	6	8	48	0	1	6	-2	C Lattice.
49-7	9	8	72	3	0	-15	9	C R: 4;4;4.

Scheme #51 , L^*_{-3} , $n = -8$, $v = 64$, $g = -3$, $n_1 = 27$, $f = -4$, $n_2 = 36$, $P_1 = \begin{bmatrix} 10 & 16 \\ 16 & 20 \end{bmatrix}$, $P_2 = \begin{bmatrix} 12 & 15 \\ 15 & 20 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{36}(r - z_2)^{27} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
51-1	9	4	144	1	0	-3	5	
51-2	9	9	64	0	2	8	-8	
51-3	10	10	64	2	1	-2	6	

Scheme #52 , L_4 , $n = 8$, $v = 64$, $g = 4$, $n_1 = 28$, $f = 5$, $n_2 = 35$, $P_1 = \begin{bmatrix} 12 & 15 \\ 15 & 20 \end{bmatrix}$, $P_2 = \begin{bmatrix} 12 & 16 \\ 16 & 18 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{28}(r - z_2)^{35} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
52-1	4	8	32	1	0	-4	4	C Lattice.
52-2	5	8	40	0	1	5	-3	C Lattice.
52-3	8	8	64	2	0	-8	8	C R: 1;1.
52-4	10	8	80	0	2	10	-6	C R: 2;2.

TABLE IV (continued)

Scheme #67 , L_2 , $n = 9$, $v = 81$, $g = 2$, $n_1 = 16$, $f = 8$, $n_2 = 64$, $P_1 = \begin{bmatrix} 7 & 8 \\ 8 & 56 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 14 \\ 14 & 49 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{16}(r - z_2)^{64} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
67-1	8	3	216	1	0	-7	2	
67-2	2	9	18	1	0	-7	2	C Lattice.
67-3	4	9	36	2	0	-14	4	C R: 2;2.
67-4	6	9	54	3	0	-21	6	C R: 2;2;2.
67-5	8	9	72	4	0	-28	8	C R: 2;2;2;2.
67-6	8	9	72	0	1	8	-1	C Lattice.
67-7	10	9	90	5	0	-35	10	C R: 2;2;2;2;2.

Scheme #68 , L^*_{-2} , $n = -9$, $v = 81$, $g = -2$, $n_1 = 20$, $f = -6$, $n_2 = 60$, $P_1 = \begin{bmatrix} 1 & 18 \\ 18 & 42 \end{bmatrix}$, $P_2 = \begin{bmatrix} 6 & 14 \\ 14 & 45 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{60}(r - z_2)^{20} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
68-1	10	3	270	1	0	-2	7	

Scheme #69 , L_3 , $n = 9$, $v = 81$, $g = 3$, $n_1 = 24$, $f = 7$, $n_2 = 56$, $P_1 = \begin{bmatrix} 9 & 14 \\ 14 & 42 \end{bmatrix}$, $P_2 = \begin{bmatrix} 6 & 18 \\ 18 & 37 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{24}(r - z_2)^{56} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
69-1	8	4	162	1	0	-6	3	
69-2	8	8	81	0	1	7	-2	
69-3	3	9	27	1	0	-6	3	C Lattice.
69-4	6	9	54	2	0	-12	6	C R: 3;3.
69-5	7	9	63	0	1	7	-2	C Lattice.
69-6	9	9	81	3	0	-18	9	C R: 3;3;3.

TABLE IV (continued)

Scheme #70 , L^*_{-3} , $n = -9$, $v = 81$, $g = -3$, $n_1 = 30$, $f = -5$, $n_2 = 50$, $P_1 = \begin{bmatrix} 9 & 20 \\ 20 & 30 \end{bmatrix}$, $P_2 = \begin{bmatrix} 12 & 18 \\ 18 & 31 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{50}(r - z_2)^{30} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
70-1	10	3	270	1	0	-3	6	
70-2	10	5	162	0	1	4	-5	

Scheme #71 , L_4 , $n = 9$, $v = 81$, $g = 4$, $n_1 = 32$, $f = 6$, $n_2 = 48$, $P_1 = \begin{bmatrix} 13 & 18 \\ 18 & 30 \end{bmatrix}$, $P_2 = \begin{bmatrix} 12 & 20 \\ 20 & 27 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{32}(r - z_2)^{48} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
71-1	4	9	36	1	0	-5	4	C Lattice.
71-2	6	9	54	0	1	6	-3	C Lattice.
71-3	8	9	72	2	0	-10	8	C R: 1;1.

Scheme #73 , L_5 , $n = 9$, $v = 81$, $g = 5$, $n_1 = 40$, $f = 5$, $n_2 = 40$, $P_1 = \begin{bmatrix} 19 & 20 \\ 20 & 20 \end{bmatrix}$, $P_2 = \begin{bmatrix} 20 & 20 \\ 20 & 19 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{40}(r - z_2)^{40} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
73-1	10	5	162	1	0	-4	5	
73-2	5	9	45	1	0	-4	5	C Lattice.
73-3	10	9	90	2	0	-8	10	C R: 2;2.

TABLE IV (continued)

Scheme #93 , L_2 , $n = 10$, $v = 100$, $g = 2$, $n_1 = 18$, $f = 9$, $n_2 = 81$, $P_1 = \begin{bmatrix} 8 & 9 \\ 9 & 72 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 16 \\ 16 & 64 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{18}(r - z_2)^{81} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
93-1	9	3	300	0	1	-8	2	
93-2	6	4	150	0	1	-8	2	
93-3	9	5	180	0	2	-16	4	
93-4	9	9	100	0	4	-32	8	
93-5	2	10	20	0	1	-8	2	C Lattice.
93-6	4	10	40	0	2	-16	4	R: 5;5.
93-7	6	10	60	0	3	-24	6	R: 5;5;5.
93-8	8	10	80	0	4	-32	8	R: 5;5;5;5.
93-9	9	10	90	1	0	9	-1	
93-10	10	10	100	0	5	-40	10	R: 5;5;5;5;5.

Scheme #94 .

No designs possible with $r \leq 10$.

Scheme #95 , L_3 , $n = 10$, $v = 100$, $g = 3$, $n_1 = 27$, $f = 8$, $n_2 = 72$, $P_1 = \begin{bmatrix} 10 & 16 \\ 16 & 56 \end{bmatrix}$, $P_2 = \begin{bmatrix} 6 & 21 \\ 21 & 50 \end{bmatrix}$,

$$NN' = rk(r - z_1)^{27}(r - z_2)^{72} .$$

#	r	k	b	λ_1	λ_2	z_1	z_2	Remarks
95-1	9	4	225	1	0	-7	3	
95-2	9	9	100	0	1	8	-2	
95-3	3	10	30	1	0	-7	3	C Lattice.
95-4	6	10	60	2	0	-14	6	C R: 3;3.
95-5	8	10	80	0	1	8	-2	
95-6	9	10	90	3	0	-21	9	C R: 3;3;3.

A.3. Construction of Two Particular Designs; Impossibility Proofs of Particular Designs.

CONSTRUCTION OF DESIGNS #6-7 and 6-13

Reference is made to these designs in Section 2.3 and Table IV. The construction of these designs involved a good deal of enumeration of possible blocks and will not be described in detail. Both designs depend on negative Latin square association scheme #6, which is constructed in Section 3.2 and is reproduced here for reference.

Treatment	First associates				
0	1	8	10	12	15
1	0	9	11	13	14
2	3	8	10	13	14
3	2	9	11	12	15
4	5	8	11	12	14
5	4	9	10	13	15
6	7	9	10	12	14
7	6	8	11	13	15
8	0	2	4	7	9
9	1	3	5	6	8
10	0	2	5	6	11
11	1	3	4	7	10
12	0	3	4	6	13
13	1	2	5	7	12
14	1	2	4	6	15
15	0	3	5	7	14

Design #6-7 has parameter values

$$v = 16 , r = 5 , k = 5 , b = 16 , \lambda_1 = 2 , \lambda_2 = 1 .$$

The blocks of the design are the following:

0	1	2	12	13	2	3	7	14	15
0	1	3	10	11	2	2	8	9	13
0	4	8	12	14	2	4	5	8	10
0	5	9	10	15	2	6	10	11	14
0	6	7	8	15	3	4	11	12	15
1	4	6	9	14	3	5	6	9	12
1	5	13	14	15	4	5	7	11	13
1	7	8	9	11	6	7	10	12	13

Design #6-13 has parameter values

$$v = 16 , r = 5 , k = 8 , b = 10 , \lambda_1 = 1 , \lambda_2 = 3 .$$

The blocks of the design are the following:

0	1	2	3	4	5	6	7
0	2	4	6	9	11	13	15
0	2	5	7	9	11	12	14
0	3	4	7	9	10	13	14
0	3	5	6	8	11	13	14
1	2	4	7	9	10	12	15
1	2	5	6	8	11	12	15
1	3	4	6	8	10	13	15
1	3	5	7	8	10	12	14
8	9	10	11	12	13	14	15

PROOF OF IMPOSSIBILITY OF DESIGN #20-2

This design has parameter values

$$v = 36 , r = 5 , k = 3 , b = 60 , \lambda_1 = 1 , \lambda_2 = 0 .$$

The design is based on an L_2 association scheme which by Theorem 4.2 is unique and may be assumed to be defined by the following array, treatments occurring in the same row or the same column being taken as first associates.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

It follows from the values of λ_1 and λ_2 that each block containing treatment 1 must contain a pair of its first associates which are first associates of each other. Notation may be chosen so that two of the blocks are 1 2 3 and 1 4 5. The pair of first associates 1, 6 must then occur together in a block. It is impossible to choose a third treatment for this block which is a common first associate of treatments 1 and 6 and has not already been used in a block with treatment 1. Therefore the design cannot be constructed.

PROOF OF IMPOSSIBILITY OF DESIGN #30-2

This design has parameter values

$$v = 49 , r = 4 , k = 4 , b = 49 , \lambda_1 = 1 , \lambda_2 = 0 .$$

The design is based on an L_2 association scheme which by Theorem 4.2 is unique and may be assumed to be assumed to be defined by the following array, treatments occurring in the same row of the same column being taken as first associates.

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

Treatments 1 to 7 are pairwise first associates, and no two treatments of this set have common first associates which are not in the set. It follows from the values of λ_1 and λ_2 that the four treatments in a block must be pairwise first associates and that no two first associates can occur together in more than one block. Notation can be chosen so that two of the blocks containing treatment 1 are 1 2 3 4 and 1 5 6 7 . The pair of first associates 2 , 5 must then occur together in a block. It is impossible to choose further treatments for this block which are common first treatments of treatments 2 and 5 and have not already been used in a block with one of them. Therefore the design cannot be constructed.

PROOF OF IMPOSSIBILITY OF DESIGN #93-1

This design has parameter values

$$v = 100 , r = 9 , k = 3 , b = 300 , \lambda_1 = 1 , \lambda_2 = 0 .$$

The design is based on an L_2 association scheme which by Theorem 4.2 is unique and may be assumed to be defined by a 10 x 10 array of the integers from 1 to 100 , treatments occurring in the same row or the same column being taken as first associates. The first row may be taken to contain the numbers from 1 to 10; these treatments will then be pairwise first associates and no two of them will have any common first associates not in this set. It follows from the values of λ_1 and λ_2 that each treatment containing treatment 1 must contain a pair of its first associates which are first associates of each other. Notation may be chosen so that four of the blocks are 1 2 3, 1 4 5, 1 6 7 and 1 8 9. The pair of first associates 1 and 10 must then occur together in a block. It is impossible to choose a third treatment for this block which is a common first associate of treatment 1 and 10 and has not already been used in a block with treatment 1. Therefore the design cannot be constructed.

DISCUSSION OF DESIGNS #7-3 AND 12-2

Design #7-3 has parameter values

$$v = 16 , r = 3 , k = 3 , b = 16 , \lambda_1 = 1 , \lambda_2 = 0 .$$

The design may be based on an L_2 association scheme defined by the array

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}$$

and if so is easily shown to be impossible, using the method applied to designs #20-2 and #93-1 . However, it is shown in Section 4.1 that the association scheme is not unique and a reference is given in Table IV to a published design which differs from this one only in the designation of first and second associates.

Design #12-2 has parameter values

$$v = 25 , r = 4 , k = 4 , b = 25 , \lambda_1 = 1 , \lambda_2 = 0 .$$

The association scheme of this design has L_3 parameter values and may be defined by a 5 x 5 array of the numbers from 1 to 25 , superimposed on a 5 x 5 Latin square. A proof that the design is impossible with this association scheme must include an investigation of different possible Latin squares, and is longer than the preceding proofs of impossibility. The proof can be carried out but will not be presented here. It is shown in Section 4.1 that this L_3 association scheme is not unique, and is pointed out in Section 2.3 that this proof of impossibility is therefore not conclusive.

A.4. List of Negative Latin Square Association Schemes.

Five association schemes of Negative Latin square type are constructed in Chapter III. The first of these, #6 of Table II, is described fully in Section 3.2 and repeated in Appendix A. #3. Another, #94 of Table II, is described in Section 3.3 and may be constructed from data given there. A simple method of constructing the remaining schemes will now be given.

ASSOCIATION SCHEME # 51

The parameter values of scheme #51 include $v = 64$ and $n_1 = 27$. The treatments will be represented by pairs (x, y) of marks of the finite field of order 8. The addition table of the field follows.

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Addition of pairs of marks is defined by

$$(x, y) + (z, w) = (x + z, y + w).$$

The first associates of any treatment (x, y) are obtained by adding (x, y) to each of the first associates of $(0, 0)$. The 27 first associates of $(0, 0)$ are listed below, with commas and parentheses omitted for brevity.

0 1	0 2	0 5	0 6	0 7	1 3	1 4
2 2	2 4	2 6	2 7	3 2	3 4	3 6
3 7	4 0	4 7	5 0	5 7	6 1	6 4
6 5	6 7	7 0	7 2	7 3	7 6	

ASSOCIATION SCHEMES #68 and #70

Each of these schemes is based on 81 treatments, which will be represented by pairs (x , y) of marks of the finite field of order 9 . The addition table of the field follows.

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	7	8	0	1	2
4	4	5	3	7	8	6	1	2	0
5	5	3	4	8	6	7	2	0	1
6	6	7	8	0	1	2	3	4	5
7	7	8	6	1	2	0	4	5	3
8	8	6	7	2	0	1	5	3	4

Addition of pairs of marks is defined as for Scheme #51.

Each treatment in scheme #68 has $n_1 = 20$ first associates. The first associates of any treatment (x , y) are obtained by adding (x , y) to each of the first associates of (0 , 0) , which are given in the following list.

0 1	0 2	0 4	0 8	1 5	1 7	2 5
2 7	3 1	3 7	4 1	4 7	5 2	5 6
6 2	6 5	7 1	7 3	8 2	8 5	

Each treatment in scheme #70 has $n_1 = 30$ first associates. The first associates of any treatment (x , y) are obtained by adding (x , y) to each of the first associates of (0 , 0) , which are given in the following list.

01	02	03	06	12	15	16	17
21	23	25	27	37	40	44	45
46	53	54	56	57	65	73	75
76	78	80	83	87	88		

A.5. Fifteen Squares with Special Orthogonality Properties.

Any two of the following 4×4 square arrays which have a common number in their identifying symbols /i,j/ are orthogonal. That is, if the two squares are superimposed every ordered pair of the letters A , B , C and D occurs in exactly one of the sixteen positions. The construction of these squares is described in Section 3.3, page 149.

A	A	A	A
B	B	B	B
C	C	C	C
D	D	D	D

/1,2/

A	B	C	D
C	D	A	B
B	A	D	C
D	C	B	A

/1,3/

A	B	C	D
D	C	B	A
C	D	A	B
B	A	D	C

/1,4/

A	B	C	D
B	A	D	C
D	C	B	A
C	D	A	B

/1,5/

A	B	C	D
A	B	C	D
A	B	C	D
A	B	C	D

/1,6/

A	B	C	D
D	C	B	A
D	C	B	A
A	B	C	D

/2,3/

A	B	C	D
C	D	A	B
A	B	C	D
C	D	A	B

/2,4/

A	B	C	D
A	B	C	D
B	A	D	C
B	A	D	C

/2,5/

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

/2,6/

A	A	B	B
A	A	B	B
C	C	D	D
C	C	D	D

/3,4/

A	B	A	B
C	D	C	D
A	B	A	B
C	D	C	D

/3,5/

A	B	B	A
C	D	D	C
D	C	C	D
B	A	A	B

/3,6/

A	B	B	A
C	D	D	C
C	D	D	C
A	B	B	A

/4,5/

A	B	A	B
C	D	C	D
B	A	B	A
D	C	D	C

/4,6/

A	A	B	B
B	B	A	A
C	C	D	D
D	D	C	C

/5,6/

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